Second Indian
Winter School in Logic
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Infinite Games

Set Theory, Part III

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Exercise 8.

If you don't know what a compact subset in a topological space is, find a definition and understand it. Then prove that if $K \subseteq \omega^\omega$ and $p \in \omega^{<\omega}$, then there is some $n \in \omega$ such that no element of K starts with the sequence $p^\frown n$.

Exercise 9.

Consider a tree $T \subseteq \omega^{<\omega}$ (*i.e.*, a set of sequences closed under subsequences). A node $t \in T$ is called an **infinite splitting node** if it has infinitely many immediate successors in the tree T. A tree is called **superperfect** if each node $t \in T$ can be extended to an infinite splitting node.

The following game $\mathrm{G}^{\mathrm{Kechris}}(A)$ is called the **Kechris game** and is due to Alexander S. Kechris. Player I plays finite sequences of integers and player II plays integers. Together, they produce an infinite sequence x of integers. Player I wins if and only if $x \in A$. Compare this game to the game $\mathrm{G}^*(A)$ from the lecture.

Prove that player I has a winning strategy in $\mathrm{G}^{\mathrm{Kechris}}(A)$ if and only if A contains a superperfect tree.

Prove that if A is a countable union of compact sets, then player II has a winning strategy. (**Hint.** Use **Exercise 8**.)

Exercise 10.

The following result is known as **Hausdorff's Theorem**: Every Borel set has the perfect set property. Assuming the determinacy of all Borel sets, prove Hausdorff's Theorem.

Hint. Use the perfect game and analyse the complexity of the payoff set of $\mathrm{G}^*(B)$ if B is a Borel set.

Exercise 11.

Consider the set $\mathrm{WO} \subseteq \omega^\omega$ of all codes for well-orders and its initial segments $\mathrm{WO}_\alpha := \{x \in \mathrm{WO}\,;\, \|x\| < \alpha\}$. Recall its boundedness theorem: If $B \subseteq \mathrm{WO}$ is Borel, then there is an $\alpha < \omega_1$ such that $B \subseteq \mathrm{WO}_\alpha$.

Let $C \subseteq \mathrm{WO}$ such that for all α, $C \cup \mathrm{WO}_\alpha$ is nonempty but at most countable. Prove that C is an uncountable set without a perfect subset. By the perfect set theorem from the lecture (under the assumption AD), this construction must use the Axiom of Choice. Where?

Again, by the perfect set theorem, if you fix a strategy σ for player I, there must be a counterstrategy τ for player II such that $\sigma * \tau \notin C$. Give a construction of this τ (depending on σ).