Infinite Games

Part III: Using Determinacy

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Overview

Tutorial 1. History and Basics.
Tutorial 2. Proving Determinacy.
Tutorial 3. Using Determinacy.

- The limits of determinacy; Projective Determinacy
- The Continuum Problem
- Uniformization

The Axiom of Determinacy.

The Mycielski-Steinhaus Axiom of Determinacy.

Jan Mycielski 

AD: “All games $\mathrm{G}(A)$ are determined.”

We have already proved that AC implies $\neg$AD, so AD is necessarily an “alternative” to AC.

The Projective Hierarchy

Reminder:

The projective hierarchy $\mathbf{\Sigma}^1_1$, $\mathbf{\Pi}^1_1$, $\mathbf{\Sigma}^1_2$, $\mathbf{\Pi}^1_2$, ... builds on top of the Borel sets. In fact, $\mathbf{\Sigma}^1_1$ and $\mathbf{\Pi}^1_1$ are the smallest families of sets closed under continuous reduction that contain all of the Borel sets.

After Martin's proof of Borel determinacy, this suggests the following question:

Are all $\mathbf{\Pi}^1_1$ sets determined?

Applications of determinacy.

If $\mathbf{\Gamma}$ is any boldface pointclass. Suppose that all sets in $\mathbf{\Gamma}$ are determined. Then:

- All sets in $\mathbf{\Gamma}$ are Lebesgue measurable. (Mycielski-Swierczkowski)
- All sets in $\mathbf{\Gamma}$ have the Baire property. (Banach-Mazur)
- All sets in $\mathbf{\Gamma}$ have the perfect set property. (Morton Davis)

Cantor's Continuum Problem (1).

First problem in Hilbert's list (1900).

The Continuum Hypothesis (CH). Every set of real numbers is either finite or countable or has the cardinality of the set of all real numbers.

Lemma. If the Axiom of Choice holds, then CH is equivalent to "there is a bijection between $\aleph_1$ and the set of real numbers (in short: $2^{\aleph_0} = \aleph_1$).

Proof. $\Leftarrow$ is obvious.

"$\Rightarrow$": By the Axiom of Choice, there is a bijection π between some ordinal α and $\mathbb{R}$. We only have to show that $\mathrm{Card}(\alpha) = \aleph_1$.

If $\mathrm{Card}(\alpha) < \aleph_1$, then the set of reals would be countable, contradicting Cantor's theorem.

If $\mathrm{Card}(\alpha) > \aleph_1$, then $\alpha \subseteq \aleph_1$, and we look at $X := \pi[\alpha] \subseteq \mathrm{R}$. Clearly, $\mathrm{Card}(X) = \aleph_1 < \mathrm{Card}(\mathbb{R})$. Contradiction. q.e.d.

Cantor's Continuum Problem (2).

One approach to solving Cantor's Continuum Problem was the perfect set property.

A tree is called perfect if any node has two incompatible extensions.

A set of real numbers has the perfect set property if it is either finite or countable or contains the branches through a perfect tree.

Cantor’s Continuum Problem (3).

Observation. If a set of reals has the perfect set property, it cannot be a counterexample to the Continuum Hypothesis.

Corollary. If all sets have the perfect set property, then CH is true.

Theorem (Bernstein). The Axiom of Choice implies that there is a set without the perfect set property.

The perfect set theorem (1).

Theorem. If all sets in Γ are determined, then all sets in Γ have the perfect set property.

Weaker Theorem. If all sets are determined, then all sets have the perfect set property.

(The difference between the “Theorem” and the “Weaker Theorem” is just an analysis of the complexity of the game.)

Proof. We construct a game that encapsules the perfect set property of a set A. For technical reasons, we play on the binary branching tree (Cantor space).

$$\begin{array}{lccccccc} \textbf{Player I} & s_0 & & s_1 & & s_2 & & \dots \\ \textbf{Player II} & & x_0 & & x_1 & & x_2 & \dots \end{array}$$

where s_i are finite sequences of bits and x_i are binary bits. We construct $x := s_0x_0s_1x_1s_2x_2\dots$, and say that I wins if $x \in A$.

The perfect set theorem (2).

$$\begin{array}{lccccccc} \textbf{PlayerI} & s_0 & & s_1 & & s_2 & & \ldots \\ \textbf{PlayerII} & & x_0 & & x_1 & & x_2 & \ldots \end{array}$$

where s_i are finite sequences of bits and x_i are binary bits. We construct $x := s_0 x_0 s_1 x_1 s_2 x_2 \ldots$, and say that I wins if $x \in A$.

- This game is called $\mathrm{G}^*(A)$, the asymmetric game, due to Morton Davis.
- If all games of the type $\mathrm{G}(A)$ are determined, then all games of the type $\mathrm{G}^*(A)$ are determined.
- Rephrasing the notion of a strategic tree, we still get: If player I was a winning strategy in $\mathrm{G}^*(A)$, then A must contain a perfect set.
- If A is countable, then player II has a winning strategy.

We need to prove the converse of the last statement.

The perfect set theorem (3).

Fix a winning strategy for player II, call it τ. If $p = \langle s_0, x_0, s_1, x_1, ..., s_n, x_n \rangle$ is a position for player II, we write $p_* t$ for $s_0 x_0 s_1 x_1 ... s_n x_n t$. If $x \in 2^\omega$, we say p kills x if for all t, we have that $p_* t \tau(p_* t) \not\subseteq x$.

Observation 1. Each p kills at most one sequence x.

Observation 2. Every $x \in A$ is killed by a sequence p.

But there are only countably many sequences, so A is a countable set. q.e.d.

Non-Extensions of Determinacy.

There is a minimal model of set theory: Gödel's $\mathbf{L}$, the constructible universe. In $\mathbf{L}$, there is a wellordering of the continuum definable in a Δ^1_2 way.

Theorem (Gödel). $\mathbf{L} \models$ "There is a $\mathbf{\Pi}^1_1$ set without the perfect set property."

Corollary. In $\mathbf{L}$, there must be a non-determined $\mathbf{\Pi}^1_1$ set.

Proof. By the perfect set theorem, if every $\mathbf{\Pi}^1_1$ set was determined, then every $\mathbf{\Pi}^1_1$ set would have the perfect set property, but that contradicts Gödel's theorem. q.e.d.

Corollary. The determinacy of all coanalytic sets cannot be proved in ZFC.

Foundations of Mathematics (1).

Gödel Incompleteness phenomenon: ZFC is not complete, i.e., there are statements independent of ZFC. Even worse, there are interesting questions independent of ZFC: the Continuum Hypothesis.

Gödel's Programme: Find further axioms for set theory that are accepted by all mathematicians than resolve all interesting questions.

Gödel's main candidate for these axioms were "Axioms of Strong Infinity", also called Large Cardinal Axioms. The most famous of these is the notion of a measurable cardinal: a cardinal κ that carries a two-valued measure measuring all subsets of κ.

Foundations of Mathematics (2).

A measurable cardinal κ is a cardinal that carries a two-valued measure measuring all subsets of κ.

These cardinals are very big. For instance, you can show that for all $\lambda < \kappa$, $2^\lambda < \kappa$. In particular, $\kappa > 2^{\aleph_0}$, $\kappa > 2^{2^{\aleph_0}}$, etc. A cardinal with this property ($\forall\lambda(2^\lambda < \kappa$) is called a **strong limit** cardinal. The first such cardinal is the limit of the sequence $\lambda_0 := \omega$, $\lambda_{n+1} := 2^{\lambda_n}$. It has cofinality ω and is thus singular. A measurable cardinal is a regular strong limit.

Proposition. If κ is measurable, then $\{\lambda < \kappa\,;\,\lambda$ is a regular strong limit cardinal$\}$ has cardinality κ.

Proposition. If λ is the least regular strong limit cardinal, then there is a model of set theory with no measurable cardinals. In particular, ZFC does not prove the existence of measurable cardinals.

Foundations of Mathematics (3).

Let us denote the statement “there is a measurable cardinal” by MC.

- ZFC (unless inconsistent) doesn’t prove that there is a measurable cardinal.
- ZFC+MC proves Cons(ZFC).
- ZFC+MC proves that all co-analytic sets are determined (Martin, 1970).

Martin's Theorem: Π^1_1 Determinacy.

ZFC+MC proves that all co-analytic sets are determined (Martin, 1970).

Let κ be a measurable cardinal. Using results from descriptive set theory, we can give a tree representation for $\mathbf{\Pi}^1_1$ sets: a set A is $\mathbf{\Pi}^1_1$ if and only if there is a tree $T \subseteq (\omega \times \kappa)^{<\omega}$ such that A is the projection of $[T]$ to ω^ω. Using the measure on κ, we can define a coherent system of measures on T that measure the set of successors of each node.

Look at the auxiliary game $\mathrm{G}([T])$. This is a game on $\omega \times \kappa$, not on ω. But it is a closed game, and thus determined by a variant of Gale-Stewart.

If player I wins $\mathrm{G}([T])$, then I wins $\mathrm{G}(A)$. If II wins $\mathrm{G}([T])$, then player II can mimic the extra moves from κ by using the measures on the nodes of T, and win $\mathrm{G}(A)$.

Large Cardinals and Determinacy (1).

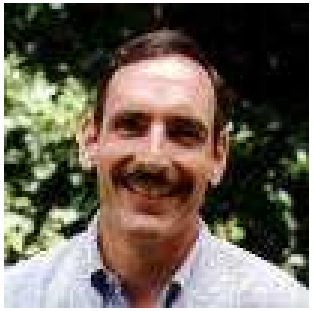
There is a level by level analysis of determinacy axioms and large cardinals:

- $\mathbf{\Pi}^1_1$ determinacy is roughly at the level of one measurable cardinal (Martin, Harrington)
- $\mathbf{\Pi}^1_{n+1}$ determinacy is roughly at the level of n Woodin cardinals (Martin-Steel, Woodin)
- There is a subtle connection between the Axiom of Choice and this project, as the lower bounds are related to whether the large cardinals involved refute the existence of definable well-orderings of the reals (“Steel games”).

  John R. Steel, Determinacy in the Mitchell models, Annals of Mathematical Logic 22 (1982), p. 109-125

Large Cardinals and Determinacy (2).

- All of the statements “roughly at the level” above can be made exact.
- The strength of the axiom of determinacy is exactly that of $\mathsf{ZFC}+$“there are infinitely many Woodin cardinals”.



Uniformization.

Reminder: The connection between set theory and infinite games started in 1968 with a paper by Blackwell.

David Blackwell, Infinite games and analytic sets, Proceedings of the National Academy of Sciences U.S.A. 58 (1967), p. 1836-1837

Blackwell's proof triggered the development of infinite game theory in foundations of mathematics.

The First Periodicity Theorem (1).

If A is a set of real numbers, we consider relations $\leq$ on A that are prewellorderings, i.e., reflexive, symmetric, transitive, linear and well-founded relations. Prewellorderings give rise to functions from A into the ordinals, traditionally called norms.

The construction of definable norms are the first step in proving general uniformization theorems ($\rightsquigarrow$ the second periodicity theorem).

Theorem (Martin, Moschovakis). *[SIMPLIFIED!]* Suppose AD (for simplicity's sake). Suppose that $\mathbf{\Delta}$ is a class closed under complementation such that every set in $\mathbf{\Delta}$ has a prewellordering in $\mathbf{\Delta}$. Then every set in $\forall\mathbf{\Delta}$ has a prewellordering in $\forall\exists\mathbf{\Delta} \cap \exists\forall\mathbf{\Delta}$.

The First Periodicity Theorem (2).

Let $A \in \forall\mathbf{\Delta}$, i.e., there is a $B \in \mathbf{\Delta}$ such that

$$x \in A \iff \forall u(\langle x, u\rangle \in B).$$

By our assumption, there is a prewellordering $\leq$ on B that is in $\mathbf{\Delta}$.

For x and y, we define the game $\mathrm{G}_{x,y}$ as follows:

Player I	u_0		u_1		u_2		$\dots$
Player II		v_0		v_1		v_2	$\dots$

Player I produces u, player II produces v, and player II wins if $\langle y, v\rangle \notin B$ or ($\langle x, u\rangle \in B$ and $\langle x, u\rangle \leq \langle y, v\rangle$).

Define $x \preceq y$ if and only if player II has a winning strategy in $\mathrm{G}_{x,y}$.

The First Periodicity Theorem (3).

Define $x \preceq y$ if and only if player II has a winning strategy in $\mathrm{G}_{x,y}$.

- $\preceq$ is reflexive.
- $\preceq$ is transitive and linear.
- $\preceq$ is well-founded.

  Suppose not, then there is a sequence $\langle x_i\,;\, i \in \omega\rangle$ such that $x_0 \succ x_1 \succ x_2 \succ \ldots$. This means that player II doesn't win $\mathrm{G}_{x_i,x_{i+1}}$, and therefore (by determinacy) that player I wins $\mathrm{G}_{x_i,x_{i+1}}$. Let σ_i be a winning strategy in that game for player I.

  We use the strategies σ_i to fill up an infinite diagram.

The First Periodicity Theorem (4).

Player II wins $\mathrm{G}_{x,y}$ if $\langle y, v\rangle \notin B$ or ($\langle x, u\rangle \in B$ and $\langle x, u\rangle \leq \langle y, v\rangle$).

Player I wins $\mathrm{G}_{x_i,x_{i+1}}$. Let σ_i be a winning strategy in that game for player I.

$$
\begin{array}{lllll}
\mathrm{G}_{x_0,x_1} & \sigma_0(\varnothing) =: u_0^0 & \sigma_0(u_0^0 u_0^1) =: u_1^0 & \sigma_0(u_0^0 u_0^1 u_1^0 u_1^1) =: u_2^0 & \sigma_0(u_0^0 u_0^1 u_1^0 u_1^1 u_2^0 u_2^1) =: u_3^0 \\
 & \nearrow & \nearrow & \nearrow & \\
\mathrm{G}_{x_1,x_2} & \sigma_1(\varnothing) =: u_0^1 & \sigma_1(u_0^1 u_0^2) =: u_1^1 & \sigma_1(u_0^1 u_0^2 u_1^1 u_1^2) =: u_2^1 & \cdots \\
 & \nearrow & \nearrow & & \\
\mathrm{G}_{x_2,x_3} & \sigma_2(\varnothing) =: u_0^2 & \sigma_2(u_0^2 u_0^3) =: u_1^2 & \cdots & \\
 & \nearrow & & & \\
\mathrm{G}_{x_3,x_4} & \sigma_3(\varnothing) =: u_0^3 & \cdots & & \\
\vdots & \vdots & & &
\end{array}
$$

So, we construct infinitely many sequences $u^i := u_0^i u_1^i u_2^i u_3^i \ldots$ where u^i is the result of playing σ_i against u^{i+1}. Since σ_i was winning, we know

- $\langle x_{i+1}, u_{i+1}\rangle \in B$, and
- if $\langle x_i, u_i\rangle \in B$, then $\langle x_{i+1}, u_{i+1}\rangle < \langle x_i, u_i\rangle$.

Therefore, $\langle x_1, u_1\rangle > \langle x_2, u_2\rangle > \langle x_3, u_3\rangle >$ ... is a strictly decreasing sequence in B.

The First Periodicity Theorem (5).

Player II wins $\mathrm{G}_{x,y}$ if $\langle y, v\rangle \notin B$ or ($\langle x, u\rangle \in B$ and $\langle x, u\rangle \leq \langle y, v\rangle$).

Let's calculate the complexity of $\preceq$:

$$
\begin{aligned}
x \preceq y \quad & \text{iff} \quad \exists\tau\forall\sigma(\sigma * \tau \text{ is a win for player II}) \\
& \text{iff} \quad \forall\sigma\exists\tau(\sigma * \tau \text{ is a win for player II})
\end{aligned}
$$

Being a win for player II is in $\mathbf{\Delta}$, and so $\preceq$ is in $\forall\exists\mathbf{\Delta} \cap \exists\forall\mathbf{\Delta}$.

q.e.d.

Summary.

What did we do?

- Infinite games play a role in set theory and the foundations of mathematics.
- Every Borel game is determined, but the proofs grow increasingly non-constructive as you go up the Borel hierarchy.
- There is a connection between the determinacy of infinite games and the axiom of choice: AC implies that there are non-determined games, and the definability of wellorderings of the real line is closely linked to how much determinacy is provable.
- Infinite games have plenty of applications in the general theory of the real line.