ON THE INTEGRALITY OF LOCALLY ALGEBRAIC REPRESENTATIONS OF $\mathrm{GL}_2(D)$

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Abstract. Emerton's theory of Jacquet modules for locally analytic representations provides necessary conditions for the existence of integral structures in locally analytic representations. These conditions are also expected to be sufficient for the integrality of generic irreducible locally algebraic representations. In this article, we prove the sufficiency of Emerton's conditions for some tamely ramified locally algebraic representations of $\mathrm{GL}_2(D)$ where D is a p-adic division algebra.

1. Introduction

Let p be a prime number, F be a finite extension of $\mathbb{Q}_p$ with residue field $\mathbb{F}_q$ and uniformizer ϖ_F, and let E be a large enough finite extension of F. Let G be the group of rational points of a connected reductive group over F and $\pi = \pi_{sm} \otimes \pi_{alg}$ be an irreducible locally algebraic representation of G over E. The question of the existence of a G-invariant lattice or a G-integral structure in π is of fundamental interest to the p-adic Langlands program.

Emerton gives necessary conditions for the existence of integral structures in terms of the *exponents* of Jacquet modules of locally algebraic representations. Let $P = M_P N_P \subseteq G$ be a parabolic subgroup with the modulus character δ_P and let J_P denote the Emerton's locally analytic Jacquet module functor. If π admits a G-integral structure, then for every parabolic P of G and χ such that $\mathrm{Hom}_{Z(M_P)}(\chi, J_P(\pi)) \neq 0$,

$$(\delta_P^{-1}\chi)(z) \text{ is integral in } E, \tag{1.1}$$

for all $z \in Z(M_P)^+$ where $Z(M_P)^+$ is the contracting monoid in the center $Z(M_P)$ of the Levi factor M_P [Eme06, Lemma 4.4.2]. The characters χ of $Z(M_P)$ occurring in $J_P(\pi)$ are called exponents. It is expected that the condition (1.1) is also sufficient for the existence of an integral structure in π when π_{sm} is *generic*. For $G = \mathrm{GL}_n(F)$, this is equivalent to Breuil-Schneider conjecture (see Hu [Hu09]). Note that for $P = G$, the condition (1.1) reads as the central character of π is integral. When π has integral central character and π_{sm} is essentially square-integrable, the Jacquet modules $J_P(\pi)$ for proper parabolic P always satisfy (1.1). In this situation, Sorensen showed using global methods that the integrality of the central character is sufficient for π to have an integral structure [Sor13]. On the other hand, when π_{sm} is a principal series representation, the Jacquet modules are no longer simple and one requires further conditions on π whose sufficiency is not easy to prove. There are only partial results available, even for $\mathrm{GL}_2(F)$, when π_{sm} is an unramified principal series and the weights of π_{alg} are small [DI13, Ass21] or when $\pi = \pi_{sm}$ is a tamely ramified principal series [Vig08]. For general split reductive groups and $F = \mathbb{Q}_p$, Große-Klönne has constructed integral structures in unramified smooth principal series representations under some technical hypothesis [GeK14].

In this article, we consider the non-quasi-split group $G = \mathrm{GL}_2(D)$ where D is a central F-division algebra of dimension d^2 and show that (1.1) is sufficient for the existence of integral structures in some tamely ramified irreducible locally algebraic representations of G. Let us spell out (1.1) for representations of $G = \mathrm{GL}_2(D)$ admitting integral structures. Let $B = TN$ be the minimal parabolic subgroup of G consisting of upper triangular matrices. One has $\delta_B(z) = q^{-d^2}$ for $z = \left(\begin{smallmatrix} \varpi_F & 0 \\ 0 & 1 \end{smallmatrix}\right) \in Z(T)^+$. Denote by $\pi(\underline{\lambda})$ the irreducible algebraic representation of $\mathrm{GL}_{2d}(\overline{F})$ with highest weight $\underline{\lambda} = (\lambda_1, \dots, \lambda_{2d})$ and by $\chi(\underline{\lambda})$ the character $(t_1, \dots, t_{2d}) \mapsto t_1^{\lambda_1} \dots t_{2d}^{\lambda_{2d}}$ of its diagonal torus.

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If $\pi = \mathrm{Ind}_B^G(\tau_1 \otimes \tau_2) \otimes \pi(\underline{\lambda})$ is a locally algebraic principal series representation, then

$$J_B(\pi) \cong (\mathrm{Ind}_B^G(\tau_1 \otimes \tau_2))_N \otimes \pi(\underline{\lambda})^N$$

and

$$(\mathrm{Ind}_B^G(\tau_1 \otimes \tau_2))_N^{ss} \cong (\tau_1 \otimes \tau_2) \oplus ((\tau_2 \otimes \tau_1) \otimes \delta_B).$$

Denoting the central characters of representations ? by $\omega_?$, the exponents χ of $J_B(\pi)$ are

$$(\omega_{\tau_1} \otimes \omega_{\tau_2})\chi(\underline{\lambda}) \text{ and } (\omega_{\tau_2} \otimes \omega_{\tau_1})\delta_B\chi(\underline{\lambda}).$$

Thus, if π has an integral structure, (1.1) says that

$$\begin{aligned}(\omega_{\tau_1}\omega_{\tau_2})(\varpi_F)\varpi_F^{\sum_{i=1}^{2d}\lambda_i} &\text{is an integral unit (the integrality of } \omega_\pi\text{), and}\\ q^{d^2}\omega_{\tau_1}(\varpi_F)\varpi_F^{\sum_{i=1}^{d}\lambda_i} &\text{ and } \omega_{\tau_2}(\varpi_F)\varpi_F^{\sum_{i=1}^{d}\lambda_i} \text{ are integral in } E. \qquad (1.2)\end{aligned}$$

Our first main result shows that the conditions (1.2) are sufficient for the existence of an integral structure in π if the smooth principal series $\mathrm{Ind}_B^G(\tau_1 \otimes \tau_2)$ is tamely ramified and $\pi(\underline{\lambda})$ is trivial, i.e., $\underline{\lambda} = \underline{0}$:

Theorem A (Theorem 3.1)**.** The integrality conditions (1.2) of Emerton are sufficient for the existence of an integral structure in a smooth tamely ramified principal series representation $\mathrm{Ind}_B^G(\tau_1 \otimes \tau_2)$ of G.

We remark that the principal series $\mathrm{Ind}_B^G(\tau_1 \otimes \tau_2)$ is not required to be irreducible in Theorem A. However, the conditions (1.2) are no longer sufficient for a reducible principal series tensored with a non-trivial algebraic representation (see erratum of [Vig08]).

A principal series of the form $\mathrm{Ind}_B^G((\tau \otimes \tau) \otimes \delta_B^{\frac{d-a}{2d}})$ is reducible with unique irreducible quotient $\mathrm{St}(\tau)$ and unique irreducible submodule $\mathrm{Sp}(\tau)$. Here a is the length of the segment that determines the Jacquet-Langlands lift of the irreducible $D^\times$-representation τ. If $\pi = \mathrm{St}(\tau) \otimes \pi(\underline{\lambda})$ (resp. $\mathrm{Sp}(\tau) \otimes \pi(\underline{\lambda})$), then

$$J_B(\pi) \cong ((\tau \otimes \tau) \otimes \delta_B^{\frac{d+a}{2d}}) \otimes \pi(\underline{\lambda})^N \quad (\text{resp. } ((\tau \otimes \tau) \otimes \delta_B^{\frac{d-a}{2d}}) \otimes \pi(\underline{\lambda})^N).$$

The exponent χ in $J_B(\pi)$ is $(\omega_\tau \otimes \omega_\tau)\delta_B^{\frac{d+a}{2d}}\chi(\underline{\lambda})$ (resp. $(\omega_\tau \otimes \omega_\tau)\delta_B^{\frac{d-a}{2d}}\chi(\underline{\lambda})$). Hence, if π has an integral structure, then (1.1)) says that

$$\begin{aligned}&\omega_\tau^2(\varpi_F)\varpi_F^{\sum_{i=1}^{2d}\lambda_i} \text{ is an integral unit and } q^{\frac{d(d-a)}{2}}\omega_\tau(\varpi_F)\varpi_F^{\sum_{i=1}^{d}\lambda_i}\\ &(\text{resp. } q^{\frac{d(d+a)}{2}}\omega_\tau(\varpi_F)\varpi_F^{\sum_{i=1}^{d}\lambda_i}) \text{ is integral in } E. \qquad (1.3)\end{aligned}$$

The first part of (1.3), i.e., the integrality of ω_π implies that $\mathrm{val}_E(\omega_\tau(\varpi_F)) = \frac{-1}{2}\sum_{i=1}^{2d}\lambda_i$. Thus $\mathrm{val}_E(\omega_\tau(\varpi_F)) + \sum_{i=1}^{d}\lambda_i = \frac{1}{2}(\sum_{i=1}^{d}\lambda_i - \sum_{i=d+1}^{2d}\lambda_i) \geq 0$. As $d - a \geq 0$, we see that the second part of (1.3) is redundant as it is implied by the first part. Hence, in this case, the integrality of ω_π is conjecturally sufficient for the existence of an integral structure in π. The sufficiency follows from Theorem 3.1 when $\mathrm{St}(\tau)$ (resp. $\mathrm{Sp}(\tau)$) is tamely ramified and $\pi(\underline{\lambda})$ is trivial (see Theorem 3.3). Our second main result shows the sufficiency of the integrality of ω_π for a locally algebraic representation $\pi = \mathrm{St}(\tau) \otimes \pi(\underline{\lambda})$ with non-trivial $\pi(\underline{\lambda})$ under the assumption that $\mathrm{St}(\tau)$ is tamely ramified and τ is of dimension at most 2:

Theorem B (Theorem 4.6)**.** Let τ be a smooth absolutely irreducible tamely ramified representation of $D^\times$ of dimension ≤ 2. Then the locally algebraic Steinberg representation $\pi = \mathrm{St}(\tau) \otimes \pi(\underline{\lambda})$ of G with integral central character admits an integral structure.

We follow local methods of Vignéras [Vig08] and Hu [Hu21] based on the theory of coefficient systems (or *diagrams*) on the Bruhat-Tits tree of G. The main idea of Vignéras is to use the realization of a locally algebraic representation π as the 0-th homology group of its fixed-point system. The question of finding an integral structure in π then amounts to the question of finding a system of lattices in the corresponding fixed-point system of finite-dimensional vector spaces. The analysis gets more involved when π_{sm} admits invariants under smaller compact open subgroups and when π_{alg} is non-trivial. This is the reason π_{sm} is tamely ramified in all of our results. Hu's argument allows us to treat some π with non-trivial π_{alg} (without explicitly working with π_{alg}) under a strong assumption on $\dim_E(\tau)$ which is necessary.

We conclude with an example of an infinite-dimensional integral locally algebraic Speh representation $\mathrm{Sp}(\tau)\otimes\pi(\underline{\lambda})$; see §4.1. It is easy to see that the tensor product of a one-dimensional Speh representation (i.e., a character) and a non-trivial irreducible algebraic representation is never integral. However, this is false for infinite-dimensional irreducible locally algebraic Speh representations. We believe that this is related to the fact that $\mathrm{Sp}(\tau)$ admits a *generalized Whittaker model* if and only if it is infinite-dimensional (see Remark 4.8). Our investigations suggest that, for the group $\mathrm{GL}_2(D)$, Emerton's integrality conditions (1.1) are sufficient for the integrality of any *infinite-dimensional* irreducible locally algebraic representation.

Organization: In §2, we discuss Vignéras' integrality criterion for the representations of $\mathrm{GL}_2(D)$. In §3, we use this criterion to show that Emerton's integrality conditions are sufficient for the integrality of smooth tamely ramified principal series representations. The §4 talks about the integrality of locally algebraic representations of $\mathrm{GL}_2(D)$ whose smooth part is either a Steinberg representation or a Speh representation. We show the sufficiency of Emerton's conditions for some locally algebraic Steinberg representations. Finally, in the subsection §4.1, we illustrate the connection between the integrality of locally algebraic representations and the *genericity* of their smooth part with an example of an infinite-dimensional integral locally algebraic Speh representation.

Notation and convention:

Let F be a non-archimedean local field of characteristic 0 with residue field $\mathbb{F}_q$ of characteristic p. Let D be the central F-division algebra of index d. Let $\mathcal{O}_F\subseteq F$ and $\mathcal{O}_D\subseteq D$ denote the respective valuation rings. Fix uniformizers $\varpi_F\in\mathcal{O}_F$ and $\varpi_D\in\mathcal{O}_D$. For a divisor d' of d, let $F_{d'}$ denote the unramified extension of F of degree d' viewed as a subfield of D. Let $|\cdot|_F$ and $|\cdot|$ denote the normalized non-archimedean absolute values on F and D respectively such that $|\varpi_F|_F=q^{-1}$ and $|\varpi_D|=q^{-d}$. We have $|\cdot|:=|\cdot|_F^d\circ\mathrm{Nrd}$ where $\mathrm{Nrd}:D\to F$ is the reduced norm. Note that $|\varpi_F|=q^{-d^2}$.

Let G be the group $\mathrm{GL}_2(D)$ of units in the matrix algebra $\mathrm{M}_2(D)$, $K=\mathrm{GL}_2(\mathcal{O}_D)$ and $I\subseteq K$ denote the standard Iwahori subgroup. We view $D^\times$ as a subgroup of G embedded diagonally in it. Let $\mathfrak{K}_0,\mathfrak{K}_1\subseteq G$ be the subgroups stabilizing respectively the standard vertex and the standard edge of the Bruhat-Tits tree of G. Note that $\mathfrak{K}_0=K\varpi_D^{\mathbb{Z}}$, and $\mathfrak{K}_1$ is generated by I and the matrix $t=\left(\begin{smallmatrix}0&1\\ \varpi_D&0\end{smallmatrix}\right)$. The groups K and I admit filtrations by pro-p-subgroups $K(n)$ and $I(n)$ respectively for $n\geq1$ where $K(n)=\left(\begin{smallmatrix}1+\varpi_D^n\mathcal{O}_D&\varpi_D^n\mathcal{O}_D\\ \varpi_D^n\mathcal{O}_D&1+\varpi_D^n\mathcal{O}_D\end{smallmatrix}\right)$ and $I(n)=\left(\begin{smallmatrix}1+\varpi_D^n\mathcal{O}_D&\varpi_D^{n-1}\mathcal{O}_D\\ \varpi_D^n\mathcal{O}_D&1+\varpi_D^n\mathcal{O}_D\end{smallmatrix}\right)$. The subgroup $I(1)\subseteq I$ is the standard pro-p-Iwahori subgroup. The groups $\mathfrak{K}_0$ and $\mathfrak{K}_1$ are the normalizers of $K(n)$ and $I(n)$ in G respectively for all $n\geq1$. Let $B\subseteq G$ be the subgroup of upper triangular matrices (the standard minimal parabolic subgroup), $N\subseteq B$ be the subgroup of upper triangular unipotent matrices (the unipotent radical), and $T\subseteq B$ be the subgroup of diagonal matrices (the Levi quotient). The modulus character δ_B of T is $|\cdot|\otimes|\cdot|^{-1}$. We let Z denote the center of G which is isomorphic to $F^\times$.

We fix a large enough finite extension E of F. The field E depends on the representation at hand and we will explain how large it should be at various places in the article when required. Its valuation ring is denoted by $\mathcal{O}$ and the residue field is denoted by $k=\mathcal{O}/\varpi\mathcal{O}$ where $\varpi\in\mathcal{O}$ is a uniformizer. The rings $R=E,\mathcal{O},k$ will serve as the coefficient rings for representations of G. The representations will be either denoted by (π,V) or just by π or V depending on the situation. Let $H\subseteq G$ is a subgroup. We write RH for the group algebra of H over R and use the phrases "RH-modules" and "representations of H over R" interchangeably. If V is an RH-module, then, for a subset $S\subseteq V$, we denote by $H\cdot S$ the RH-submodule of V generated by S.

We fix an isomorphism between $\mathbb{C}$ and the algebraic closure $\overline{E}$ of E. For a smooth representation π over E, we write $\pi_{\mathbb{C}}=\pi\otimes_E\mathbb{C}$ for its scalar extension via the embedding $E\hookrightarrow\mathbb{C}$ induced by the fixed isomorphism. We also call π an E-model of $\pi_{\mathbb{C}}$. By [CEG+16, Section 3.13], all the results about $\pi_{\mathbb{C}}$ are valid for π (over a large enough E). If a representation π admits a central character, it is denoted by ω_π.

A smooth representation of G (resp. of $D^\times$) will be called tamely ramified if it has a non-zero vector fixed by the subgroup $K(1)$ (resp. by $D(1) = 1 + \varpi_D \mathcal{O}_D$). For a divisor d' of d, let $D_{d'}$ be the centralizer of $F_{d'}$ in D which is a central $F_{d'}$-division algebra of index d/d'. Let $\theta : F_{d'}^\times \to E^\times$ be a character which is trivial on the subgroup of integral units congruent to 1 modulo the maximal ideal and whose all Galois conjugates are distinct. Here one requires E to contain all d roots of unity. Composing it with the reduced norm $\mathrm{Nrd} : D_{d'}^\times \to F_{d'}^\times$ and extending it to $D(1)D_{d'}^\times$ by declaring it to be trivial on $D(1)$, we get a character $\theta : D(1)D_{d'}^\times \to E^\times$. The representation $\mathrm{Ind}_{D(1)D_{d'}^\times}^{D^\times}\theta$ is absolutely irreducible and tamely ramified. In fact, all smooth tamely ramified absolutely irreducible representations of $D^\times$ over E are obtained in this fashion [SZ05].

2. Coefficient systems and Vignéras' integrality criterion

We begin by recalling some definitions. Let $H \subseteq G$ be an open subgroup. A locally algebraic representation of H over E is a representation of the form $\pi = \pi_{sm} \otimes \pi_{alg}$, where π_{sm} is a smooth representation of H over E and π_{alg} is the restriction to H of a finite-dimensional rational representation of G over E. If π_{sm} has a name "X", then π will be called by the name "locally algebraic X". The representation π is irreducible if and only if π_{sm} and π_{alg} are irreducible [STP01, Appendix, Theorem 1]. An H-integral structure is an H-stable free $\mathcal{O}$-submodule $\pi^\circ \subseteq \pi$ which spans π over E. The integral structure is also called an H-lattice since it is a lattice stable under the action of H. If an H-integral structure exists, we say that π is H-integral, or just integral if the group is clear from the context. We are interested in the integrality of irreducible locally algebraic representations of G.

A diagram
$$\mathcal{D}_1 \xrightarrow{r} \mathcal{D}_0$$
is a data consisting of continuous (smooth when $R = k$) $R\mathfrak{K}_i$-modules $\mathcal{D}_i$ and a map r equivariant for the action of $\mathfrak{K}_0 \cap \mathfrak{K}_1 = I\varpi_D^{\mathbb{Z}}$. Such a diagram gives rise to a G-equivariant coefficient system on the Bruhat-Tits tree of G. Conversely, the restriction of a G-equivariant coefficient system to the subtree consisting of the standard edge and the standard vertex is a diagram. Associated to a diagram $\mathcal{D}$ (or to a coefficient system), one has oriented chain homology groups $H_i(\mathcal{D})$, $i = 0, 1$, which are continuous RG-modules, see [Vig08, page 3].

Let $\pi = \pi_{sm} \otimes \pi_{alg}$ be a locally algebraic representation of G over E. Assume that π_{sm} is generated by its subspace $\pi_{sm}^{I(n)}$ of $I(n)$-invariants for some positive integer n and $\dim_E(\pi_{sm}^{I(n)}) < \infty$. Let $V_1 = \pi_{sm}^{I(n)} \otimes \pi_{alg}$ and $V_0 = \pi_{sm}^{K(n)} \otimes \pi_{alg}$ and consider the diagram
$$\mathcal{D}(\pi) = V_1 \hookrightarrow V_0$$
of $E\mathfrak{K}_i$-modules V_i. It follows from [SS97, Theorem II.3.1] and [Vig08, Proposition 0.4] (with the same exact proof for $\mathrm{GL}_2(D)$) that the representation $H_0(\mathcal{D}(\pi))$ of G is isomorphic to π.

Theorem 2.1 (Vignéras)**.** *π is integral if and only if V_0 contains a $\mathfrak{K}_0$-lattice M_0 such that $M_1 = M_0 \cap V_1$ is a $\mathfrak{K}_1$-lattice of V_1, i.e. M_1 is stable under the action of t. In this situation, $H_0(M_1 \hookrightarrow M_0)$ is a G-integral structure of π.*

Proof. See [Vig08, Corollary 0.2 and Proposition 0.4]. □

Suppose V_1 contains a $\mathfrak{K}_1$-lattice L_1. Starting from L_1, define inductively an increasing sequence of $\mathfrak{K}_1$-lattices of V_1 as follows:
$$L_1^{(0)} := L_1,$$
$$L_1^{(i+1)} := \sum_{i=0}^{2d-1} t^i \left(\left(\mathfrak{K}_0 \cdot L_1^{(i)} \right) \cap V_1 \right) \quad \text{for } i \geq 0.$$

Corollary 2.2. *π is integral if and only if V_1 contains a $\mathfrak{K}_1$-lattice L_1 such that the increasing sequence $(L_1^{(i)})_i$ of $\mathfrak{K}_1$-lattices of V_1 becomes stationary.*

Proof. If π is integral, then the $\mathfrak{K}_1$-lattice M_1 as in Theorem 2.1 satisfies $M_1^{(0)} = M_1^{(1)}$. Conversely, if V_1 contains a $\mathfrak{K}_1$-lattice L_1 such that $L_1^{(i_0)} = L_1^{(i_0+1)}$ for some positive integer i_0, then $M_0 = \mathfrak{K}_0 \cdot L_1^{(i_0)}$ is a $\mathfrak{K}_0$-lattice of V_0 such that $M_1 = M_0 \cap V_1 = L_1^{(i_0)}$. □

Remark 2.3. If π is integral, then for any $\mathfrak{K}_1$-lattice L_1 of V_1, the sequence $(L_1^{(i)})_i$ of $\mathfrak{K}_1$-lattices becomes stationary. Indeed, we know that there is a $\mathfrak{K}_1$-lattice M_1 of V_1 such that the the sequence $(M_1^{(i)})_i$ becomes stationary. We may assume $L_1 \subseteq M_1$. Since $[M_1 : L_1]$ is finite, it is clear that the increasing sequence $(L_1^{(i)})_i$ also becomes stationary.

The integrality criterion in Corollary 2.2 will be used in the following sections to show that Emerton's conditions are sufficient for the existence of integral structures in π for which π_{sm} is tamely ramified.

3. Integrality of smooth principal series

Let (τ_1, W_1) and (τ_2, W_2) be two smooth absolutely irreducible tamely ramified representations of $D^\times$ over E. In this section, $\pi = \pi_{sm} = \mathrm{Ind}_B^G(\tau_1 \otimes \tau_2)$. Note that $\pi^{I(1)} \neq 0$. In fact, π is generated by $\pi^{I(1)}$ as a G-representation. In order to describe the spaces $V_0 = \pi^{K(1)}$ and $V_1 = \pi^{I(1)}$, we define some explicit elements of the principal series π.

Let $s = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $u_\lambda = \left(\begin{smallmatrix} 1 & [\lambda] \\ 0 & 1 \end{smallmatrix}\right)$ where $[\lambda] \in \mathcal{O}_D$ is the Teichmüller lift of $\lambda \in \mathbb{F}_{q^d}$. For $h \in G$ and $v \in W_1 \otimes W_2$, we denote by f_v^h the unique function in $\pi^{I(1)}$ supported on $BhI(1)$ such that $f_v^h(h) = v$. Similarly, we denote by g_v^h the unique function in $\pi^{K(1)}$ supported on $BhK(1)$ such that $g_v^h(h) = v$. Note that $f_v^1 = g_v^1$ because $BI(1) = BK(1)$, and $f_v^s = \sum_{\lambda \in \mathbb{F}_{q^d}} g_v^{su_\lambda}$ because $BsI(1) = \bigsqcup_{\lambda \in \mathbb{F}_{q^d}} Bsu_\lambda K(1)$. If $M \subseteq W_1 \otimes W_2$ is an $\mathcal{O}$-submodule, it is convenient to write f_M^h for the set $\{f_v^h : v \in M\}$ and similarly g_M^h for the set $\{g_v^h : v \in M\}$. As

$$G = BI(1) \sqcup BsI(1) = BK(1) \sqcup \bigsqcup_{\lambda \in \mathbb{F}_{q^d}} Bsu_\lambda K(1),$$

we have

$$V_1 = f^1_{W_1 \otimes W_2} \oplus f^s_{W_1 \otimes W_2} \ \text{ and } \ V_0 = g^1_{W_1 \otimes W_2} \oplus \bigoplus_{\lambda \in \mathbb{F}_{q^d}} g^{su_\lambda}_{W_1 \otimes W_2} .$$

It is easy to check $tf_v^1 = f^s_{(\tau_1(\varpi_D) \otimes \mathrm{Id})(v)}$ and $tf_v^s = f^1_{(\mathrm{Id} \otimes \tau_2(\varpi_D))(v)}$. By letting $K(1)$ act trivially on $W_1 \otimes W_2$, one can extend the action of $B \cap \mathfrak{K}_0$ on $W_1 \otimes W_2$ to $I\varpi_D^{\mathbb{Z}} = (B \cap \mathfrak{K}_0)K(1)$. Then, as $E\mathfrak{K}_0$-modules, $V_0 \cong \mathrm{Ind}_{I\varpi_D^{\mathbb{Z}}}^{\mathfrak{K}_0}(\tau_1 \otimes \tau_2)$.

Let $T_0 = T \cap \mathfrak{K}_0$. The central character ω_π of π equals $\omega_{\tau_1}\omega_{\tau_2}$. When ω_π is integral, there exists a T_0-lattice $\mathcal{L} \subseteq W_1 \otimes W_2$. The main result of this section is the following:

Theorem 3.1. *The tamely ramified principal series representation $\pi = \mathrm{Ind}_B^G(\tau_1 \otimes \tau_2)$ with integral central character is integral if and only if $q^{d^2}\omega_{\tau_1}(\varpi_F) \in \mathcal{O}$ and $\omega_{\tau_2}(\varpi_F) \in \mathcal{O}$.*

Proof. ($\Longrightarrow$) Though the necessity is known due to Emerton, we provide a proof to set up the notation for the next part. Let

$$L_0 := \mathrm{Ind}_{I\varpi_D^{\mathbb{Z}}}^{\mathfrak{K}_0} \mathcal{L} = g^1_{\mathcal{L}} \oplus \bigoplus_{\lambda \in \mathbb{F}_{q^d}} g^{su_\lambda}_{\mathcal{L}}$$

be a $\mathfrak{K}_0$-lattice of V_0. Then,

$$L_0 \cap V_1 = L_0^{I(1)} = f^1_{\mathcal{L}} \oplus f^s_{\mathcal{L}}.$$

Let

$$L_1 = L_1^{(0)} := L_0 \cap V_1 + t(L_0 \cap V_1).$$

As $\mathcal{L}$ is stable under the diagonal action of ϖ_D, L_1 is stable under the action of $\mathfrak{K}_1$ and so it is a $\mathfrak{K}_1$-lattice of V_1. One computes that

$$L_1 = f^1_{\mathcal{L} + (\mathrm{Id} \otimes \tau_2(\varpi_D))\mathcal{L}} \oplus f^s_{\mathcal{L} + (\tau_1(\varpi_D) \otimes \mathrm{Id})\mathcal{L}}.$$

Thus,

$$\mathfrak{K}_0 \cdot L_1 = \mathfrak{K}_0 \cdot f^1_{\mathcal{L} + (\mathrm{Id} \otimes \tau_2(\varpi_D))\mathcal{L}} + \mathfrak{K}_0 \cdot f^s_{\mathcal{L} + (\tau_1(\varpi_D) \otimes \mathrm{Id})\mathcal{L}}. \tag{3.1}$$

Since $u_{-\lambda}s\cdot f_v^1 = u_{-\lambda}s\cdot g_v^1 = g_v^{su_\lambda}$, the first summand $\mathfrak{K}_0\cdot f^1_{\mathcal{L}+(\mathrm{Id}\otimes\tau_2(\varpi_D))\mathcal{L}}$ in (3.1)) is

$$g^1_{\mathcal{L}+(\mathrm{Id}\otimes\tau_2(\varpi_D))\mathcal{L}} \oplus \bigoplus_{\lambda\in\mathbb{F}_{q^d}} g^{su_\lambda}_{\mathcal{L}+(\mathrm{Id}\otimes\tau_2(\varpi_D))\mathcal{L}}.$$

To describe the second summand $\mathfrak{K}_0\cdot f^s_{\mathcal{L}+(\tau_1(\varpi_D)\otimes\mathrm{Id})\mathcal{L}}$, let

$$F_v^x := u_x s\cdot f_v^s \quad \text{for } x\in\mathbb{F}_{q^d} \text{ and } v\in\mathcal{L}+(\tau_1(\varpi_D)\otimes\mathrm{Id})\mathcal{L}.$$

The lattice $f^s_{\mathcal{L}+(\tau_1(\varpi_D)\otimes\mathrm{Id})\mathcal{L}}$ is stable under the action of $I\varpi_D^{\mathbb{Z}}$. The set $\{1, u_x s : x\in\mathbb{F}_{q^d}\}$ forms a set of representatives of $\mathfrak{K}_0/I\varpi_D^{\mathbb{Z}}$. Thus

$$\mathfrak{K}_0\cdot f^s_{\mathcal{L}+(\tau_1(\varpi_D)\otimes\mathrm{Id})\mathcal{L}} = f^s_{\mathcal{L}+(\tau_1(\varpi_D)\otimes\mathrm{Id})\mathcal{L}} + \langle {\sum_{x,v}}' F_v^x\rangle, \tag{3.2}$$

where ${\sum}'_{x,v}$ denotes a sum over finitely many pairs (x,v) with $x\in\mathbb{F}_{q^d}$ and $v\in\mathcal{L}+(\tau_1(\varpi_D)\otimes\mathrm{Id})\mathcal{L}$ and $\langle {\sum}'_{x,v} F_v^x\rangle$ is the $\mathcal{O}$-module of all such sums. Using $su_cs = \begin{pmatrix} -1/[c] & 1 \\ 0 & [c]\end{pmatrix} su_{1/c}$ for $c\neq 0$, we get

$$F_v^x = u_x s\cdot \sum_{\lambda\in\mathbb{F}_{q^d}} g_v^{su_\lambda} = g_v^1 + \omega_{\tau_1}(-1)\sum_{\lambda\in\mathbb{F}_{q^d}^\times} g^{su_{(1/\lambda)-x}}_{\xi_\lambda(v)}$$

where $\xi_\lambda = \tau_1(\lambda)\otimes\tau_2(1/\lambda)$. For a fixed but arbitrary $v\in\mathcal{L}+(\tau_1(\varpi_D)\otimes\mathrm{Id})\mathcal{L}$, consider

$$\begin{aligned}\sum_{x\in\mathbb{F}_{q^d}} F_v^x &= q^d g_v^1 + \omega_{\tau_1}(-1)\sum_{x\in\mathbb{F}_{q^d}}\sum_{\lambda\in\mathbb{F}_{q^d}^\times} g^{su_{(1/\lambda)-x}}_{\xi_\lambda(v)}\\ &= f^1_{q^dv} + \omega_{\tau_1}(-1)\sum_{\lambda\in\mathbb{F}_{q^d}^\times}\sum_{x\in\mathbb{F}_{q^d}} g^{su_{(1/\lambda)-x}}_{\xi_\lambda(v)}\\ &= f^1_{q^dv} + \omega_{\tau_1}(-1)\sum_{\lambda\in\mathbb{F}_{q^d}^\times} f^s_{\xi_\lambda(v)}.\end{aligned}$$

Since $\xi_\lambda(v)\in\mathcal{L}+(\tau_1(\varpi_D)\otimes\mathrm{Id})\mathcal{L}$, it follows that $f^1_{q^dv}\in\mathfrak{K}_0\cdot f^s_{\mathcal{L}+(\tau_1(\varpi_D)\otimes\mathrm{Id})\mathcal{L}}$. Therefore, we have

$$f^1_{(\mathrm{Id}\otimes\tau_2(\varpi_D))\mathcal{L}} \subseteq (\mathfrak{K}_0\cdot L_1)^{I(1)} \subseteq L_1^{(1)} \text{ and } f^1_{q^d(\tau_1(\varpi_D)\otimes\mathrm{Id})\mathcal{L}} \subseteq (\mathfrak{K}_0\cdot L_1)^{I(1)} \subseteq L_1^{(1)}.$$

Then by the same arguments as above,

$$f^1_{(\mathrm{Id}\otimes\tau_2(\varpi_D))^i\mathcal{L}} \subseteq L_1^{(i)} \text{ and } f^1_{(q^d(\tau_1(\varpi_D)\otimes\mathrm{Id}))^i\mathcal{L}} \subseteq L_1^{(i)}.$$

By Remark 2.3, the existence of an integral structure in π implies that the sequence of $\mathfrak{K}_1$-lattices $L_1^{(i)}$ stabilizes. This implies that the linear maps $\mathrm{Id}\otimes\tau_2(\varpi_D)$ and $q^d(\tau_1(\varpi_D)\otimes\mathrm{Id})$ stabilize some lattices in $W_1\otimes W_2$. Taking the d-th power of these maps, we get $\omega_{\tau_2}(\varpi_F)\in\mathcal{O}$ and $q^{d^2}\omega_{\tau_1}(\varpi_F)\in\mathcal{O}$.

($\Longleftarrow$) The assumptions on π that its central character is integral and

$$\omega_{\tau_2}(\varpi_F), q^{d^2}\omega_{\tau_1}(\varpi_F)\in\mathcal{O}$$

imply that there exists a T_0-lattice $\mathcal{L}\subseteq W_1\otimes W_2$ such that

$$(\mathrm{Id}\otimes\tau_2(\varpi_D))\mathcal{L}\subseteq\mathcal{L} \text{ and } q^d(\tau_1(\varpi_D)\otimes\mathrm{Id})\mathcal{L}\subseteq\mathcal{L}. \tag{3.3}$$

Using $\mathcal{L}$, we define the lattices L_0 and L_1 as before. Because of (3.3),

$$L_1 = f^1_{\mathcal{L}}\oplus f^s_{\mathcal{L}+(\tau_1(\varpi_D)\otimes\mathrm{Id})\mathcal{L}}.$$

We will prove that $L_1^{(1)} = L_1^{(0)} = L_1$ which implies the integrality of π by Corollary 2.2. This is equivalent to proving that

$$(\mathfrak{K}_0\cdot L_1)\cap V_1 = (\mathfrak{K}_0\cdot L_1)^{I(1)} = L_1.$$

By (3.1) and (3.2), it is enough to show that if

$$l + {\sum_{x,v}}' F_v^x \in (\mathfrak{K}_0\cdot L_1)^{I(1)} \text{ with } l\in L_0,$$

then $l+\sum'_{x,v} F^x_v \in L_1$. Since q^d-1 is invertible in $\mathcal{O}$, we can choose an $\mathcal{O}$-basis $\{v_1,\dots,v_n\}$ of the lattice $\mathcal{L}+(\tau_1(\varpi_D)\otimes \mathrm{Id})\mathcal{L}$ that is an eigenbasis for the operators ξ_λ and such that the scalar multiples of v_i's form an $\mathcal{O}$-basis of the sub-lattice $\mathcal{L}$. Let $\psi_i(\lambda)$ be the eigenvalue for the action of $\xi_{1/\lambda}$ on v_i. Then ψ_i defines a character on $\mathbb{F}_{q^d}^\times$. Extend ψ_i to a function on $\mathbb{F}_{q^d}$ by defining $\psi_i(0)=0$.

Let $l+\sum'_{x,v} F^x_v \in (\mathfrak{K}_0\cdot L_1)^{I(1)}$ with $l\in L_0$. We write

$$\begin{aligned}\sum\nolimits'_{x,v} F^x_v &= \sum_{x\in\mathbb{F}_{q^d}}\left(\sum_{i=1}^n a_{x,i}F^x_{v_i}\right)\\ &= \sum_{i=1}^n\sum_{x\in\mathbb{F}_{q^d}} a_{x,i}F^x_{v_i}\\ &= \sum_{i=1}^n\left(\left(\sum_{x\in\mathbb{F}_{q^d}} a_{x,i}\right)g^1_{v_i}+\omega_{\tau_1}(-1)\sum_{x\in\mathbb{F}_{q^d}} a_{x,i}\sum_{\lambda\in\mathbb{F}_{q^d}^\times} g^{su_{(1/\lambda)-x}}_{\xi_\lambda(v_i)}\right)\\ &= \sum_{i=1}^n\left(f^1_{\left(\sum_{x\in\mathbb{F}_{q^d}} a_{x,i}\right)(v_i)}+\omega_{\tau_1}(-1)\sum_{x,\lambda\in\mathbb{F}_{q^d}} a_{x,i}\psi_i(\lambda)g^{su_{\lambda-x}}_{v_i}\right).\end{aligned}$$

Let us write $a_{x,i}=a_i(-x)$ to view it as a function on $\mathbb{F}_{q^d}$, and let

$$S_1=\sum_{i=1}^n f^1_{\left(\sum_{x\in\mathbb{F}_{q^d}} a_i(-x)\right)(v_i)}\quad\text{and}\quad S_2=\omega_{\tau_1}(-1)\sum_{i=1}^n\sum_{x,\lambda\in\mathbb{F}_{q^d}} a_i(-x)\psi_i(\lambda)g^{su_{\lambda-x}}_{v_i}.$$

Thus

$$\sum\nolimits'_{x,v} F^x_v = S_1+S_2.$$

We may take $l\in\bigoplus_{\lambda\in\mathbb{F}_{q^d}} g^{su_\lambda}_{\mathcal{L}}$ and thus write

$$l=\sum_{i=1}^n\sum_{\lambda\in\mathbb{F}_{q^d}} b_i(\lambda)g^{su_\lambda}_{v_i}.$$

Recall that

$$V_1=f^1_{W_1\otimes W_2}\oplus f^s_{W_1\otimes W_2}\quad\text{and}\quad L_1=f^1_{\mathcal{L}}\oplus f^s_{\mathcal{L}+(\tau_1(\varpi_D)\otimes\mathrm{Id})\mathcal{L}}.$$

Note that S_1 is invariant under the action of $I(1)$. Therefore, $l+S_2$ is also invariant under the action of $I(1)$. Further, the function $l+S_2$ is *not* supported on $BI(1)$. Hence $l+S_2\in f^s_{W_1\otimes W_2}$. Writing

$$l+S_2=\sum_{y\in\mathbb{F}_{q^d}} c_1 g^{su_y}_{v_1}+\ldots+\sum_{y\in\mathbb{F}_{q^d}} c_n g^{su_y}_{v_n}$$

gives that the function

$$y\mapsto \omega_{\tau_1}(-1)\left(\sum_{x\in\mathbb{F}_{q^d}} a_i(-x)\psi_i(x+y)\right)+b_i(y)\quad\text{on }\mathbb{F}_{q^d}$$

is the constant function c_i for all $1\le i\le n$. Thus $c_i\in\mathcal{O}$ for all i, and $l+S_2=f^s_{\sum_i c_iv_i}\in f^s_{\mathcal{L}+(\tau_1(\varpi_D)\otimes\mathrm{Id})\mathcal{L}}\subseteq L_1$.

To show that $S_1\in L_1$, we use Fourier theoretic methods as in [Vig08, §3]. We assume that our coefficient field E is large enough so that the Fourier transform $\widehat{?}$ of $\mathcal{O}$-valued functions ? on $\mathbb{F}_{q^d}$ is well-defined. Following the notation in [Vig08, §3], we let Δ denote the constant function 1 on $\mathbb{F}_{q^d}$, δ_0 denote the characteristic function of 0, and use $*$ to denote the convolution product of two functions. In this notation, $S_1=f^1_{\sum_{i=1}^n\widehat{a_i}(0)v_i}$. We show that $\widehat{a_i}(0)v_i\in\mathcal{L}$ for each $1\le i\le n$. Indeed, for each i, we have from the previous paragraph

$$c_i\Delta=\omega_{\tau_1}(-1)(a_i*\psi_i)+b_i. \tag{3.4}$$

If ψ_i is trivial, then it follows that $c_i = \omega_{\tau_1}(-1)(\widehat{a_i}(0) - a_i(y)) + b_i(y)$ for all $y \in \mathbb{F}_{q^d}$. Hence

$$(\widehat{a_i}(0) - \omega_{\tau_1}(-1)c_i)v_i + \omega_{\tau_1}(-1)b_i(y)v_i = a_i(y)v_i \text{ for all } y \in \mathbb{F}_{q^d}.$$

Adding over $y \in \mathbb{F}_{q^d}$ gives

$$(\widehat{a_i}(0) - \omega_{\tau_1}(-1)c_i)q^d v_i + \omega_{\tau_1}(-1)\widehat{b_i}(0)v_i = \widehat{a_i}(0)v_i.$$

By (3.3), $(\widehat{a_i}(0) - \omega_{\tau_1}(-1)c_i)q^d v_i \in \mathcal{L}$. Further, recall that $\sum_{i=1}^n b_i(y)v_i \in \mathcal{L}$. The choice of the basis $\{v_i\}$ implies that each $b_i(y)v_i \in \mathcal{L}$. Thus $\widehat{b_i}(0)v_i \in \mathcal{L}$. Hence $\widehat{a_i}(0)v_i \in \mathcal{L}$. If ψ_i is non-trivial, then applying the Fourier transform on both sides of (3.4) gives

$$c_i q^d \delta_0 = \omega_{\tau_1}(-1)\widehat{a_i}\widehat{\psi_i} + \widehat{b_i}.$$

Multiplying both sides by $\widehat{\psi_i^{-1}}$ and using that $\widehat{\psi_i}\widehat{\psi_i^{-1}} = \psi_i(-1)q^d(\Delta - \delta_0)$, we get

$$c_i q^d \delta_0 \widehat{\psi_i^{-1}} = \omega_{\tau_1}(-1)\psi_i(-1)q^d(\Delta - \delta_0)\widehat{a_i} + \widehat{b_i}\widehat{\psi_i^{-1}}.$$

Thus

$$c_i q^d \delta_0 \widehat{\psi_i^{-1}} = \omega_{\tau_1}(-1)\psi_i(-1)q^d \widehat{a_i} - \omega_{\tau_1}(-1)\psi_i(-1)q^d\delta_0\widehat{a_i} + \widehat{b_i}\widehat{\psi_i^{-1}}.$$

Rearranging the terms, we have

$$q^d\widehat{a_i} = q^d\widehat{a_i}(0)\delta_0 + \widehat{\psi_i^{-1}}\widehat{\phi} = q^d\widehat{a_i}(0)\delta_0 + \widehat{\psi_i^{-1} * \phi}$$

where $\phi = \omega_{\tau_1}(-1)\psi_i(-1)(c_i\Delta - b_i)$. By the Fourier transform again, we get

$$q^d a_i = \widehat{a_i}(0)\Delta + \psi_i^{-1} * \phi.$$

Hence

$$\widehat{a_i}(0)v_i = a_i(0)q^d v_i - (\psi_i^{-1} * \phi)(0)v_i.$$

Note that

$$\begin{aligned}
(\psi_i^{-1} * \phi)(0)v_i &= \omega_{\tau_1}(-1)\psi_i(-1)\left(\sum_{x\in\mathbb{F}_{q^d}} \psi_i^{-1}(-x)(c_i - b_i(x))\right)v_i \\
&= \omega_{\tau_1}(-1)\psi_i(-1)\left(c_i\widehat{\psi_i^{-1}}(0) - \sum_{x\in\mathbb{F}_{q^d}} \psi_i^{-1}(-x)b_i(x)\right)v_i \\
&= -\omega_{\tau_1}(-1)\psi_i(-1)\sum_{x\in\mathbb{F}_{q^d}} \psi_i^{-1}(-x)b_i(x)v_i \in \mathcal{L}.
\end{aligned}$$

Also $q^d v_i \in \mathcal{L}$ by (3.3)). Therefore $\widehat{a_i}(0)v_i \in \mathcal{L}$.

It follows that $S_1 = f^1_{\sum_{i=1}^n \widehat{a_i}(0)v_i} \in f^1_{\mathcal{L}} \subseteq L_1$. Therefore, $l + \sum'_{x,v} F^x_v = l + S_1 + S_2 \in L_1$. ☐

Let E be large enough to contain (a fixed choice of) $\sqrt{q}$, and let

$$\tau_1 \times \tau_2 := \mathrm{Ind}_B^G\left(\tau_1|\cdot|^{\frac{1}{2}} \otimes \tau_2|\cdot|^{-\frac{1}{2}}\right)$$

be the normalized parabolic induction over E. The integrality criterion in Theorem 3.1 is symmetric for the normalized induction:

Theorem 3.2. *Let τ_1 and τ_2 be smooth irreducible tamely ramified representations of $D^\times$. The representation $\tau_1 \times \tau_2$ with integral central character admits an integral structure if and only if $\omega_{\tau_1}(\varpi_F)q^{\frac{d^2}{2}}, \omega_{\tau_2}(\varpi_F)q^{\frac{d^2}{2}} \in \mathcal{O}$.*

As a consequence of Theorem 3.2, we obtain that when $\tau_1 \times \tau_2$ with integral central character is reducible, its irreducible subquotients are always integral. Indeed, enlarging E if necessary, we assume that all irreducible subquotients of $\tau_1 \times \tau_2$ are defined over E. We recall a result of Tadić which says that $(\tau_1 \times \tau_2)_{\mathbb{C}}$, which is isomorphic to $(\tau_1)_{\mathbb{C}} \times (\tau_2)_{\mathbb{C}}$, is reducible if and only if $(\tau_2)_{\mathbb{C}} \cong (\tau_1)_{\mathbb{C}}|\cdot|_{\mathbb{C}}^{\pm\frac{a}{d}}$, and in this case, it is multiplicity-free and has length 2. Here a is the length of the segment of the essentially square-integrable representation of $\mathrm{GL}_d(F)$ associated to $(\tau_1)_{\mathbb{C}}$ under the Jacquet-Langlands correspondence. It follows that $\tau_1 \times \tau_2$ is reducible over E if and only if $\tau_2 \cong \tau_1|\cdot|^{\pm\frac{a}{d}}$ as E-linear representations (again after enlarging E if necessary). If τ_1 is tamely ramified of dimension d', then $d = ad'$.

Let $\tau = \tau_1|\cdot|^{\frac{a}{2d}}$. Denoting by $\mathrm{St}(\tau)$ and $\mathrm{Sp}(\tau)$ the E-models of the *Steinberg* $\mathrm{St}(\tau_{\mathbb{C}})$ and the *Speh* $\mathrm{Sp}(\tau_{\mathbb{C}})$ representation respectively, one has the following short exact sequence of smooth E-linear representations

$$0 \longrightarrow \mathrm{Sp}(\tau) \longrightarrow \tau|\cdot|^{-\frac{a}{2d}} \times \tau|\cdot|^{\frac{a}{2d}} \longrightarrow \mathrm{St}(\tau) \longrightarrow 0 \tag{3.5}$$

(see [Rag07, Theorem 2.2]). We remark that $\mathrm{Sp}(\tau)$ is infinite-dimensional if and only if τ has dimension > 1 [Rag07, Remark 2.4].

Theorem 3.3. *Let τ be a smooth absolutely irreducible tamely ramified representation of $D^\times$ over E. The representation $\mathrm{St}(\tau)$ with integral central character admits an integral structure. The representation $\mathrm{Sp}(\tau)$ with integral central character admits an integral structure.*

Proof. Let $\Pi = \tau|\cdot|^{-\frac{a}{2d}} \times \tau|\cdot|^{\frac{a}{2d}}$. From the sequence (3.5), we see that $\omega_{\mathrm{St}(\tau)} = \omega_{\mathrm{Sp}(\tau)} = \omega_\Pi = \omega_\tau^2$. If $\omega_{\mathrm{St}(\tau)}$ is integral, then $\omega_\tau(\varpi_F) \in \mathcal{O}^\times$ and thus ω_Π is integral. Further, note that

$$\omega_\tau(\varpi_F)q^{\frac{d(d+a)}{2}}, \omega_\tau(\varpi_F)q^{\frac{d(d-a)}{2}} \in \mathcal{O}.$$

Hence, by Theorem 3.2, Π has an integral structure and thus its quotient $\mathrm{St}(\tau)$ also has an integral structure [Vig96, II.4.14(a)]. One similarly shows that $\mathrm{Sp}(\tau)$ with integral central character $\omega_{\mathrm{Sp}(\tau)}$ has an integral structure. □

Corollary 3.4. $\mathrm{St}(\tau)$ *is integral if and only if* $\mathrm{Sp}(\tau)$ *is integral.* □

4. Integrality of locally algebraic representations

In this section, $\pi = \pi_{sm} \otimes \pi_{alg}$ where π_{alg} is a non-trivial irreducible algebraic representation of G over E. We begin with a simple generalization of [Hu21, Proposition 2.2] of Hu on diagrams of k-vector spaces with trivial 0-th homology. We say that a diagram $\mathcal{D}_1 \xrightarrow{r} \mathcal{D}_0$ admits a central character if Z acts on $\mathcal{D}_0$ and $\mathcal{D}_1$ by a character.

Lemma 4.1. *Let $\mathcal{D}$ be a diagram $\mathcal{D}_1 \xrightarrow{r} \mathcal{D}_0$ of smooth k-representations admitting a central character such that $H_0(\mathcal{D}) = 0$ and $\mathcal{D}_1$ is an irreducible representation of $\mathfrak{K}_1$. Then*

$$\dim_k\mathcal{D}_0 \leq \frac{q^d+1}{2}\dim_k\mathcal{D}_1.$$

Moreover, if $\dim_k\mathcal{D}_0 = \frac{q^d+1}{2}\dim_k\mathcal{D}_1$, then $\mathcal{D}_0 \cong \mathrm{Ind}_{I\varpi_D^{\mathbb{Z}}}^{\mathfrak{K}_0} r(\mathcal{D}_1)$.

Proof. Since $H_0(\mathcal{D}) = 0$, $\mathrm{Ker}(r) \neq 0$. Pick a non-zero $I/I(1)$-eigenvector $v \in \mathrm{Ker}(r)$. Let $\mathcal{D}' \subseteq \mathcal{D}$ be the subdiagram $\mathcal{D}'_1 \xrightarrow{r'} \mathcal{D}'_0$ where

$$\mathcal{D}'_1 = \mathfrak{K}_1 \cdot v,\ \mathcal{D}'_0 = \mathfrak{K}_0 \cdot r(\mathcal{D}'_1), \text{ and } r' = r|_{\mathcal{D}'_1}. \tag{4.1}$$

Since $\mathcal{D}_1$ is irreducible, $\mathcal{D}_1/\mathcal{D}'_1 = 0$. Further, $H_0(\mathcal{D}) = 0$ implies that $H_0(\mathcal{D}/\mathcal{D}') = 0$. Therefore, $\mathcal{D}_0/\mathcal{D}'_0 = 0$. Consequently, $\mathcal{D}' = \mathcal{D}$, and thus there is a surjection

$$\mathrm{Ind}_{I\varpi_D^{\mathbb{Z}}}^{\mathfrak{K}_0} r(\mathcal{D}_1) \twoheadrightarrow \mathcal{D}_0. \tag{4.2}$$

Let $\mathcal{D}_1^0 \subseteq \mathcal{D}_1$ be the k-span of vectors $\varpi_D^i v$ for $0 \leq i \leq d-1$. Then $\mathcal{D}_1 = \mathcal{D}_1^0 + t\mathcal{D}_1^0$. Note that t is a linear isomorphism. So

$$\frac{\dim_k\mathcal{D}_1}{2} \leq \dim_k\mathcal{D}_1^0.$$

Moreover, as $r(v) = 0$ and r is $\varpi_D^{\mathbb{Z}}$-linear, we have $\mathcal{D}_1^0 \subseteq \mathrm{Ker}(r)$. Thus

$$\frac{\dim_k\mathcal{D}_1}{2} \leq \dim_k\mathcal{D}_1^0 \leq \dim_k\mathrm{Ker}(r).$$

Therefore, $\dim_k\mathcal{D}_1 = \dim_k r(\mathcal{D}_1) + \dim_k\mathrm{Ker}(r) \geq \dim_k r(\mathcal{D}_1) + \frac{\dim_k\mathcal{D}_1}{2}$. Hence $\dim_k r(\mathcal{D}_1) \leq \frac{\dim_k\mathcal{D}_1}{2}$. Now it follows from (4.2) that

$$\dim_k\mathcal{D}_0 \leq \frac{q^d+1}{2}\dim_k\mathcal{D}_1$$

because $[\mathfrak{K}_0 : I\varpi_D^{\mathbb{Z}}] = q^d + 1$.

If $\dim_k \mathcal{D}_0 = \frac{q^d+1}{2}\dim_k \mathcal{D}_1$, then from (4.2), we get that $\frac{\dim_k \mathcal{D}_1}{2} \leq \dim_k r(\mathcal{D}_1)$. By the previous paragraph, this implies $\frac{\dim_k \mathcal{D}_1}{2} = \dim_k r(\mathcal{D}_1)$ and thus $\mathcal{D}_0 \cong \mathrm{Ind}_{I\varpi_D^{\mathbb{Z}}}^{\mathfrak{K}_0} r(\mathcal{D}_1)$. □

Remark 4.2. In the above lemma, if $\mathcal{D}_1$ is not irreducible, then the diagram $\mathcal{D}$ has a filtration by subdiagrams whose graded pieces are the diagrams of the form (4.1). Thus the same dimension relation as in the lemma holds if $\dim_k \mathcal{D}_1 < \infty$. Further, if $\dim_k \mathcal{D}_0 = \frac{q^d+1}{2}\dim_k \mathcal{D}_1$, then for any graded piece $\mathcal{Q}_1 \xrightarrow{\overline{r}} \mathcal{Q}_0$ of $\mathcal{D}$, $\mathcal{Q}_0 \cong \mathrm{Ind}_{I\varpi_D^{\mathbb{Z}}}^{\mathfrak{K}_0} \overline{r}(\mathcal{Q}_1)$.

The next lemma is well-known:

Lemma 4.3. *Suppose a group H acting on a finite-dimensional $\mathbb{Q}_p$-vector space V stabilizes a lattice in V. Then H stabilizes finitely many homothety classes of lattices in V if and only if its action on V is irreducible.*

Proof. The group H acts irreducibly on V if and only if its image $\overline{H}$ in $\mathrm{GL}(V)$ is not contained a proper parabolic subgroup of $\mathrm{GL}(V) \cong \mathrm{GL}_n(\mathbb{Q}_p)$. Suppose $\overline{H}$ is contained a proper parabolic subgroup of $\mathrm{GL}_n(\mathbb{Q}_p)$. Since H stabilizes a lattice, without loss of generality, we may assume that $\overline{H}$ is a subgroup of a standard proper parabolic subgroup of $\mathrm{GL}_n(\mathbb{Z}_p)$ corresponding to the partition $n = n_1 + \ldots + n_k$. For $m \in \mathbb{N}$, consider the lattice L_m given by the direct sum of n_1 copies of $\frac{1}{p^m}\mathbb{Z}_p$ and $n_2 + \ldots + n_k$ copies of $\mathbb{Z}_p$. Then H stabilizes the infinite family $\{[L_m]\}_{m\in\mathbb{N}}$ of homothety classes of lattices.

Conversely, assume that H stabilizes an infinite family of homothety classes of lattices. Fix a set $\{g_\alpha\}$ of representatives for $\mathrm{GL}_n(\mathbb{Q}_p)/\mathbb{Q}_p^\times \mathrm{GL}_n(\mathbb{Z}_p)$ such that $g_\alpha = (g_{ij}^\alpha) \in \mathrm{M}_n(\mathbb{Z}_p)$. If $L_0 = \mathbb{Z}_p \oplus \ldots \oplus \mathbb{Z}_p$ denotes the standard lattice in $\mathbb{Q}_p^n$, then

$$g_\alpha L_0 = \bigoplus_{i=1}^n p^{\min_j\{\mathrm{val}_p(g_{ij}^\alpha)\}}\mathbb{Z}_p.$$

By assumption, $\overline{H}$ stabilizes a family $\{g_\alpha L_0 : \alpha \in \mathcal{I}\}$ of lattices where $\mathcal{I}$ is not finite. Thus there exists i, $1 \leq i \leq n$, such that

$$\max_{\alpha\in\mathcal{I}}\{\min_j\{\mathrm{val}_p(g_{ij}^\alpha)\}\}$$

is not bounded. Consequently, we find that $\bigcap_{\alpha\in\mathcal{I}} g_\alpha L_0$ is contained in a proper subspace of V stable under the action of H, i.e., the H-action on V is reducible. □

Let $\tau = \mathrm{Ind}_{D(1)D_{d'}^\times}^{D^\times}\theta$ be a smooth absolutely irreducible tamely ramified representation of $D^\times$ of dimension d'.

Lemma 4.4. *(i) As $I/I(1)$-representations,*

$$\mathrm{St}(\tau)^{I(1)} \cong \mathrm{Sp}(\tau)^{I(1)} \cong (\theta \oplus \theta^q \oplus \ldots \oplus \theta^{q^{d'-1}}) \otimes (\theta \oplus \theta^q \oplus \ldots \oplus \theta^{q^{d'-1}}).$$

(ii) The representations $\mathrm{St}(\tau)^{I(1)}$ and $\mathrm{Sp}(\tau)^{I(1)}$ are irreducible as $\mathfrak{K}_1$-representations if and only if $d' = 1, 2$.

Proof. We prove lemma for $\mathrm{St}(\tau)$; the proof for $\mathrm{Sp}(\tau)$ is similar.

(i) By [MP96, Proposition 6.7], the natural T-equivariant surjective map

$$\mathrm{St}(\tau_{\mathbb{C}}) \twoheadrightarrow \mathrm{St}(\tau_{\mathbb{C}})_N$$

from the Steinberg representation to its smooth Jacquet module induces a $(T \cap I)$-equivariant isomorphism

$$\mathrm{St}(\tau_{\mathbb{C}})^{I(1)} \to (\mathrm{St}(\tau_{\mathbb{C}})_N)^{T\cap I(1)}.$$

By [Rag07, Theorem 2.2 (ii)], $\mathrm{St}(\tau_{\mathbb{C}})_N \cong \tau_{\mathbb{C}}|\cdot|_{\mathbb{C}}^{\frac{a+d}{2d}} \otimes \tau_{\mathbb{C}}|\cdot|_{\mathbb{C}}^{-\frac{a+d}{2d}}$ as T-representations. So the group $T \cap I(1) = D(1) \times D(1)$ acts trivially on $\mathrm{St}(\tau_{\mathbb{C}})_N$ because $\tau_{\mathbb{C}}$ is tamely ramified, i.e.,

$$(\mathrm{St}(\tau_{\mathbb{C}})_N)^{T\cap I(1)} = \mathrm{St}(\tau_{\mathbb{C}})_N.$$

Since all the representations are defined over E, it follows that

$$\mathrm{St}(\tau)^{I(1)} \cong \tau|\cdot|^{\frac{a+d}{2d}} \otimes \tau|\cdot|^{-\frac{a+d}{2d}}$$

as E-linear $I/I(1)$-representations. Now, (i) follows from the isomorphisms

$$\tau|\cdot|^{\frac{a+d}{2d}} \cong \tau|\cdot|^{-\frac{a+d}{2d}} \cong \theta \oplus \theta^q \oplus \ldots \oplus \theta^{q^{d'-1}}$$

as $\mathcal{O}_D^\times/D(1)$-representations.

(ii) The $I/I(1)$-representation $\mathrm{St}(\tau)^{I(1)}$ is a sum of d'^2 distinct $I/I(1)$-characters because the Galois conjugates of θ are distinct. Recall that $\mathfrak{K}_1$ is generated by I and t. So the $\mathfrak{K}_1$-subrepresentation of $\mathrm{St}(\tau)^{I(1)}$ generated by a non-zero $I/I(1)$-eigenvector has dimension at most $2d'$. Hence $\mathrm{St}(\tau)^{I(1)}$ is reducible if $d' > 2$. Conversely, if $d' = 2$, it is easy to check that any of the four $I/I(1)$-characters in $\mathrm{St}(\tau)^{I(1)}$ generate the whole representation under the action of t. ☐

By Frobenius reciprocity, there are non-zero maps of K-representations

$$\mathrm{Ind}_I^K(\theta^{q^i} \otimes \theta^{q^j}) \to \mathrm{St}(\tau)^{K(1)} \text{ and } \mathrm{Ind}_I^K(\theta^{q^i} \otimes \theta^{q^j}) \to \mathrm{Sp}(\tau)^{K(1)}$$

for all $0 \le i, j \le d' - 1$. The representations $\mathrm{Ind}_I^K(\theta^{q^i} \otimes \theta^{q^j})$ are irreducible if $i \ne j$. Thus,

$$\mathrm{Ind}_I^K(\theta^{q^i} \otimes \theta^{q^j}) \subseteq \mathrm{St}(\tau)^{K(1)} \text{ and } \mathrm{Ind}_I^K(\theta^{q^i} \otimes \theta^{q^j}) \subseteq \mathrm{Sp}(\tau)^{K(1)} \tag{4.3}$$

for all $0 \le i < j \le d'-1$. If $i = j$, then $\mathrm{Ind}_I^K(\theta^{q^i} \otimes \theta^{q^i})$ is a sum of 2 irreducible subrepresentations $\theta^{q^i} \circ \overline{\det}$ and $\mathrm{st}(\theta^{q^i})$. Here, $\overline{\det}$ is the composition of the determinant character of $\mathrm{GL}_2(\mathbb{F}_{q^d})$ and the natural surjection $K \twoheadrightarrow \mathrm{GL}_2(\mathbb{F}_{q^d})$.

Lemma 4.5. *As K-representations*

$$\mathrm{St}(\tau)^{K(1)} \cong \bigoplus_{i<j} \mathrm{Ind}_I^K(\theta^{q^i} \otimes \theta^{q^j}) \oplus \bigoplus_i \mathrm{st}(\theta^{q^i}) \text{ and}$$

$$\mathrm{Sp}(\tau)^{K(1)} \cong \bigoplus_{i<j} \mathrm{Ind}_I^K(\theta^{q^i} \otimes \theta^{q^j}) \oplus \bigoplus_i (\theta^{q^i} \circ \overline{\det}).$$

Proof. It suffices to show that $\theta \circ \overline{\det} \subseteq \mathrm{Sp}(\tau)^{K(1)}$. Indeed, if $\theta \circ \overline{\det} \subseteq \mathrm{Sp}(\tau)^{K(1)}$, then the diagonal action of ϖ_D gives $\theta^{q^i} \circ \overline{\det} \subseteq \mathrm{Sp}(\tau)^{K(1)}$ for all i, and the multiplicity-freeness of $\mathrm{St}(\tau)^{I(1)}$ and $\mathrm{Sp}(\tau)^{I(1)}$ then implies that $\mathrm{st}(\theta^{q^i}) \subseteq \mathrm{St}(\tau)^{K(1)}$ for all i. We may use [MS14, Proposition 7.21(1)] to conclude that $\theta \circ \overline{\det} \subseteq \mathrm{Sp}(\tau)^{K(1)}$. ☐

Theorem 4.6. *Let τ be a smooth absolutely irreducible tamely ramified representation of $D^\times$ of dimension $d' \le 2$. Let $\pi = \mathrm{St}(\tau) \otimes \pi_{alg}$ be an irreducible locally algebraic representation with integral central character. Then π admits an integral structure.*

Proof. Let $V_1 = \mathrm{St}(\tau)^{I(1)} \otimes \pi_{alg}$ and $V_0 = \mathrm{St}(\tau)^{K(1)} \otimes \pi_{alg}$. The group $\mathfrak{K}_1$ is isomorphic to $I \rtimes t^{\mathbb{Z}}$. Since I is compact, $t^{2d} \in Z$, and Z acts on V_1 by an integral character, it follows that V_1 contains a $\mathfrak{K}_1$-lattice L_1. Moreover, as $\mathrm{St}(\tau)^{I(1)}$ is an irreducible $\mathfrak{K}_1$-representation by Lemma 4.4 (ii) and π_{alg} is also an irreducible $\mathfrak{K}_1$-representation, V_1 is an irreducible locally algebraic representation of $\mathfrak{K}_1$. Thus V_1 contains finitely many homothety classes of $\mathfrak{K}_1$-lattices by Lemma 4.3.

Suppose π is not integral. Then, by Corollary 2.2, the increasing sequence of $\mathfrak{K}_1$-lattices $(L_1^{(i)})_i$ of V_1 does not become stationary. By the previous paragraph, there is $i_0 > 0$ such that $L_1^{(i_0)}$ and L_1 are in the same homothety class, i.e., $L_1^{(i_0)} = \varpi^j L_1$ for some $j < 0$. Let $L_0 = \mathfrak{K}_0 \cdot L_1$ and $L_0^{(i_0)} = \mathfrak{K}_0 \cdot L_1^{(i_0)} = \varpi^j L_0$. Let

$$\mathcal{D}_{\mathcal{O}} = L_1 \hookrightarrow L_0 \text{ and } \mathcal{D}_{\mathcal{O}}^{(i_0)} = L_1^{(i_0)} \hookrightarrow L_0^{(i_0)}$$

be the corresponding diagrams of free $\mathcal{O}\mathfrak{K}_i$-modules. The diagram $\mathcal{D}_{\mathcal{O}}^{(i_0)}$ is equal to the diagram $\varpi^j \mathcal{D}_{\mathcal{O}}$. Thus the natural surjective map $H_0(\mathcal{D}_{\mathcal{O}}) \twoheadrightarrow H_0(\mathcal{D}_{\mathcal{O}}^{(i_0)})$ gives $H_0(\varpi^j \mathcal{D}_{\mathcal{O}}/\mathcal{D}_{\mathcal{O}}) = 0$. By dévissage, we have $H_0(\mathcal{D}_k) = 0$ where $\mathcal{D}_k = \mathcal{D}_{\mathcal{O}} \otimes_{\mathcal{O}} k = \varpi^{-1}\mathcal{D}_{\mathcal{O}}/\mathcal{D}_{\mathcal{O}}$. By Lemma 4.1 and Remark 4.2, we get that $\dim_k(L_0 \otimes_{\mathcal{O}} k) \le \frac{q^d+1}{2} \dim_k(L_1 \otimes_{\mathcal{O}} k)$. Since $\dim_k(L_0 \otimes_{\mathcal{O}} k) = \dim_E V_0$ and $\dim_k(L_1 \otimes_{\mathcal{O}} k) = \dim_E V_1$, we obtain $\dim_E V_0 \le \frac{q^d+1}{2} \dim_E V_1$. However, it follows from Lemma 4.4 (i) that

$$\dim_E \mathrm{St}(\tau)^{I(1)} = d'^2,$$

and from Lemma 4.5 that

$$\dim_E \mathrm{St}(\tau)^{K(1)} = \frac{1}{2}(d'^2 - d')(q^d + 1) + d'q^d.$$

This implies that $\dim_E V_0 > \frac{q^d+1}{2}\dim_E V_1$. A contradiction. □

4.1. **An example of an integral locally algebraic Speh representation.** For simplicity, we take D to be the quaternionic division algebra in this subsection. Let $\tau = \mathrm{Ind}_{D(1)D_2^\times}^{D^\times}\theta$ be a smooth absolutely irreducible tamely ramified representation of $D^\times$ over E. Note that τ has dimension 2 and hence $\mathrm{Sp}(\tau)$ is infinite-dimensional. Consider the following irreducible locally algebraic representation

$$\pi := \mathrm{Sp}(\tau) \otimes (\mathrm{Sym}^1 E^4 \otimes \det^{-\frac{1}{4}})$$

of G. Here, G acts on the algebraic representation via $G \hookrightarrow \mathrm{GL}_4(F_2) \hookrightarrow \mathrm{GL}_4(E)$ induced by the map $D \to \mathrm{M}_2(F_2)$, $\alpha + \beta\varpi_D \mapsto \left(\begin{smallmatrix} \alpha & \beta\varpi_F \\ \sigma(\beta) & \sigma(\alpha)\end{smallmatrix}\right)$ where σ is the unique non-trivial automorphism in $\mathrm{Gal}(F_2/F)$. We assume that ω_τ is integral so that the central character ω_π of π is integral. We now show that π *is* integral which, in particular, implies that Emerton's conditions are sufficient for the integrality of π. We take $s = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0\end{smallmatrix}\right)$ in this subsection so that $\overline{\det}(s) = 1$.

Proposition 4.7. *The representation π admits an integral structure.*

Proof. Recall from Lemma 4.5 that

$$\mathrm{Sp}(\tau)^{K(1)} = (\theta \circ \overline{\det}) \oplus (\theta^q \circ \overline{\det}) \oplus \mathrm{Ind}_I^K(\theta \otimes \theta^q),$$
$$\mathrm{Sp}(\tau)^{I(1)} = (\theta \otimes \theta) \oplus (\theta^q \otimes \theta^q) \oplus (\theta \otimes \theta^q) \oplus (\theta^q \otimes \theta).$$

Let e_1 be a non-zero vector in the underlying space of the character $\theta \circ \overline{\det}$. Then $e_2 := t^2 e_1$ and $e_0 := te_1$ are the non-zero vectors in the underlying spaces of the characters $\theta^q \circ \det$ and $\theta \otimes \theta^q$ of K and I respectively. Note that $(\theta \circ \overline{\det})|_I = \theta \otimes \theta$ and $(\theta^q \circ \overline{\det})|_I = \theta^q \otimes \theta^q$. The K-representation $\mathrm{Ind}_I^K Ee_0$ is stable under the action of t^2. Thus $t^2 e_0 = \varepsilon\varpi^\nu f_0$ for some $\varepsilon \in \mathcal{O}^\times$ and $\nu \in \mathbb{Z}$ where $f_0 = \sum_x u_x s e_0 \in (\mathrm{Ind}_I^K Ee_0)^{I(1)}$ is a function supported on $IsI(1)$. Let $q = \varepsilon'\varpi^{\nu'}$ where $\varepsilon' \in \mathcal{O}^\times$ and $\nu' \in \mathbb{Z}$. The evaluation of the $I(1)$-invariant function $\sum_x u_x s f_0$ on 1 is q^2 and on s is 0. Thus $\sum_x u_x s f_0 = q^2 e_0$. Using that $t^2 \circ (\sum_x u_x s) = (\sum_x u_x s) \circ t^2$ and that the action of t^4 is by multiplication by a unit, one obtains that $\nu = -\nu'$.

Consider the K-lattice

$$M_0 = \mathrm{Sym}^1\mathcal{O}^4 \otimes \det^{-\frac{1}{4}} = \mathcal{O}X_1 \oplus \mathcal{O}X_2 \oplus \mathcal{O}Y_1 \oplus \mathcal{O}Y_2$$

in the representation $\mathrm{Sym}^1 E^4 \otimes \det^{-\frac{1}{4}}$. Then $M_1 = (M_0 + tM_0) + t^2(M_0 + tM_0)$ is a $\mathfrak{K}_1$-lattice in $\mathrm{Sym}^1 E^4 \otimes \det^{-\frac{1}{4}}$. One computes that

$$\begin{aligned} M_1 &= \varpi_F^{\frac{-1}{4}}\mathcal{O}X_1 \oplus \varpi_F^{\frac{-3}{4}}\mathcal{O}X_2 \oplus \varpi_F^{\frac{0}{4}}\mathcal{O}Y_1 \oplus \varpi_F^{\frac{-2}{4}}\mathcal{O}Y_2 \\ &= \varpi^{\frac{-e}{4}}\mathcal{O}X_1 \oplus \varpi^{\frac{-3e}{4}}\mathcal{O}X_2 \oplus \varpi^0\mathcal{O}Y_1 \oplus \varpi^{\frac{-2e}{4}}\mathcal{O}Y_2 \end{aligned}$$

where e is the ramification index $e(E/F)$. Note that 4 divides e because E is taken large enough to contain $\varpi_F^{\frac{1}{4}}$. Hence $\nu' \geq e \geq 4$ and $\nu \leq -4$.

Consider the following $\mathfrak{K}_1$-lattice of V_1:

$$L_1^{(0)} = L_1 := (\varpi^0\mathcal{O}e_1 \oplus \varpi^0\mathcal{O}e_2 \oplus \varpi^0\mathcal{O}e_0 \oplus \varpi^\nu\mathcal{O}f_0) \otimes M_1.$$

For the ease of computation, we write $L_1^{(0)}$ as follows:

$$\begin{aligned} L_1^{(0)} &= (0e_1 \oplus 0e_2) \otimes (\tfrac{-e}{4}X_1 \oplus \tfrac{-3e}{4}X_2 \oplus 0Y_1 \oplus \tfrac{-2e}{4}Y_2) \\ &\quad \oplus (0e_0 \oplus \nu f_0) \otimes (\tfrac{-e}{4}X_1 \oplus \tfrac{-3e}{4}X_2 \oplus 0Y_1 \oplus \tfrac{-2e}{4}Y_2). \end{aligned}$$

Let us record the actions of $u_x s$, t, and t^2:

$$\begin{aligned}
&u_xs((ae_1 \oplus be_2) \otimes (n_1X_1 \oplus n_2X_2 \oplus m_1Y_1 \oplus m_2Y_2))\\
&= (ae_1 \oplus be_2) \otimes (n_1([x]X_1 + Y_1) \oplus n_2([x^q]X_2 + Y_2) \oplus m_1X_1 \oplus m_2X_2),\\
&t((ae_1 \oplus be_2) \otimes (n_1X_1 \oplus n_2X_2 \oplus m_1Y_1 \oplus m_2Y_2))\\
&= (ae_0 \oplus (b+\nu)f_0) \otimes ((n_1 + \tfrac{-e}{4})Y_2 \oplus (n_2 + \tfrac{3e}{4})Y_1 \oplus (m_1 + \tfrac{-e}{4})X_1 \oplus (m_2 + \tfrac{-e}{4})X_2),\\
&t^2((ae_1 \oplus be_2) \otimes (n_1X_1 \oplus n_2X_2 \oplus m_1Y_1 \oplus m_2Y_2))\\
&= (be_1 \oplus ae_2) \otimes ((n_1 + \tfrac{-2e}{4})X_2 \oplus (n_2 + \tfrac{2e}{4})X_1 \oplus (m_1 + \tfrac{-2e}{4})Y_2 \oplus (m_2 + \tfrac{2e}{4})Y_1),\\
&t((ae_0 \oplus bf_0) \otimes (n_1X_1 \oplus n_2X_2 \oplus m_1Y_1 \oplus m_2Y_2))\\
&= ((b-\nu)e_1 \oplus ae_2) \otimes ((n_1 + \tfrac{-e}{4})Y_2 \oplus (n_2 + \tfrac{3e}{4})Y_1 \oplus (m_1 + \tfrac{-e}{4})X_1 \oplus (m_2 + \tfrac{-e}{4})X_2),\\
&t^2((ae_0 \oplus bf_0) \otimes (n_1X_1 \oplus n_2X_2 \oplus m_1Y_1 \oplus m_2Y_2))\\
&= ((b-\nu)e_0 \oplus (a+\nu)f_0) \otimes ((n_1 + \tfrac{-2e}{4})Y_2 \oplus (n_2 + \tfrac{2e}{4})Y_1 \oplus (m_1 + \tfrac{-2e}{4})X_1 \oplus (m_2 + \tfrac{2e}{4})X_2).
\end{aligned}$$

We thus have

$$\begin{aligned}
&\sum_x u_xs((ae_0 \oplus bf_0) \otimes (n_1X_1 \oplus n_2X_2 \oplus m_1Y_1 \oplus m_2Y_2))\\
&= ((b-2\nu)e_0 \oplus af_0) \otimes ((m_1 - 2\nu)X_1 \oplus (m_2 - 2\nu)X_2) \oplus (n_1 - 2\nu)Y_1 \oplus (n_2 - 2\nu)Y_2.
\end{aligned}$$

We now compute $L_1^{(1)}$ in two steps:

(i)

$$\begin{aligned}
(\mathfrak{K}_0 \cdot L_1^{(0)}) \cap V_1 = {} & (0e_1 \oplus 0e_2) \otimes (\tfrac{-e}{4}X_1 \oplus \tfrac{-3e}{4}X_2 \oplus \tfrac{-e}{4}Y_1 \oplus \tfrac{-3e}{4}Y_2)\\
&\oplus (0e_0 \oplus \nu f_0) \otimes (\tfrac{-e}{4}X_1 \oplus \tfrac{-3e}{4}X_2 \oplus 0Y_1 \oplus \tfrac{-2e}{4}Y_2).
\end{aligned}$$

In the above computation, we used $s(e_i \otimes X_j) = e_i \otimes Y_j$ for the first half of the lattice $(\mathfrak{K}_0 \cdot L_1^{(0)}) \cap V_1$. That the second half of the lattice $(\mathfrak{K}_0 \cdot L_1^{(0)}) \cap V_1$ is the same as that of $L_1^{(0)}$ follows because $\nu \leq -4$ and thus the contribution from the action of $\sum_x u_xs$ is already in the lattice $L_1^{(0)}$. Consequently, we have

$$\begin{aligned}
t((\mathfrak{K}_0 \cdot L_1^{(0)}) \cap V_1) = {} & (0e_0 \oplus \nu f_0) \otimes (\tfrac{-2e}{4}Y_2 \oplus 0Y_1 \oplus \tfrac{-2e}{4}X_1 \oplus \tfrac{-4e}{4}X_2)\\
&\oplus (0e_1 \oplus 0e_2) \otimes (\tfrac{-2e}{4}Y_2 \oplus 0Y_1 \oplus \tfrac{-e}{4}X_1 \oplus \tfrac{-3e}{4}X_2).
\end{aligned}$$

(ii)

$$\begin{aligned}
(\mathfrak{K}_0 \cdot L_1^{(0)}) \cap V_1 + t((\mathfrak{K}_0 \cdot L_1^{(0)}) \cap V_1) = {} & (0e_1 \oplus 0e_2) \otimes (\tfrac{-e}{4}X_1 \oplus \tfrac{-3e}{4}X_2 \oplus \tfrac{-e}{4}Y_1 \oplus \tfrac{-3e}{4}Y_2)\\
&\oplus (0e_0 \oplus \nu f_0) \otimes (\tfrac{-2e}{4}X_1 \oplus \tfrac{-4e}{4}X_2 \oplus 0Y_1 \oplus \tfrac{-2e}{4}Y_2).
\end{aligned}$$

Thus

$$\begin{aligned}
t^2((\mathfrak{K}_0 \cdot L_1^{(0)}) \cap V_1 + t((\mathfrak{K}_0 \cdot L_1^{(0)}) \cap V_1)) = {} & (0e_1 \oplus 0e_2) \otimes (\tfrac{-3e}{4}X_2 \oplus \tfrac{-e}{4}X_1 \oplus \tfrac{-3e}{4}Y_2 \oplus \tfrac{-e}{4}Y_1)\\
&\oplus (0e_0 \oplus \nu f_0) \otimes (\tfrac{-4e}{4}X_2 \oplus \tfrac{-2e}{4}X_1 \oplus \tfrac{-2e}{4}Y_2 \oplus 0Y_1).
\end{aligned}$$

It follows from step (ii) that

$$\begin{aligned}
L_1^{(1)} &= (\mathfrak{K}_0 \cdot L_1^{(0)}) \cap V_1 + t((\mathfrak{K}_0 \cdot L_1^{(0)}) \cap V_1) + t^2((\mathfrak{K}_0 \cdot L_1^{(0)}) \cap V_1 + t((\mathfrak{K}_0 \cdot L_1^{(0)}) \cap V_1))\\
&= (0e_1 \oplus 0e_2) \otimes (\tfrac{-e}{4}X_1 \oplus \tfrac{-3e}{4}X_2 \oplus \tfrac{-e}{4}Y_1 \oplus \tfrac{-3e}{4}Y_2)\\
&\oplus (0e_0 \oplus \nu f_0) \otimes (\tfrac{-2e}{4}X_1 \oplus \tfrac{-4e}{4}X_2 \oplus 0Y_1 \oplus \tfrac{-2e}{4}Y_2).
\end{aligned}$$

Consequently

$$\begin{aligned}
(\mathfrak{K}_0 \cdot L_1^{(1)}) \cap V_1 = {} & (0e_1 \oplus 0e_2) \otimes (\tfrac{-e}{4}X_1 \oplus \tfrac{-3e}{4}X_2 \oplus \tfrac{-e}{4}Y_1 \oplus \tfrac{-3e}{4}Y_2)\\
&\oplus (0e_0 \oplus \nu f_0) \otimes (\tfrac{-2e}{4}X_1 \oplus \tfrac{-4e}{4}X_2 \oplus 0Y_1 \oplus \tfrac{-2e}{4}Y_2) = L_1^{(1)}.
\end{aligned}$$

Hence $L_1^{(2)} = L_1^{(1)}$ and π is integral by Corollary 2.2. □

Remark 4.8. The genericity of π_{sm} is important in the formulation of the Breuil-Schneider conjecture which predicts the existence of integral structures in locally algebraic representations of $\mathrm{GL}_n(F)$ [BS07, p. 16]. In view of this, the integrality of the infinite-dimensional locally algebraic Speh representation in Proposition 4.7 seems to be related to the fact that $\mathrm{Sp}(\tau)_{\mathbb{C}}$ has a non-zero twisted Jacquet module if and only if $\mathrm{Sp}(\tau)_{\mathbb{C}}$ is infinite-dimensional, equivalently, if $\dim_E(\tau) > 1$. Indeed, it follows from [PR00, Theorem 3.1] that for any smooth irreducible infinite-dimensional complex representation V of $G = \mathrm{GL}_2(D)$, there is a short exact sequence of B-representations

$$0 \longrightarrow C_c^{\infty}(D^{\times}, V_{N,\psi}) \longrightarrow V \longrightarrow V_N \longrightarrow 0$$

(see also [Rag07, Theorem 2.1]). If $V = \mathrm{Sp}(\tau)_{\mathbb{C}}$ is infinite-dimensional, then $V_{N,\psi} \neq 0$ because V_N is finite-dimensional. On the other hand, if $V = \mathrm{Sp}(\tau)_{\mathbb{C}}$ is finite-dimensional, then it has dimension 1 and thus τ has dimension 1. By [PR00, Theorem 2.1], we have $\dim_{\mathbb{C}}(V_{N,\psi}) + \dim_{\mathbb{C}}((\mathrm{St}(\tau)_{\mathbb{C}})_{N,\psi}) = 1$. As $\mathrm{St}(\tau)_{\mathbb{C}}$ is infinite-dimensional, $(\mathrm{St}(\tau)_{\mathbb{C}})_{N,\psi} \neq 0$. Hence $V_{N,\psi} = 0$.

References

[Ass21] Eran Assaf, *Existence of invariant norms in p-adic representations of $GL_2(F)$ of large weights*, J. Number Theory **224** (2021), 95–141. MR 4221529

[BS07] Christophe Breuil and Peter Schneider, *First steps towards p-adic Langlands functoriality*, J. Reine Angew. Math. **610** (2007), 149–180. MR 2359853

[CEG+16] Ana Caraiani, Matthew Emerton, Toby Gee, David Geraghty, Vytautas Paškūnas, and Sug Woo Shin, *Patching and the p-adic local Langlands correspondence*, Camb. J. Math. **4** (2016), no. 2, 197–287. MR 3529394

[DI13] Marco De Ieso, *Existence de normes invariantes pour* GL_2, J. Number Theory **133** (2013), no. 8, 2729–2755. MR 3045213

[Eme06] Matthew Emerton, *Jacquet modules of locally analytic representations of p-adic reductive groups. I. Construction and first properties*, Ann. Sci. École Norm. Sup. (4) **39** (2006), no. 5, 775–839. MR 2292633

[GeK14] Elmar Groß e Klönne, *On the universal module of p-adic spherical Hecke algebras*, Amer. J. Math. **136** (2014), no. 3, 599–652. MR 3214272

[Hu09] Yongquan Hu, *Normes invariantes et existence de filtrations admissibles*, J. Reine Angew. Math. **634** (2009), 107–141. MR 2560407

[Hu21] Yong Quan Hu, *A note on integral structures in some locally algebraic representations of* GL_2, Acta Math. Sin. (Engl. Ser.) **37** (2021), no. 1, 59–72. MR 4204535

[MP96] Allen Moy and Gopal Prasad, *Jacquet functors and unrefined minimal K-types*, Comment. Math. Helv. **71** (1996), no. 1, 98–121. MR 1371680

[MS14] Alberto Mínguez and Vincent Sécherre, *Représentations lisses modulo l de* $\mathrm{GL}_m(D)$, Duke Math. J. **163** (2014), no. 4, 795–887. MR 3178433

[PR00] Dipendra Prasad and A. Raghuram, *Kirillov theory for* $\mathrm{GL}_2(D)$ *where D is a division algebra over a non-Archimedean local field*, Duke Math. J. **104** (2000), no. 1, 19–44. MR 1769724

[Rag07] A. Raghuram, *On the restriction to* $D^* \times D^*$ *of representations of p-adic* $\mathrm{GL}_2(D)$, Canad. J. Math. **59** (2007), no. 5, 1050–1068. MR 2354402

[Sor13] Claus M. Sorensen, *A proof of the Breuil-Schneider conjecture in the indecomposable case*, Ann. of Math. (2) **177** (2013), no. 1, 367–382. MR 2999043

[SS97] Peter Schneider and Ulrich Stuhler, *Representation theory and sheaves on the Bruhat-Tits building*, Inst. Hautes Études Sci. Publ. Math. (1997), no. 85, 97–191. MR 1471867

[STP01] P. Schneider, J. Teitelbaum, and Dipendra Prasad, $U(\mathfrak{g})$*-finite locally analytic representations*, Represent. Theory **5** (2001), 111–128, With an appendix by Dipendra Prasad. MR 1835001

[SZ05] Allan J. Silberger and Ernst-Wilhelm Zink, *An explicit matching theorem for level zero discrete series of unit groups of p-adic simple algebras*, J. Reine Angew. Math. **585** (2005), 173–235. MR 2164626

[Vig96] Marie-France Vignéras, *Représentations l-modulaires d'un groupe réductif p-adique avec* $l \neq p$, Progress in Mathematics, vol. 137, Birkhäuser Boston, Inc., Boston, MA, 1996. MR 1395151

[Vig08] ———, *A criterion for integral structures and coefficient systems on the tree of* $\mathrm{PGL}(2, F)$, Pure Appl. Math. Q. **4** (2008), no. 4, Special Issue: In honor of Jean-Pierre Serre. Part 1, 1291–1316. MR 2441702

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