MTH201 2021-2022 SEM 1; MIDSEM EXAM

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Instructions:

(1) You can only use concepts and results, introduced and proved respectively, in this course to answer the following questions.
(2) Partial scores will be given only for a complete and correct statement, if they are relavent to the solution of the final problem.
(3) No partial scores will be given for questions with less than 4 points.
(4) I reserve the right to give zero score if two candidates produce the exact same answers.
(5) You are suggested to restrict your answers to one side of a page for each question.
(6) Please remember that there are 2 pages in this question paper.
(7) Note that the set $\{1, 2, \dots, n\}$ is denoted by $[n]$. For any finite set X, we denote by $|X|$, the cardinality of X. I will continue to use notations from my lecture notes.
(8) For me null space of a matrix $M \in M_{m\times n}(F)$ is the kernel of the map $T_M : M_{n\times 1}(F) \to M_{m\times 1}(F)$, given $X \mapsto MX$.

1. Fields and vector space definitions

Let F be any field and let E be a field containing F. Let V be a finite dimensional vector space over a field F. Let $V^\vee$ be the dual vector space, i.e., the space $\mathrm{Hom}_F(V, F)$.

1.1. **(2 points).** Is there a unique $\mathbb{C}$ vector space structure on $\mathbb{R}^2$ such that restriction of scalar multiplication to real numbers is the standard $\mathbb{R}^2$.

1.2. **(2+3 points).** Let V_E be the space $\mathrm{Hom}_F(V^\vee, E)$. For any $\alpha \in E$ and $l \in V_E$, we define $(\alpha * l) \in V_E$, by setting

$$(\alpha * l)(v) = \alpha l(v), \text{ for all } v \in V^\vee.$$

Show that the scalar multiplication $*$ makes V_E a E-vector space. Show that V_E is a finite dimensional E-vector space whose dimension is equal to $\dim_F(V)$.

1.3. **(3 points).** Does there exists $M \in M_{7\times 7}(\mathbb{R})$ such that $M^2 = -7I_7$. Here, I_7 is the identity matrix in $M_{7\times 7}(\mathbb{R})$.

2. Linear transformations

2.1. **(10 points).** Let S be a subset of $M_{n\times n}(\mathbb{Q})$. Show that there exists a non-zero $X \in M_{n\times 1}(\mathbb{Q})$ such that $MX = X$ for all $M \in S$ if and only if there exists a non-zero $Y \in M_{n\times 1}(\mathbb{R})$ such that $MY = Y$ for all $M \in S$.

2.2. **(5 points).** Let $a_0, a_1, a_2, \dots, a_n$ be non-zero rational numbers. Let $q \in \mathbb{Q}[t]$ be a polynomial. Show that there exists $p \in \mathbb{Q}[X]$ such that

$$a_0 p + a_1 \frac{dp}{dt} + a_2 \frac{d^2 p}{d^2 t} + \cdots + a_n \frac{d^n p}{d^n t} = q.$$

3. Bilinear forms and combinatorics

3.1. **(2 points).** Let $\mathbb{F}_2$ be the field with exactly 2 elements. Let V be an n-dimensional vector space over the field $\mathbb{F}_2$. Show that there exists a non-degenerate symmetric bilinear form on V. Let $B : V \times V \to \mathbb{F}_2$ be a non-degenerate symmetric bilinear form. If U is a subspace of V such that $B(v, w) = 0$, for all $v, w \in U$, then show that $\dim(U) \le \dim(V)/2$.

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3.2. **(6 points).** Consider the following set:

$$Y = \{X_1, X_2, \ldots, X_m : X_i \subseteq [n] \ \forall i \in [m]\}.$$

If $|X_i \cap X_j|$ is even for all pairs (i, j) with $i, j \in [m]$, then show that $m \leq 2^{[n/2]}$, where $[x]$ is the greatest integer less than or equal to x.

3.3. **(2 points).** Let $M \in M_{m \times n}(\mathbb{R})$ and let $M^t M = J + D$, where J is a matrix with all its entries equal to 1, and D is a diagonal matrix with all positive entries. Show that the null space of M is the zero subspace.

3.4. **(5 points).** Let $Y = \{X_1, X_2, \ldots, X_m : X_i \subseteq [n] \ \forall i \in [m]\}$. Assume that for any $i \neq j$, we have $|X_i \cap X_j| = c$, where $1 \leq c < n$. Show that $m \leq n$. (Hint: Consider an $n \times m$ matrix $M = (m_{ij})$ where $m_{ij} = 1$ if $i \in X_j$ and $m_{ij} = 0$ if $i \notin X_j$).