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Infinite Games

Set Theory, Part III

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Exercise 5.
If $A \subseteq \omega^\omega$ and $p \in \omega^{<\omega}$ we can consider the **subgame starting at** p as $A_p := \{x \in \omega^\omega\,;\, p^\frown x \in A\}$. If σ is a strategy for player I and $p \in \omega^{<\omega}$ is a position of odd length $2n+1$, we say that **player I deviated from strategy** σ **in the position** p if $p(2n) \neq \sigma(p{\upharpoonright}2n)$.
Construct a set C such that player I has a winning strategy in $\mathrm{G}(C)$, but for all positions in which player I deviated from σ, C_p is a non-determined set.

Exercise 6.
Consider two arbitrary open sets P and Q. In this exercise, we wish to investigate the game-theoretic properties of $P \backslash Q$. Consider the following two auxiliary games:

(1) In the game $\mathrm{G}_{3,1}(P)$, player I plays triples of natural numbers $\langle x_0, x_1, x_2\rangle$, $\langle x_4, x_5, x_6\rangle$, ..., $\langle x_{4n}, x_{4n+1}, x_{4n+2}\rangle$ and player I produces natural numbers x_{4n+3}. Together, they produce a function x, and player I wins if $x \in P$.
(2) In the game $\mathrm{G}_{1,3}(Q)$, player II plays triples of natural numbers $\langle x_1, x_2, x_3\rangle$, $\langle x_5, x_6, x_7\rangle$, ..., $\langle x_{4n+1}, x_{4n+2}, x_{4n+3}\rangle$ and player II produces natural numbers x_{4n}. Together, they produce a function x, and player II wins if $x \in Q$.

Argue that both of these games can be given in the form of a game $\mathrm{G}(A)$ by giving an appropriate payoff set A. (**Hint.** Use a bijection between $\omega \times \omega \times \omega$ and ω.) Prove that both of these games are open.

Prove that if player I has no winning strategy in $\mathrm{G}_{3,1}(P)$, then player II wins $\mathrm{G}(P \backslash Q)$. If player I has a winning strategy in both $\mathrm{G}_{3,1}(P)$ and $\mathrm{G}_{1,3}(Q)$, then player I wins $\mathrm{G}(P \backslash Q)$.

Exercise 7.
Consider the following unravelling of a graph game: if G is a (countable) graph with distinguished vertex s, let $\pi : G \to \omega$ be an injection. Call a (finite or infinite) sequence $x \in \omega^\omega$ an s**-path through** G if $x(0) = \pi(s)$, for all n such that $x(n)$ is defined, there is a vertex v such that $x(n) = \pi(v)$, and if $x(n) = \pi(v)$ and $x(n+1) = \pi(v^*)$, then there is an edge from v to v^*.
Using the idea of an s-path through G, define a set $A_{G,s}$ such that the graph game on G starting at s is equivalent to $\mathrm{G}(A_{G,S})$ in the following sense: for every G and s, if player I (II) has a winning strategy in the graph game on G starting at s if and only if player I (II) has a winning strategy in $\mathrm{G}(A_{G,s})$.
Prove that the winning condition "the last player to make a legal moves wins" corresponds to an open set and the winning condition of "vertex v is visited infinitely many times" corresponds to a $\mathbf{\Pi}^0_2$ set.