Infinite Games

Part II: Proving Determinacy

Second Winter School on Logic

14-26 January 2008

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Is every set determined?

Is the class of determined sets trivial? It could be that every set is determined...

Theorem (Banach-Mazur; Gale-Stewart). If there is a wellordering of the set of real numbers, then there is a non-determined set.

In particular, AC implies that there is a non-determined set. The "Axiom of Determinacy" proposed by Mycielski and Steinhaus in 1962 is therefore an *alternative* to the Axiom of Choice.
The connection between non-determinacy and the axiom of choice remains intricate: we'll discuss this further when we look at the limits of determinacy (Lecture 3).

Proof (1).

Using the well-ordering of the set of real numbers, we give a well-ordered list of all strategic trees $\langle T_\alpha\,;\,\alpha < 2^{\aleph_0}\rangle$.

Remember that “player I has a winning strategy in $\mathrm{G}(A)$” means that for some σ, the tree $T^{\sigma,\mathrm{I}}$ must be contained in A; for “player II has a winning strategy”, some tree $T^{\tau,\mathrm{II}}$ must be contained in the complement of A.

We shall make sure that neither of these can be the case. Using transfinite recursion, we define two sets A and B:

$$A_0 := B_0 := \varnothing$$

$$A_\lambda := \bigcup_{\alpha<\lambda} A_\alpha \quad B_\lambda := \bigcup_{\alpha<\lambda} B_\alpha$$

Proof (2).

$$A_0 := B_0 := \varnothing$$

$$A_\lambda := \bigcup_{\alpha<\lambda} A_\alpha \quad B_\lambda := \bigcup_{\alpha<\lambda} B_\alpha$$

In each successor step, both A and B will gain exactly one element, thus making sure that $\mathrm{Card}(A_\alpha) = \mathrm{Card}(B_\alpha) = \mathrm{Card}(\alpha)$.

For each α, $[T_\alpha]$ has cardinality $2^{\aleph_0} > \mathrm{Card}(\alpha)$, and therefore $[T_\alpha]\backslash(A_\alpha \cup B_\alpha)$ has uncountably many elements. Pick two of them; call them a_α and b_α.

Then let $A_{\alpha+1} := A_\alpha \cup \{a_\alpha\}$ and $B_{\alpha+1} := B_\alpha \cup \{b_\alpha\}$.

Finally

$$A := \bigcup_{\alpha<2^{\aleph_0}} A_\alpha \text{ and } B := \bigcup_{\alpha<2^{\aleph_0}} B_\alpha.$$

Proof (3).

$$A_0 := B_0 := \varnothing \quad A_\lambda := \bigcup_{\alpha<\lambda} A_\alpha \quad B_\lambda := \bigcup_{\alpha<\lambda} B_\alpha$$

$$A_{\alpha+1} := A_\alpha \cup \{a_\alpha\} \text{ and } B_{\alpha+1} := B_\alpha \cup \{b_\alpha\}$$

$$A := \bigcup_{\alpha<2^{\aleph_0}} A_\alpha \text{ and } B := \bigcup_{\alpha<2^{\aleph_0}} B_\alpha .$$

Note that $A \cap B = \varnothing$.

We **claim** that A is not determined. Suppose it was, then either there is some $T^{\sigma,\mathrm{I}} \subseteq A$ or some $T^{\tau,\mathrm{II}} \subseteq \omega^\omega \backslash A$.

In Case 1, find α such that $T^{\sigma,\mathrm{I}} = T_\alpha$. Then $b_\alpha \in [T_\alpha] \cap B$, and so $b_\alpha \notin A$. Contradiction.

In Case 2, find α such that $T^{\tau,\mathrm{II}} = T_\alpha$. Then $a_\alpha \in [T_\alpha] \cap A$, and so $[T_\alpha]$ is not disjoint from A. Contradiction. q.e.d.

Gale-Stewart I (1).

Theorem (Gale-Stewart 1953). If A is a clopen payoff set, then $\mathrm{G}(A)$ is determined.

Proof. If A is clopen, then both A and the complement of A are unions of basic open sets. So, there are sets X and Y of finite sequences such that $A = \{x\,;\, \exists s \in X(s \subseteq x)\}$ and $\omega^\omega \backslash A = \{x\,;\, \exists s \in Y(s \subseteq x)\}$.

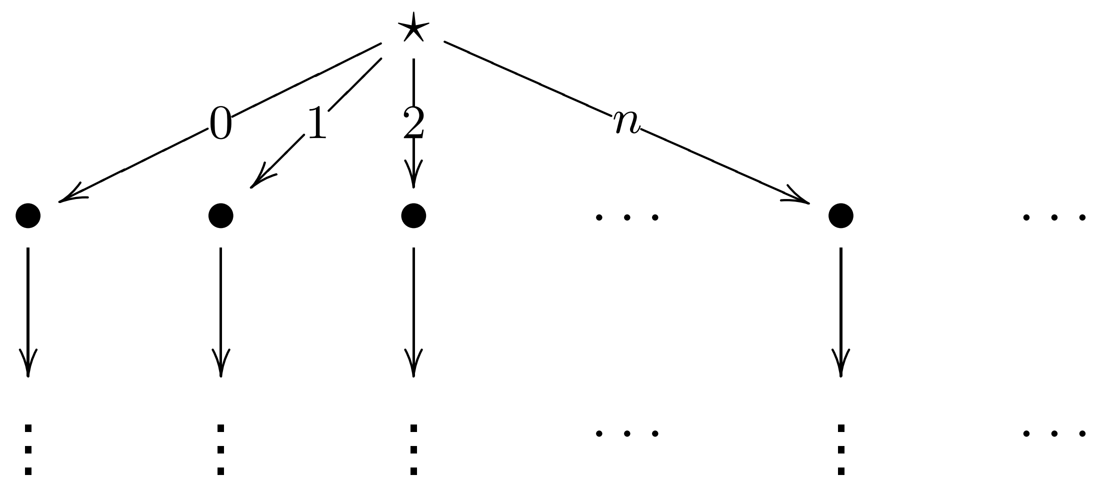
The set $X \cup Y$ has the following “barrier property”: If $x \in \omega^\omega$ then there is some $s \in X \cup Y$ such that $s \subseteq x$.

We label the elements of X (and all their extensions) by I and the elements of Y (and all their extensions) by II and start our backward induction:

- If s is a move for player I (player II) and there is at least one successor labelled I (II), then we label s also with I (II).
- If s is a move for player I (player II) and all successors are labelled II (I), then we label s also with II (I).

Not quite so easy.

Consider the following game: Player I plays a natural number n, after that players I and II alternate, and player II wins if and only if he plays a 0 in his nth move. Obviously, player II has a winning strategy, but let's do the recursion for the labelling:



So, $\star$ won't get labelled in a finite amount of time as we need that all successors are labelled.

Gale-Stewart I (2).

So, we need to extend this process into the transfinite. We start with our initial labelling ℓ_0 and define a recursion as follows:

- $\ell_\lambda := \bigcup_{\alpha<\lambda} \ell_\alpha$
- $\ell_{\alpha+1}(s) := \mathrm{I}$ (II) if s is a move for player I (II) and at least one successor t of s has the property $\ell_\alpha(s) = \mathrm{I}$ (II).
- $\ell_{\alpha+1}(s) := \mathrm{I}$ (II) if s is a move for player II (I) and all t of s have the property $\ell_\alpha(s) = \mathrm{I}$ (II).

Note that the domains of the partial labellings are increasing, i.e., $\mathrm{dom}(\ell_\alpha) \subseteq \mathrm{dom}(\ell_{\alpha+1})$. As a consequence, there must be a countable ordinal ζ that is a fixed point of this procedure, i.e., $\ell_\zeta = \ell_{\zeta+1}$.

Gale-Stewart I (3).

Barrier Property: If $x \in \omega^\omega$ then there is some $s \in X \cup Y$ such that $s \subseteq x$.

$\ell_{\alpha+1}(s) := \mathrm{I}$ (II) if s is a move for player I (II) and at least one successor t of s has the property $\ell_\alpha(s) = \mathrm{I}$ (II).

$\ell_{\alpha+1}(s) := \mathrm{I}$ (II) if s is a move for player II (I) and all t of s have the property $\ell_\alpha(s) = \mathrm{I}$ (II).

Claim 1. If $s \notin \mathrm{dom}(\ell_\zeta)$, then there is a successor t of s such that $t \notin \mathrm{dom}(\ell_\zeta)$.

Claim 2. ℓ_ζ is a total function.

[Suppose not, then $\ell_\zeta(s)$ is not defined. By Claim 1, there must be an infinite sequence x such that $\ell_\zeta(x{\upharpoonright}n)$ is not defined for all $n \geq \mathrm{lh}(s)$. But by the barrier property, there must be some n such that $x{\upharpoonright}n \in X \cup Y$. Contradiction!]

Claim 3. If $\ell_\zeta(\varnothing) = \mathrm{I}$ (II), then there is a strategy for player I (II) that guarantees that all positions of the run of the game are labelled I (II).

Gale-Stewart I (4).

Claim 4. Any infinite sequence whose positions are all labelled I (II) is a win for player I (II).

[Again, this is an application of the barrier property: There is some n such that $x{\restriction}n \in X \cup Y$.]

We have established in a constructive way that $\mathrm{G}(A)$ is determined. By Claim 2, $\ell_\zeta(\varnothing)$ is defined and thus is either I or II. By Claim 3, the player who owns the label has a strategy to stay on his labels; and by Claim 4, this is a winning strategy. q.e.d.

Gale-Stewart II (1).

Theorem (Gale-Stewart). If A is open, then $\mathrm{G}(A)$ is determined.

Proof. Now, $A = \bigcup_{s \in X}[s]$, but the complement may not be open. We just do the same procedure with the limited information we have at hand. We let $\ell_0(t) = \mathrm{I}$ if there is an $s \in X$ and $t \supseteq s$, and then run the Gale-Stewart procedure:

- $\ell_\lambda := \bigcup_{\alpha<\lambda} \ell_\alpha$
- $\ell_{\alpha+1}(s) := \mathrm{I}$ if s is a move for player I and at least one successor t of s has the property $\ell_\alpha(s) = \mathrm{I}$.
- $\ell_{\alpha+1}(s) := \mathrm{I}$ if s is a move for player II and all t of s have the property $\ell_\alpha(s) = \mathrm{I}$.

Gale-Stewart II (2).

$\ell_{\alpha+1}(s) := \mathrm{I}$ if s is a move for player I and at least one successor t of s has the property $\ell_\alpha(s) = \mathrm{I}$.
$\ell_{\alpha+1}(s) := \mathrm{I}$ if s is a move for player II and all t of s have the property $\ell_\alpha(s) = \mathrm{I}$.

Again, the procedure reaches a fixed point ℓ_ζ, and again, we have Claim 1.

Claim 1. If $s \notin \mathrm{dom}(\ell_\zeta)$, then there is a successor t of s such that $t \notin \mathrm{dom}(\ell_\zeta)$.

Even stronger now: If player I has to move at s, and $s \notin \mathrm{dom}(\ell_\zeta)$, then no successors of s are in $\mathrm{dom}(\ell_\zeta)$.

But we cannot deduce that ℓ_ζ is total, as this relied on the barrier property. Define

$$\ell^*(s) := \begin{cases} \ell_\zeta(s) & \text{if } s \in \mathrm{dom}(\ell_\zeta), \text{ and} \\ \mathrm{II} & \text{if } s \notin \mathrm{dom}(\ell_\zeta). \end{cases}$$

Gale-Stewart II (3).

Claim 1*. If $s \notin \mathrm{dom}(\ell_\zeta)$, then there is a successor t of s such that $t \notin \mathrm{dom}(\ell_\zeta)$. If player I has to move at s, then no successors of s are in $\mathrm{dom}(\ell_\zeta)$.

$$\ell^*(s) := \begin{cases} \ell_\zeta(s) & \text{if } s \in \mathrm{dom}(\ell_\zeta), \text{ and} \\ \mathrm{II} & \text{if } s \notin \mathrm{dom}(\ell_\zeta). \end{cases}$$

With this, we again have

Claim 3. If $\ell^*(\varnothing) = \mathrm{I}$ (II), then there is a strategy for player I (II) that guarantees that all positions of the run of the game are labelled I (II).

Are all such strategies winning? Yes for player II: If a strategy stays on label II producing x, in particular it never hits an element of X, and thus $x \notin A$, so player II wins.

Again, not quite so easy.

Consider the game $A = \{x\,;\,\exists n(x(2n) \neq 0)\}$. The set A is open. All nodes $s1$ of odd length are labelled I in the initial labelling ℓ_0. Then all nodes s of even length get labelled I in ℓ_1, and thus all nodes of odd length get labelled I in ℓ_2 which is the fixed point of the procedure.

Therefore, the strategy “play 0” for player I has the property that it stays on label I. But obviously, it is a losing strategy.

Gale-Stewart II (4).

Claim 3. If $\ell^*(\varnothing) = \mathrm{I}$ (II), then there is a strategy for player I (II) that guarantees that all positions of the run of the game are labelled I (II).

$\ell_{\alpha+1}(s) := \mathrm{I}$ if s is a move for player I and at least one successor t of s has the property $\ell_\alpha(s) = \mathrm{I}$.

$\ell_{\alpha+1}(s) := \mathrm{I}$ if s is a move for player II and all t of s have the property $\ell_\alpha(s) = \mathrm{I}$.

We have to introduce the **index** of a position: this is the least α such that $\ell_\alpha(s)$ is defined (if there is such an α).

We observe that if if $\ell^*(s) = \mathrm{I}$ and player I has to play, then there is a successor of lower index with label I (unless the index of s is 0), and if player II has to play, then all successors are of lower index (unless the index of s is 0).

So, if $\ell^*(\varnothing) = \mathrm{I}$, then player I has a strategy that forces the labels to be I and that forces the sequence of indices to be a decreasing sequence of ordinals (i.e., either $\mathrm{ind}(x{\upharpoonright}n+1) < \mathrm{ind}(x{\upharpoonright}n)$ or $\mathrm{ind}(x{\upharpoonright}n+1) = \mathrm{ind}(x{\upharpoonright}n) = 0$).

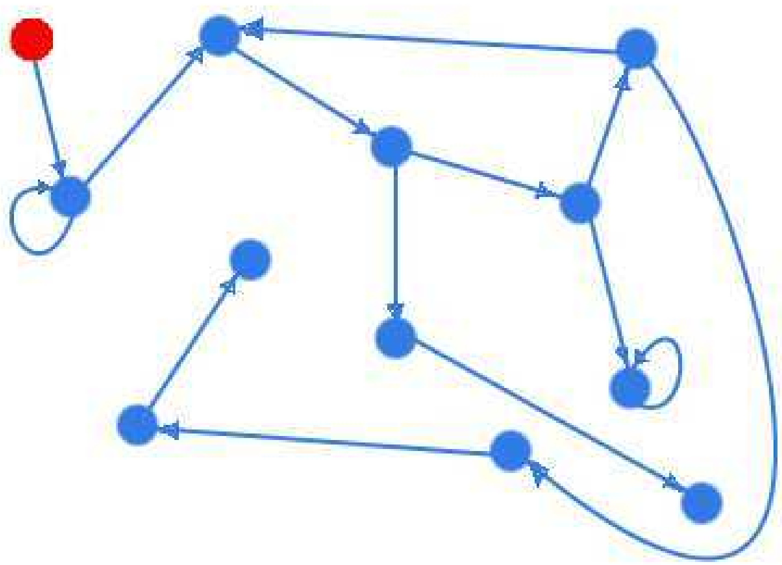
Gale-Stewart II (5).

So, if $\ell^*(\varnothing) = \mathrm{I}$, then player I has a strategy that forces the labels to be I and that forces the sequence of indices to be a decreasing sequence of ordinals (i.e., either $\mathrm{ind}(x{\upharpoonright}n+1) < \mathrm{ind}(x{\upharpoonright}n)$ or $\mathrm{ind}(x{\upharpoonright}n+1) = \mathrm{ind}(x{\upharpoonright}n) = 0$.

Now let x be a play according to that strategy. Since there is no infinite decreasing sequence of ordinals, we know that there must be some n such that the index of $x{\upharpoonright}n$ is 0, but then $\ell_0(x{\upharpoonright}n) = \mathrm{I}$. But that means that $x{\upharpoonright}n \in X$, and that $x \in A$. So the strategy is a winning strategy. q.e.d.

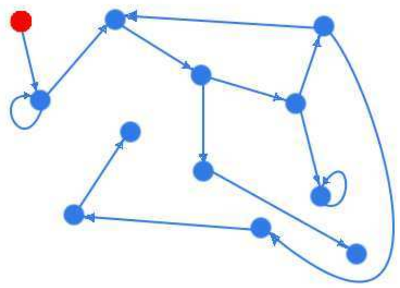
Graph games (1).

Take a directed graph, specify a vertex as the initial node and play a game in which players I and II push a token along the edges.



Such a game can easily be transferred into a game on Baire space by just labelling the vertices of the graph with natural numbers (the “tree unravelling” of a graph).

Graph games (2).

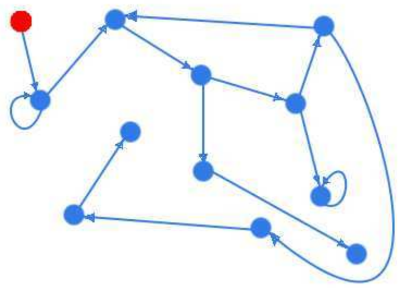


Typical winning conditions:

- “the player who makes the last (legal) move wins”.
- Player I wins if vertex v is visited.
- Player I wins if vertex v is visited n times.
- Player I wins if vertex v is visited infinitely many times.

If you unravel the trees of these games, the first three conditions give open payoffs.

Graph games (3).



"Player I wins if vertex v is visited infinitely many times."

Remember our example: $P := \bigcap_{n\in\omega} P_n$ is the set of all sequences that contain infinitely many zeros. This was a set which was neither open nor closed.

Similarly, the unravelled game for our graph game will produce a $\mathbf{\Pi}^0_2$ set which is neither closed nor open.

Thus: The Gale-Stewart theorem is not enough to deal with these games.

Extensions of Gale-Stewart.

- **Philip Wolfe** (1955). Every Σ^0_2 set is determined.
- **Morton Davis** (1963). Every Σ^0_3 set is determined.
- **Jeff Paris** (1972). Every Σ^0_4 set is determined.
- **Tony Martin** (1975). Every Borel set is determined.
- Where are the limits of determinacy?