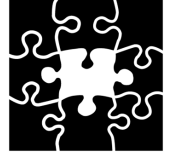




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Infinite Games

Set Theory, Part III

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Exercise 5.

If $A \subseteq \omega^\omega$ and $p \in \omega^{<\omega}$ we can consider the **subgame starting at p** as $A_p := \{x \in \omega^\omega ; p \frown x \in A\}$. If σ is a strategy for player I and $p \in \omega^{<\omega}$ is a position of odd length $2n + 1$, we say that **player I deviated from strategy σ in the position p** if $p(2n) \neq \sigma(p \upharpoonright 2n)$. Construct a set C such that player I has a winning strategy in $G(C)$, but for all positions in which player I deviated from σ , C_p is a non-determined set.

Exercise 6.

Consider two arbitrary open sets P and Q . In this exercise, we wish to investigate the game-theoretic properties of $P \setminus Q$. Consider the following two auxiliary games:

- (1) In the game $G_{3,1}(P)$, player I plays triples of natural numbers $\langle x_0, x_1, x_2 \rangle, \langle x_4, x_5, x_6 \rangle, \dots, \langle x_{4n}, x_{4n+1}, x_{4n+2} \rangle$ and player I produces natural numbers x_{4n+3} . Together, they produce a function x , and player I wins if $x \in P$.
- (2) In the game $G_{1,3}(Q)$, player II plays triples of natural numbers $\langle x_1, x_2, x_3 \rangle, \langle x_5, x_6, x_7 \rangle, \dots, \langle x_{4n+1}, x_{4n+2}, x_{4n+3} \rangle$ and player II produces natural numbers x_{4n} . Together, they produce a function x , and player II wins if $x \in Q$.

Argue that both of these games can be given in the form of a game $G(A)$ by giving an appropriate payoff set A . (**Hint.** Use a bijection between $\omega \times \omega \times \omega$ and ω .) Prove that both of these games are open.

Prove that if player I has no winning strategy in $G_{3,1}(P)$, then player II wins $G(P \setminus Q)$. If player I has a winning strategy in both $G_{3,1}(P)$ and $G_{1,3}(Q)$, then player I wins $G(P \setminus Q)$.

Exercise 7.

Consider the following unravelling of a graph game: if G is a (countable) graph with distinguished vertex s , let $\pi : G \rightarrow \omega$ be an injection. Call a (finite or infinite) sequence $x \in \omega^\omega$ an **s -path through G** if $x(0) = \pi(s)$, for all n such that $x(n)$ is defined, there is a vertex v such that $x(n) = \pi(v)$, and if $x(n) = \pi(v)$ and $x(n+1) = \pi(v^*)$, then there is an edge from v to v^* .

Using the idea of an s -path through G , define a set $A_{G,s}$ such that the graph game on G starting at s is equivalent to $G(A_{G,s})$ in the following sense: for every G and s , if player I (II) has a winning strategy in the graph game on G starting at s if and only if player I (II) has a winning strategy in $G(A_{G,s})$.

Prove that the winning condition “the last player to make a legal moves wins” corresponds to an open set and the winning condition of “vertex v is visited infinitely many times” corresponds to a Π_2^0 set.