Second Indian Winter School in Logic
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Infinite Games

Set Theory, Part III

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Exercise 1.

Fix a payoff set A and consider $\mathrm{G}(A)$. For a position $p \in \omega^{<\omega}$, we say that **the game is essentially over** if either all extensions $x \supseteq p$ are in A or all extensions $x \supseteq p$ are in the complement of A.
Let $P := \{x \in \omega^\omega\,;\,\text{there are infinitely many } n \text{ such that } x(n) = 0\}$. Prove that player I has a winning strategy in $\mathrm{G}(P)$, but for no position $p \in \omega^{<\omega}$, the game is essentially over.

Exercise 2.

Recall the topology of Baire space ω^ω: a set A is open if there is a collection of finite sequences $S \subseteq \omega^{<\omega}$ such that $x \in A$ if and only if there is some $s \in S$ such that $s \subseteq x$. A function from ω^ω to ω^ω if preimages of open sets are open.
Consider functions $\varphi : \omega^{<\omega} \to \omega^{<\omega}$. We say that such a function φ is **monotone** if for $s \subseteq t$, we have $\varphi(s) \subseteq \varphi(t)$; we say that it is **infinitary** if for any $x \in \omega^\omega$ and $n \in \omega$ there is a k such that the sequence $\varphi(x{\restriction}k)$ has length $\geq n$. If φ is monotone and infinitary, then $\widehat{\varphi}(x) := \bigcup\{\varphi(x{\restriction}n)\,;\,n \in \omega\}$ defines a function from ω^ω, called the **lifting** of φ.
Prove that a function $f : \omega^\omega \to \omega^\omega$ is continuous if and only if $f = \widehat{\varphi}$ for an infinitary monotone function φ.

How is this directly related to the characterization of continuous functions mentioned in the lecture?

Exercise 3.

Use the idea of the Wadge game to prove that there is no continuous reduction of the set $P := \{x \in \omega^\omega\,;\,\text{there are infinitely many } n \text{ such that } x(n) = 0\}$ to an open set, *i.e.*, a continuous function $f : \omega^\omega \to \omega^\omega$ such that $f^{-1}[A] = P$ for some open set A.

Hint. Fix an open set A by giving $S \subseteq \omega^{<\omega}$ as in **Exercise 2** and provide a winning strategy for the game $\mathrm{G}(P, A)$.

Exercise 4.

Consider the **multitape game** (due to Brian T. Semmes). In this game, player II has an array of infinitely many tapes that he can continue to write on. Let's call the tapes t_n. Player II has the following possible moves: $\mathsf{write}(n)$, $\mathsf{jumpto}(m)$, and pass. The move $\mathsf{write}(n)$ writes the number n on the current tape (always at the end of the string that has already been produced), the move $\mathsf{jumpto}(m)$ jumps to the mth tape, and the move pass doesn't do anything.
After infinitely many moves, each of the t_n contains a string of natural numbers, possibly empty, and possibly finite. We call a strategy τ for player II in this game a **multitape strategy** if it ensures that for any play x for player I, $x * \tau$ has at least one tape with infinitely many entries. If τ is a multitape strategy, let $n_{x,\tau} := \min(n\,;\,t_n \text{ has infinitely many entries}\}$, and $\mathrm{outcome}(x, \tau)$ be the content of the tape $t_{n_{x,\tau}}$.

We define a game $\mathrm{G}_{\mathrm{multitape}}(A, B)$ where player I plays integers and player II plays according to a multitape strategy, and a play of x versus a strategy τ is a win for player II if

$$x \in A \iff \mathrm{outcome}(x, \tau) \in B.$$

Consider the set P from **Exercise 1** and **Exercise 3** and prove that player II has a winning strategy in $\mathrm{G}_{\mathrm{multitape}}(P, A)$ for $A := \{x \in \omega^\omega\,;\, x(0) = 0\}$. Compare with **Exercise 3**. What does this tell us about the set of functions generated by the multitape game?