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QUOTIENTS OF COMMUTING SCHEMES ASSOCIATED TO SYMMETRIC PAIRS

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Abstract. Let $\mathfrak{g}$ be a classical Lie algebra over an algebraically closed field k of characteristic zero. Let θ be an involution of $\mathfrak{g}$, and let $\mathfrak{g}_0$ and $\mathfrak{g}_1$ be 1 and -1 eigenspaces of θ. Let G be a classical group with Lie algebra $\mathfrak{g}$ and let G_0 be the connected subgroup of G with $\mathrm{Lie}(G_0) = \mathfrak{g}_0$. For $d \geq 2$, let $\mathfrak{C}^d(\mathfrak{g}_1)$ be the d-th commuting scheme associated with the symmetric pair $(\mathfrak{g}, \mathfrak{g}_0)$. In this article, we study the reducedness of the quotient scheme $\mathfrak{C}^d(\mathfrak{g}_1)//G_0$ via the Chevalley restriction map. As a part of the proof, we describe a generating set for the algebra $k[\mathfrak{g}_1^d]^{G_0}$, which is of independent interest.

1. Introduction

Let $\mathfrak{g}$ be a reductive Lie algebra over an algebraically closed field k of characteristic zero, and let τ be an involution of $\mathfrak{g}$. The eigenspaces of τ determine a $\mathbb{Z}_2$-grading of $\mathfrak{g}$, we have $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$. Note that $\mathfrak{g}_1$ is a $\mathfrak{g}_0$-module. We say that $(\mathfrak{g}, \mathfrak{g}_0)$ is a symmetric pair (see [TY05, Chapter 37] and [Hel78, Chapter X] for a general treatment of symmetric pairs). Let G be a reductive group with Lie algebra $\mathfrak{g}$ and let G_0 be the connected subgroup of G with Lie algebra $\mathfrak{g}_0$.

For $d \geq 2$, the d-th commuting scheme associated with the involution τ, or with the symmetric pair $(\mathfrak{g}, \mathfrak{g}_0)$, is the subscheme $\mathfrak{C}^d(\mathfrak{g}_1) \subset \mathfrak{g}_1^d$ defined by the equations

$$[X_i, X_j] = 0,\ 1 \leq i, j \leq d,$$

where $(X_1, X_2, \ldots, X_d)$ represents a system of coordinates for $\mathfrak{g}_1^d$. The scheme $\mathfrak{C}^d(\mathfrak{g}_1)$ naturally generalises the commuting scheme $\mathfrak{C}^d(\mathfrak{g})$, defined by the similar set of equations

$$[Y_i, Y_j] = 0,\ 1 \leq i, j \leq d,$$

where $(Y_1, Y_2, \ldots, Y_d)$ represents a system of coordinates for $\mathfrak{g}^d$. In this article we are interested in the reducedness of the quotient scheme $\mathfrak{C}^d(\mathfrak{g}_1)//G_0$. Gan and Ginzburg, for $d = 2$, and Vaccarino for $d \geq 2$, showed that the quotient scheme $\mathfrak{C}^d(\mathfrak{gl}_n)//\mathrm{GL}_n$ is reduced and normal ([GG06, Section 2.7] and ([Vac07]). In [CN20] Chen and Ngô conjectured that the quotient scheme $\mathfrak{C}^d(\mathfrak{g})//G$ is reduced and normal for any reductive Lie algebra $\mathfrak{g}$ and they proved the conjecture for $\mathfrak{g} = \mathfrak{gl}_n$ and $\mathfrak{sp}_{2n}$ in [CN20] and [CC21] respectively. In a recent preprint, Losev proved that the almost commuting scheme, denoted by X_n, associated to the symplectic Lie algebra is irreducible, reduced complete intersection; and the categorical quotient $X_n//\mathrm{Sp}_{2n}(\mathbb{C})$ is isomorphic to $\mathfrak{C}^2(\mathfrak{sp}_{2n})/\mathrm{Sp}_{2n}(\mathbb{C})$ (see [Los21]).

The geometry of ordinary commuting varieties has been studied extensively. Richardson in the article [Ric79, Theorem A], proved that the scheme $\mathfrak{C}^2(\mathfrak{g})$ is irreducible. It is a well known conjecture that the commuting scheme $\mathfrak{C}^2(\mathfrak{g})$ is reduced. For $d \geq 4$, the scheme $\mathfrak{C}^d(\mathfrak{gl}_n)$ is irreducible if and only if $n \leq 3$ (see [Ger61]). So, the scheme $\mathfrak{C}^d(\mathfrak{g})$ is not necessarily irreducible, for large enough d. For $d \geq 3$ it is not expected that $\mathfrak{C}^d(\mathfrak{g})$ is reduced. However, the categorical quotient $\mathfrak{C}^d(\mathfrak{g})//G := \mathrm{Spec}(k[\mathfrak{C}^d(\mathfrak{g})]^G)$ behaves better.

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Chen and Ngô, prove the reducedness of the quotient schemes $\mathfrak{C}^d(\mathfrak{gl}_n)//\mathrm{GL}_n$ and $\mathfrak{C}^d(\mathfrak{sp}_{2n})//\mathrm{Sp}_{2n}$ by constructing a section to the Chevalley restriction morphism. Already in the Lie algebra case, a weaker version of the Chevalley restriction theorem for the commuting scheme $\mathfrak{C}^d(\mathfrak{g})$ is proved by Hunziker in [Hun97]. Hunziker proved that the morphism $i : \mathfrak{h}^d//W \to \mathfrak{C}^d(\mathfrak{g})//G$ of quotient schemes induced by the inclusion map $\mathfrak{h}^d \to \mathfrak{C}^d(\mathfrak{g})$ is a finite bijective morphism, where $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$. In other words, i is a universal homeomorphism of schemes. In particular, $\mathfrak{h}^d//W$ is the normalisation of the underlying reduced subscheme $(\mathfrak{C}^d(\mathfrak{g})//G)^{red}$. Independently, Joseph in the article [Jos97] also proved that the Chevalley restriction map is an isomorphism of underlying reduced schemes. This also follows from a result of Luna (see [Lun75]). Since $\mathfrak{h}^d//W$ is irreducible, the categorical quotient $\mathfrak{C}^d(\mathfrak{g})//G$ is also irreducible. However for a general simple Lie algebra $\mathfrak{g}$, the reducedness of the categorical quotient $\mathfrak{C}^d(\mathfrak{g})//G$ is still not known.

The irreducibility problem for the commuting varieties associated to symmetric pairs was first considered by Panyushev in [Pan94], and he observed that $\mathfrak{C}^2(\mathfrak{g}_1)$ can be reducible. He showed that $\mathfrak{C}^2(\mathfrak{g}_1)$ is irreducible if and only if $\mathfrak{C}^2(\mathfrak{g}_1) = \overline{G_0(\mathfrak{c} \times \mathfrak{c})}$. Using this he concluded that if $(\mathfrak{g}, \mathfrak{g}_0)$ is a symmetric pair of maximal rank (i.e., $\mathrm{rk}((\mathfrak{g}, \mathfrak{g}_0)) = rk(\mathfrak{g})$), then $\mathfrak{C}^2(\mathfrak{g}_1)$ is an irreducible normal complete intersection and the ideal of $\mathfrak{C}^2(\mathfrak{g}_1)$ in $k[\mathfrak{g}_1 \times \mathfrak{g}_1]$ is generated by quadrics. In [Pan94] Panyushev showed that for a symmetric pair $(\mathfrak{g}, \mathfrak{g}_0)$ of maximal rank, the quotient variety $\mathfrak{C}^2(\mathfrak{g}_1)//G_0$ is isomorphic to $\mathfrak{h}^2//W$, where $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ and W is the Weyl group of $\mathfrak{g}$. Later (see [PY07] and [Pan04]) he extended the irreducibility results for some more symmetric pairs including the pairs that we are interested in this paper. See also [SY02] and [SY06] for some more results in this direction.

Let us recall Chevalley restriction for symmetric pairs. Let $\mathfrak{c} \subset \mathfrak{g}_1$ be a maximal subspace consisting of pairwise commuting semisimple elements. Any such subspace is called a Cartan subspace. All Cartan subspaces are G_0-conjugate. The number $\dim_k \mathfrak{c}$ is called the rank of the symmetric pair. Let $N(\mathfrak{c})$ and $Z(\mathfrak{c})$ be the normaliser and the centraliser of $\mathfrak{c}$ in G_0 respectively. Then $Z(\mathfrak{c})$ is a normal subgroup of $N(\mathfrak{c})$ and the quotient $W_{\mathfrak{c}} := N(\mathfrak{c})/Z(\mathfrak{c})$ is a finite group, called the little Weyl group of the pair (G, G_0). From [Vin76], it is known that the action of $W_{\mathfrak{c}}$ on $\mathfrak{c}$ is generated by transformations fixing a hyperplane in $\mathfrak{c}$, that the restriction map $k[\mathfrak{g}_1]^{G_0} \to k[\mathfrak{c}]^{W_{\mathfrak{c}}}$ is an isomorphism. It was shown by Tevelev that the chevalley restriction map $k[\mathfrak{g}_1^d]^{G_0} \to k[\mathfrak{c}^d]^{W_{\mathfrak{c}}}$ is surjective (see [Tev00]).

In this note, following the work of Chen and Ngô in [CC21] we consider all classical symmetric pairs (defined below in 1.1) except the pair $(\mathfrak{so}_{2n}(k), \mathfrak{gl}_n(k))$, and show that the induced morphism $\mathfrak{i} : \mathfrak{c}^d//W_{\mathfrak{c}} \to \mathfrak{C}^d(\mathfrak{g}_1)//G_0$ is an isomorphism of affine schemes. Since $\mathfrak{c}^d//W_{\mathfrak{c}}$ is normal and reduced, the isomorphism implies that the quotient scheme $\mathfrak{C}^d(\mathfrak{g}_1)//G_0$ is normal and reduced. The main idea of the proofs is to construct a section (called the spectral data map) of the Chevalley restriction map using either the determinant map or the Pfaffian norm map (see Chen and Ngô [CC21]). In order to verify that the spectral data map is a section of Chevalley restriction, we need a convenient set of generators for the algebra $k[\mathfrak{g}_1^d]^{G_0}$. We use fundamental theorems of classical invariant theory to produce a set of generators for $k[\mathfrak{g}_1^d]^{G_0}$. Then, the explicit form of the restriction to Cartan subspace of these generators already show the surjectivity of Chevalley restriction map. The injectivity is then verified by formal properties of the spectral data map.

1.1. Classical Symmetric Pairs.

Let $\mathfrak{g}$ be a classical Lie algebra and let τ be an involution of $\mathfrak{g}$. Let $\mathfrak{g}_0$ and $\mathfrak{g}_1$ be the 1 and -1 eigenspaces of τ. The pair $(\mathfrak{g}, \mathfrak{g}_0)$ is called a classical symmetric pair. The following is the classification of classical symmetric pairs: (see for example [Hel78, Chapter X])

(1) Bilinear Form: $(AI) : (\mathfrak{gl}_n(k), \mathfrak{so}_n(k))$, $(AII) : (\mathfrak{gl}_{2n}(k), \mathfrak{sp}_{2n}(k))$,
(2) Polarization: $(DIII) : (\mathfrak{so}_{2n}(k), \mathfrak{gl}_n(k))$, $(CI) : (\mathfrak{sp}_{2n}(k), \mathfrak{gl}_n(k))$,

(3) Direct Sum: $(AIII) : (\mathfrak{gl}_{m+n}(k), \mathfrak{gl}_m(k) \times \mathfrak{gl}_n(k))$, $(BDI) : (\mathfrak{so}_{m+n}(k), \mathfrak{so}_m(k) \times \mathfrak{so}_n(k))$, $(CII) : (\mathfrak{sp}_{2(m+n)}(k), \mathfrak{sp}_{2m}(k) \times \mathfrak{sp}_{2n}(k))$.

See Section 5 for the detailed structure of these symmetric pairs. The main theorem of this paper is the following:

Theorem 1.1. *Let $(\mathfrak{g}, \mathfrak{g}_0)$ be a classical symmetric pair except the pair $(\mathfrak{so}_{2n}(k), \mathfrak{gl}_n(k))$. Then the Chevalley restriction map $\mathfrak{i} : \mathfrak{c}^d//W_{\mathfrak{c}} \to \mathfrak{C}^d(\mathfrak{g}_1)//G_0$ is an isomorphism of affine schemes.*

Our techniques for the construction of a section for the Chevalley restriction map does not work in the case where $(\mathfrak{g}, \mathfrak{g}_0)$ is equal to $(\mathfrak{so}_{2n}(k), \mathfrak{gl}_n(k))$. The reason being that the rank of symmetric pair is half of the rank of rank of $\mathfrak{g}_0$. See remark 6.1 for details.

Since $\mathfrak{c}^d//W_{\mathfrak{c}}$ is normal and reduced, as an immediate corollary we get the following:

Corollary 1.2. *Let $(\mathfrak{g}, \mathfrak{g}_0)$ be a classical symmetric pair except the pair $(\mathfrak{so}_{2n}(k), \mathfrak{gl}_n(k))$. Then the categorical quotient scheme $\mathfrak{C}^d(\mathfrak{g}_1)//G_0$ is normal and reduced.*

We will explain the contents of each section. In section 2, we recall the notion of polynomial laws and Roby's results. In section 3, we recall Deligne's construction of the spectral data map. In Section 4, we describe the ring of G_0 invariants of $k[\mathfrak{g}_1^d]$. For many pairs $(\mathfrak{g}, \mathfrak{g}_1)$ these results are new and of independent interest. In section 5, we give a proof of the main theorem by constructing the respective spectral data map for each symmetric pair and verify that this spectral data map is a section of the Chevalley restriction map.

2. Polynomial Laws

2.1. Let A be a commutative ring and let Alg_A be the category of A-algebras. Let V be an A-module and let V_A be the functor from Alg_A to the category of A-modules given by $R \mapsto V \otimes_A R$. For two A-modules V and W, we denote by $P(V, W)$, the set of natural transformations between the functors V_A and W_A. If $W = A$, then we denote by $P(V)$ the set $P(V, A)$. The set $P(V)$ is called the set of *polynomial laws* on V. Let S_A be the polynomial ring $A[X_1, X_2, \dots, X_n]$. For $f \in P(V)$ we get a map $f_{S_A} : V \otimes_A S_A \to S_A$. Thus for any $f \in P(V)$, and a finite set of elements $v_1, \dots, v_n$, we associate a polynomial $P_f \in S_A$ given by $f_{S_A}(v_1X_1 + v_2X_2 + \cdots + v_nX_n)$. A polynomial law $f \in P(V)$ is called homogenous of degree d if $f(uv) = u^n f(v)$, for all A-algebras R, for all $u \in R^\times$ and $v \in V \otimes_A R$. We denote by $P_n(V)$ the set of all degree n homogenous polynomial laws on V. For a general reference for polynomial laws we refer to [Rob63].

2.2. Let V be an A-module, and let $T^n(V)$ be the n-fold tensor product of the A-module V. We denote by $TS^n(V)$, the A-submodule fixed by the action of S_n on $T^n(V)$ and by $S^n(V)$, the S_n-coinvariants of $T^n(V)$. Roby (see [Rob63]) showed that the homogenous polynomial laws on V of degree n are in canonical bijection with homogenous degree 1 polynomial laws on $TS^n(V)$ given by the relation

$$f(v) = h(v^{\otimes n}), f \in P_n(V), h \in P_1(TS^n(V)),$$

where $P_1(TS^n(V))$ is the space of degree 1 polynomial laws on $TS^n(V)$. Moreover, if V is an A-algebra, which is free as an A-module, and f is a multiplicative homogenous polynomial law of degree n on V, then the degree 1 polynomial law h associated to f is a homomorphism of algebras

$$TS^n(V) \to A.$$

3. Spectral data map

In this section, we first recall Deligne's construction of spectral data map for $\mathrm{GL}_n(k)$ (see [Del73, Section 6.3.1]) which plays a pivotal role in the proof of the main theorem. As the name suggests it assigns the set of common eigenvalues of a set of commuting matrices. In fact, we construct the spectral data map for the relevant symmetric pairs and show that it is actually a section of the Chevalley restriction map. For convenience, we recall the Chevalley restriction map here. Let G be a reductive group over k with Lie algebra $\mathfrak{g}$. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ and let W be the Weyl group of $\mathfrak{g}$. The diagonal adjoint action of G on $\mathfrak{g}^d$ leaves the commuting scheme $\mathfrak{C}^d(\mathfrak{g})$ invariant. The inclusion $\mathfrak{h}^d \to \mathfrak{g}^d$ factors through the commuting scheme $\mathfrak{C}^d(\mathfrak{g})$ and it induces a homomorphism of k-algebras $\mathfrak{i} : k[\mathfrak{C}^d(\mathfrak{g})]^G \to k[\mathfrak{h}^d]^W$ because the restriction of a G-invariant function to $\mathfrak{h}^d$ is also W-invariant. In other words, we have a morphism of affine schemes (which we still denote by $\mathfrak{i}$):

$$\mathfrak{i} : \mathfrak{h}^d//W \to \mathfrak{C}^d(\mathfrak{g})//G.$$

3.1. Let V be an n-dimensional k-vector space and and let R be any k-algebra. Let $\mathfrak{C}^d(\mathfrak{gl}(V))$ be the pair wise commuting scheme of $\mathfrak{gl}(V)$ whose R-points are given by the set

$$\{(x_1, \dots, x_d) \in \mathfrak{gl}(V \otimes R)^d : [x_i, x_j] = 0, \text{ for all } i, j \in [d]\}.$$

Let A be a k-algebra representing the functor $\mathfrak{C}^d(\mathfrak{gl}(V))$. Let $(x_1, x_2, \dots, x_d)$ be the universal point in $\mathfrak{C}^d(A)$. We then have a homomorphism of R-algebras:

$$p : k[X_1, \dots, X_d] \otimes R \to \mathfrak{gl}(A \otimes R)$$

given by $p(X_1, \dots, X_d) = (x_1, \dots, x_d)$. The map $\det \circ p$ gives a homogenous polynomial law on the algebra $k[X_1, \dots, X_d]$. Since $\det \circ p$ is multiplicative, using Roby's result we get an algebra homomorphism

$$\tilde{\mathfrak{s}} : (k[X_1, X_2, \dots X_d]^{\otimes n})^{S_n} \to A$$

such that $\det p((f)) = \tilde{\mathfrak{s}}(f^{\otimes n})$ for all $f \in k[X_1, X_2, \dots X_d]$. We note that the image of the map $\tilde{\mathfrak{s}}$ belongs to the algebra of $GL(V)$-invariants $A^{\mathrm{GL}(V)}$. Thus we obtain a map of schemes

$$\mathfrak{s} : \mathfrak{C}^d(\mathfrak{gl}(V))//\mathrm{GL}(V) \to \mathfrak{h}^d//S_n.$$

Here, $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{gl}(V)$. This map $\mathfrak{s}$ is called the spectral data map. Note that in this case the Weyl group is S_n and the Cartan subalgebra is the standard representation of S_n. The compositions $\mathfrak{s} \circ \mathfrak{i}$ and $\mathfrak{i} \circ \mathfrak{s}$ are verified to be identity on a generating set given by trace functions due to Procesi (see [Pro76]). As a result it was concluded that the quotient scheme $\mathfrak{C}^d(\mathfrak{gl}(V))//GL(V)$ is normal and reduced (see [CN20] for details).

4. Invariants

Let $(\mathfrak{g}, \mathfrak{g}_0)$ be a classical symmetric pair so that $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$. We require a generating set for the k-algebra $k[\mathfrak{g}_1^d]^{G_0}$. In some of the cases, these invariants are known from the work of Procesi (see [Pro76]). In this section, we consider those cases which are not available (to the best of our knowledge) in the literature. These invariants will be used to check that the spectral data map is a section of the Chevalley restriction map.

4.1. Let $W(n, m, d)$ be the space $M^d_{n\times m}(k) \oplus M^d_{m\times n}(k)$. Let G be the group $\mathrm{GL}_n(k) \times \mathrm{GL}_m(k)$ via left and right action on $W(n, m, d)$. We define

$$\mu : W(n, m, d) \to M^{d^2}_{n\times n}(k); ((M_1, \dots, M_d), (N_1, \dots, N_d)) \mapsto (A_{ij}),$$

where $A_{ij} = M_i N_j$. Note that the map $\mu^* : k[M^{d^2}_{n\times n}(k)] \to k[M^d_{n\times m}(k) \bigoplus M^d_{m\times n}(k)]^{\mathrm{GL}_m(k)}$ is surjective from the first fundamental theorem for $GL_n(k)$ (see Theorem 5.2.1 of [GW98]). Hence, we get that the map

$$\mu^* : k[M^{d^2}_{n\times n}(k)]^{\mathrm{GL}_n(k)} \to k[M^d_{n\times m}(k) \oplus M^d_{m\times n}(k)]^G \tag{4.1}$$

is surjective. Since $k[M^{d^2}_{n\times n}(k)]^{\mathrm{GL}_n(k)}$ is generated by monomials in A_i, $i \in [d^2]$ (see Theorem 3.4 of [Pro76]), from (4.1) we get that polynomials of the form

$$\mathrm{Tr}(M_1 M_2 \dots M_l)$$

where $M_i = Q_{n_i} R_{m_i}$ with $Q_{n_i}, R^t_{m_i} \in M_{n\times m}(k)$ and $n_i, m_i \in [d]$ for all $i \in [l]$ generate the algebra $k[W(n,m,d)]^G$.

4.2. Let $V(n,m,d)$ be the space $M^d_{n\times m}(k)$, and let G be the group $\mathrm{SO}_n(k) \times \mathrm{SO}_m(k)$. The group G acts on $V(m,n,d)$ by setting

$$(A,B)(M_1, M_2, \dots, M_d) = (AM_1B^t, AM_2B^t, \dots, AM_dB^t).$$

Let $\mu : M^d_{n\times m}(k) \to (\mathrm{Sym}^2 k^m)^d \bigoplus_{i>j} M_{m\times m}(k)$ be the map

$$(M_1, M_2, \dots, M_d) \mapsto (A_{ij} : i \geq j),$$

where $A_{ij} = M_i^t M_j$. The map $\mu^* : k[(\mathrm{Sym}^2 k^m)^d \bigoplus_{i>j} M_{m\times m}(k)] \to k[M^d_{n\times m}(k)]^{\mathrm{O}_n(k)}$ is a surjective map from the first fundamental theorem for orthogonal group (see Theorem 5.2.2 of [GW98]). Since $\mathrm{SO}_m(k)$ is reductive, we get that the map

$$\mu^* : k[(\mathrm{Sym}^2 k^m)^d \bigoplus_{i>j} M_{m\times m}(k)]^{\mathrm{SO}_m(k)} \to k[M^d_{n\times m}(k)]^{\mathrm{O}_n(k)\times SO_m(k)} \tag{4.2}$$

is surjective. The algebra $k[(\mathrm{Sym}^2 k^m)^d \bigoplus_{i>j} M_{m\times m}(k)]^{\mathrm{SO}_m(k)}$ is spanned by polynomials of the form

$$\mathrm{Tr}(A_{n_1m_1} A_{n_2m_2} \dots A_{n_lm_l}),$$

and

$$\widetilde{\mathrm{Pf}}(A_{n_1m_1} A_{n_2m_2} \dots A_{n_{l'}m_{l'}}),$$

where $\widetilde{Pf}$ is the complete polarisation of the pfaffian, and A_{ii} is a symmetric matrix and A_{ij} for $i \neq j$ is any $m \times m$ matrix; for details see [ATZ95]. Hence, the algebra

$$k[M^d_{n\times m}(k)]^{\mathrm{O}_n(k)\times SO_m(k)}$$

is generated by polynomials of the form

$$\mathrm{Tr}(A_{n_1m_1} A_{n_2m_2} \dots A_{n_lm_l}), \tag{4.3}$$

$$\widetilde{\mathrm{Pf}}(A_{n_1m_1} A_{n_2m_2} \dots A_{n'_lm'_l}), \tag{4.4}$$

where $A_{n_im_i} = M^t_{n_i} M_{m_j}$. If n is odd then we get that a system of generators of $k[M^d_{n\times m}(k)]^{\mathrm{SO}_n(k)\times \mathrm{SO}_m(k)}$ is given by the invariants in (4.3).

Remark 4.1. If $M_i^t M_j = M_j^t M_i$ and $M_j M_i^t = M_i M_j^t$ for all $i, j \in [d]$. Then the matrix

$$M^t_{n_1} M_{m_1} M^t_{n_2} M_{m_2} \dots M^t_{n_l} M_{m_l}$$

is symmetric. Hence the restriction of the invariant (4.4) to the subvariety of $M^d_{n\times m}(k)$ defined by the relations $M_i^t M_j = M_j^t M_i$ and $M_j M_i^t = M_i M_j^t$ is zero.

If n is even and $n = m$, then the algebra $k[M^d_{n\times m}(k)]^{\mathrm{SO}_n(k)\times \mathrm{SO}_m(k)}$ might be strictly bigger than $k[M^d_{n\times m}(k)]^{\mathrm{O}_n(k)\times \mathrm{SO}_m(k)}$; for instance the elements

$$\det(T_1 \otimes A_1 + T_2 \otimes A_2 + \cdots + T_d \otimes A_d)$$

for any $(T_1, T_2, \ldots, T_d) \in M^d_{l\times l}(k)$, where $l \geq 1$, are invariant for $\mathrm{SO}_n(k) \times \mathrm{SO}_m(k)$.

4.3. Let $W_{(d_1,d_2)}$ be the space $(\mathrm{Sym}^2(V))^{d_1} \oplus (\mathrm{Sym}^2(V^*))^{d_2}$, and let G be the group $\mathrm{GL}(V)$. For the definitions of full polarization and restitution, we refer to Section 3.2.2 of [Pro07]. The algebra $A = k[W_{(d_1,d_2)}]^G$ is graded by $\mathbb{N}^{d_1+d_2}$. Let F be a non-zero multihomogeneous invariant in A. Let $P(F)$ be the full polarization of F, and note that $P(F)$ is a multilinear invariant on $W_{d_1',d_2'}$ for some integers d_1' and d_2' depending on the degree of F. We embed $W_{d_1',d_2'}$ in $W'_{d_1',d_2'} := (V\otimes V)^{d_1'} \oplus (V^*\otimes V^*)^{d_2'}$. The space of multilinear invariants of $W'_{d_1',d_2'}$ is equal to $(V^{\otimes 2d_1'} \otimes V^{\otimes 2d_2'})^G$. We then have $d_1' = d_2' = d'$ and the space $(V^{2d'} \otimes V^{2d'})^G$ is spanned by the complete contractions (see Corollary 5.3.2 of [GW98]). So $(V^{2d'} \otimes V^{2d'})^G$ is spanned by monomials of the form

$$u_\sigma = \prod_{i=1}^{2d'} u_{i,\sigma(i)},$$

where $\sigma \in S_{2d'}$ and

$$u_{ij}(v_1 \otimes v_2 \otimes \cdots \otimes v_{2d'} \otimes v_1^* \otimes v_2^* \otimes \cdots \otimes v_{2d'}^*) = v_j^*(v_i).$$

Since G is reductive, the restrictions of u_σ from $W'_{d',d'}$ to the space of multilinear invariants of $W_{d',d'}$ generate the multilinear G-invariants. Since every multihomogeneous invariant is the restitution of some multilinear invariant, the restitution of u_σ, $\sigma \in S_{2d'}$ generate $k[W_{(d',d')}]^G$. The restitution of u_σ is of the form

$$\mathrm{Tr}(M_1M_2\ldots M_k),$$

with M_i is of the from $Q_{n_i}R_{m_i}$, where $(Q_1, \ldots, Q_d)$ and $(R_1, \ldots, R_d)$ are tuples of symmetric matrices and $n_i, m_i \in [d]$ (see [GW98, Exercise 1 of 5.3.3]).

4.4. Let $V(2p, 2q, d)$ be the space $M^d_{2p\times 2q}(k)$ and let G be the group $\mathrm{Sp}_{2p}(k) \times \mathrm{Sp}_{2q}(k)$. We define the action of G on $V(2p, 2q, d)$ by setting:

$$(A, B)(M_1, M_2, \ldots, M_d) = (AM_1B^{-1}, AM_2B^{-1}, \ldots, AM_dB^{-1}).$$

Let T_r be the $2r \times 2r$ matrix given by $\mathrm{diag}(\mu, \mu, \ldots, \mu)$, where μ is the matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Let $\mu : V(2p, 2q, d) \twoheadrightarrow M^{d^2}_{2q,2q}(k)$ be given by $(M_1, M_2 \ldots, M_d) \mapsto (A_{ij})$, where $A_{ij} = M_i^tT_pM_jT_q$. The map μ induces a surjective map $\mu^* : k[M^{d^2}_{2q\times 2q}(k)] \twoheadrightarrow k[V(2p, 2q, d)]^{\mathrm{Sp}_{2p}(k)}$. Hence, we get a surjective map

$$\mu^* : k[M^{d^2}_{2q\times 2q}]^{\mathrm{Sp}_{2q}(k)} \twoheadrightarrow k[V(2p, 2q, d)]^G.$$

Note that the algebra $k[M^{d^2}_{2q\times 2q}(k)]^{\mathrm{Sp}_{2q}(k)}$ is spanned by polynomials:

$$\mathrm{Tr}(M)$$

where M is a monomial in A_{ij} or A^t_{ij} with A_{ij} is a variable matrix of size $2q \times 2q$, for all $i, j \in [d]$ (see Theorem 10.1 of [Pro76]). Hence, the k-algebra $k[V(2p, 2q, d)]^G$ is spanned by elements of the form

$$\mathrm{Tr}(M_{n_1m_1}M_{n_2m_2}\ldots M_{n_lm_l})$$

where $M_{ij} = M_jT_pM_i^tT_q$ and $n_i, m_i \in [d]$.

5. Construction of Spectral data map and proof of the main theorem

In this section we construct the spectral data map for the relevant classical symmetric pairs and we show that in each case the spectral data map is a section of the Chevalley restriction map.

5.1. **The symmetric pair AI.** Let V be a k-vector space of dimension n, and let $\omega : V \times V \to k$ be a non-degenerate symmetric bilinear form on V. Let $\tau : \mathfrak{gl}(V) \to \mathfrak{gl}(V)$ be the involution defined by $\tau(X) = -X^*$, where X^* is given by:

$$\omega(Xv, w) = \omega(v, X^* w).$$

Let $\mathfrak{g}_0$ and $\mathfrak{g}_1$ be the 1 and -1 eigenspaces of τ. Then the algebra $\mathfrak{g}_0$ can be identified with the Lie algebra of the connected component G_0 of $\mathrm{O}(V, \omega)$ containing the identity element. With respect to an orthogonal basis $(v_1, v_2, \ldots, v_n)$ of V a Cartan subspace $\mathfrak{c} \subset \mathfrak{g}_1$ is given by the following subspace:

$$\{\mathrm{diag}(b_1, b_2, \ldots, b_n) : b_i \in k, i \in [n]\}.$$

The little Weyl group $W_\mathfrak{c}$ is isomorphic to S_n–which is identified with the group of permutations of the coordinates of $\mathfrak{c}$.

Let $\mathfrak{C}^d(\mathfrak{g}_1)$ be the d-fold commuting scheme associated with the symmetric pair $(\mathfrak{gl}(V), \mathfrak{g}_0)$. We have the Chevalley restriction map

$$\mathfrak{i} : \mathfrak{c}^d // W_\mathfrak{c} \to \mathfrak{C}^d(\mathfrak{g}_1) // G_0,$$

which is induced from the inclusion $\mathfrak{c}^d \to \mathfrak{g}_1^d$. We will construct a section of this map.

Let R be any k-algebra, and let A be the coordinate ring of $\mathfrak{C}_d(\mathfrak{g}_1)$. Let $(x_1, x_2, \ldots, x_d)$ be the universal point of $\mathfrak{C}_d(\mathfrak{g}_1)(A)$. For any k-algebra R, we define the map

$$p : R[X_1, X_2, \ldots, X_d] \to \mathfrak{gl}_n(A \otimes R), \;\; X_i \mapsto x_i, i \in [d].$$

Note that the map $\det \circ\, p$ is a degree n multiplicative map. Hence, by Roby's theorem we get an algebra map

$$\tilde{\mathfrak{s}} : TS^n(k[X_1, X_2, \ldots, X_d]) \to A$$

such that

$$\tilde{\mathfrak{s}}(\theta^{\otimes n}) = \det \circ\, p(\theta)$$

for all $\theta \in k[X_1, X_2, \ldots, X_d]$. Since det is G_0-invariant, the image of the map $\tilde{\mathfrak{s}}$ is is contained in A^{G_0}.

Let $(y_1, \ldots, y_d)$ be the tautological point of $\mathfrak{c}(B)$, where $B = k[\mathfrak{c}^d]$. Note that $B = k[\mathfrak{c}^d]$ is a polynomial algebra in the variables $b_j(y_i)$, $1 \le i \le d$ and $1 \le j \le n$. Let

$$\beta : TS^n(k[X_1, X_2, \ldots, X_d]) \to k[\mathfrak{c}^d]$$

be the isomorphism given by $\beta(X_{i,j}) = b_j(y_i)$, where $X_{i,j}$ is the variable X_i in the j-th copy, for $i \in [d]$ and $j \in [n]$. We get a map

$$\mathfrak{s} : k[\mathfrak{c}^d]^{W_\mathfrak{c}} \to A^{G_0} \tag{5.1}$$

such that $\mathfrak{s} \circ \beta(\theta^{\otimes n}) = \det \circ p(\theta)$. Thus we obtain a map of schemes $\mathfrak{s} : \mathfrak{C}^d(\mathfrak{g}_1)//G_0 \to \mathfrak{c}^d//W_\mathfrak{c}$. We show that the map $\mathfrak{s} : \mathfrak{C}^d(\mathfrak{g}_1)//G_0 \to \mathfrak{c}^d//W_\mathfrak{c}$ is the inverse of the Chevalley restriction map $\mathfrak{i} : \mathfrak{c}^d//W_\mathfrak{c} \to \mathfrak{C}^d(\mathfrak{g}_1)//G_0$.

For the diagonal action of G_0 on the d-copies of $\mathfrak{gl}(V)$, the ring of invariants $k[\mathfrak{gl}(V)^d]^{G_0}$ is generated by the elements $Tr(M)$ and some polarized Pfaffians, where M is a monomial in X_j, X_j^*, $j = 1, 2, \cdots, d$ (see [ATZ95]). Recall that the space $\mathfrak{g}_1$ is the -1 eigen space of the map $\theta(X) = -X^*$. So the elements in $\mathfrak{g}_1$ satisfy $X^* = X$ and hence the polarized Pfaffians become zero when we restrict them to $\mathfrak{g}_1^d$.

Since $\mathfrak{C}^d(\mathfrak{g}_1)$ is a closed subscheme of $\mathfrak{gl}(V)^d$ and G_0 is (linearly) reductive, the restriction map $k[\mathfrak{gl}(V)^d] \to k[\mathfrak{C}^d(\mathfrak{g}_1)]$ induces a surjective map $k[\mathfrak{gl}(V)^d]^{G_0} \to k[\mathfrak{C}^d(\mathfrak{g}_1)]^{G_0}$. So the restrictions of the functions $Tr(M)$, where M is a monomial in X_j, $j = 1, 2, \cdots, d$ form a generating set for $k[\mathfrak{C}^d(\mathfrak{g}_1)]^{G_0}$. Let $(x_1, x_2, \cdots, x_d)$ be the universal point of the commuting scheme $\mathfrak{C}^d(\mathfrak{g}_1)(A)$. Then the set $\{\phi_{\underline{a}} = \mathrm{Tr}(x_1^{a_1} x_2^{a_2} \ldots x_d^{a_d}) : \underline{a} = (a_1, \ldots, a_d) \in \mathbb{Z}^d_{\geq 0}\}$ is a generating set for $k[\mathfrak{C}^d(\mathfrak{g}_1)]^G$.

Note that

$$\mathfrak{i}(\phi_{\underline{a}}) = \sum_{j=1}^{n} \prod_{i} b_j(y_i)^{a_i}.$$

From the above equation, we get that $\mathfrak{i}(\phi_{\underline{a}})$ generate the k-algebra $k[\mathfrak{c}^d]^{W_\mathfrak{c}}$. We set $\psi_{\underline{a}} = \mathfrak{i}(\phi_{\underline{a}})$. Let $\theta_{\underline{a}} \in k[t] \otimes [X_1, X_2, \cdots, X_d]$ be the polynomial $t - X_1^{a_1} X_2^{a_2} \ldots X_n^{a_n}$. Then $p(\theta_{\underline{a}}) = tI - x_1^{a_1} x_2^{a_2} \ldots x_d^{a_d} \in \mathfrak{g}_1(A \otimes k[t])$, where I is the identity matrix. Using (5.1) we have

$$\det(p(\theta_{\underline{a}})) = \mathfrak{s}(\beta(\theta_{\underline{a}}^{\otimes n})).$$

By the definition of the map β we have $\beta(\theta_{\underline{a}}^{\otimes n}) = \prod_{j=1}^{n} (t - \prod_{i=1}^{d} b_j(y_i)^{a_i})$. Hence, we get that

$$\det(tI - x_1^{a_1} x_2^{a_2} \ldots x_d^{a_d}) = \mathfrak{s}(\prod_{j=1}^{n} (t - \prod_{i=1}^{d} b_j(y_i)^{a_i})). \tag{5.2}$$

Note that $\det(tI - x_1^{a_1} x_2^{a_2} \ldots x_d^{a_d})$ is the characteristic polynomial of the matrix $x_1^{a_1} x_2^{a_2} \ldots x_d^{a_d}$ and hence

$$\mathfrak{s}(\det(tI - x_1^{a_1} x_2^{a_2} \ldots x_d^{a_d})) = t^n - \mathfrak{s}(\mathrm{Tr}(x_1^{a_1} x_2^{a_2} \ldots x_d^{a_d}))t^{n-1} + \text{ lower degree terms in } t. \tag{5.3}$$

Comparing the coefficients of t^{2n-1} in equations (5.2) and (5.3) we get that $\mathfrak{s}(\mathfrak{i}(\phi_{\underline{a}})) = \phi_{\underline{a}}$. Thus, we conclude that $\mathfrak{s} \circ \mathfrak{i}$ and $\mathfrak{i} \circ \mathfrak{s}$ are both identities.

Hence, we get that the Chevalley restriction map $\mathfrak{i} : \mathfrak{c}^d // W_\mathfrak{c} \to \mathfrak{C}^d(\mathfrak{g}_1) // G$ is an isomorphism.

5.2. **The symmetric pair** AII. Let V be a k-vector space of dimension $2n$, and let $\omega : V \times V \to k$ be a non-degenerate skew symmetric bilinear form on V. Let $\tau : \mathfrak{gl}(V) \to \mathfrak{gl}(V)$ be the involution defined by $\tau(X) = -X^*$, where X^* is given by:

$$\omega(Xv, w) = \omega(v, X^* w).$$

Let $\mathfrak{g}_0$ and $\mathfrak{g}_1$ be the 1 and -1 eigenspaces of τ. Then the algebra $\mathfrak{g}_0$ can be identified with the Lie algebra of the connected component G_0 of $\mathrm{Sp}(V, \omega)$ containing the identity. Let

$$(w_{-n}, w_{-n+1}, \ldots, w_{-1}, w_1, \cdots, w_{n-1}, w_n)$$

be a Witt basis of V, and in this basis a Cartan subspace $\mathfrak{c} \subset \mathfrak{g}_1$ is given by the following subspace:

$$\{\mathrm{diag}(b_n, \ldots, b_1, b_1, \ldots, b_n) : b_i \in k, i \in [n]\}.$$

The little Weyl group $W_\mathfrak{c}$ is isomorphic to S_n-which is identified with the group of permutations of the coordinates of $\mathfrak{c}$.

In [CC21] for any k-algebra R, the authors defined a map

$$N_+ : \mathfrak{g}_1(R) \to R, \tag{5.4}$$

called it the Pfaffian norm map which satisfies that $\det(x) = N_+(x)^2$ for any $x \in \mathfrak{g}_1(R)$. More interestingly, they showed that the map N_+ is multiplicative on the coordinate ring of the commuting subscheme $\mathfrak{C}(\mathfrak{g}_1)$ of $\mathfrak{g}_1 \times \mathfrak{g}_1$.

Let A be the coordinate ring of $\mathfrak{C}^d(\mathfrak{g}_1)$, and let $(x_1, x_2, \ldots, x_d)$ be the universal point of this scheme. For any k-algebra R, we define the map

$$p : R[X_1, X_2, \ldots, X_d] \to \mathfrak{gl}_{2n}(A \otimes R), X_i \mapsto x_i, i \in [d].$$

The image of p is contained in $\mathfrak{g}_1(A \otimes R)$. Then we have the map

$$R[X_1, X_2, \ldots, X_d] \xrightarrow{p} \mathfrak{g}_1(A \otimes R) \xrightarrow{N_+} A \otimes R. \tag{5.5}$$

Since N_+ is multiplicative, the composition $N_+ \circ p$ is multiplicative and homogeneous of degree n. Thus by Roby's theorem we get a homomorphism of k-algebras:

$$\tilde{\mathfrak{s}} : TS^n(k[X_1, \ldots, X_d]) \to A$$

such that

$$\tilde{\mathfrak{s}}(q^{\otimes n}) = N_+(p(\theta)), \text{ for all } \theta \in k[X_1, \ldots, X_d]. \tag{5.6}$$

Since N_+ is a G_0-invariant map, the map $\tilde{\mathfrak{s}}$ is also G_0-invariant and hence we get that the image of $\tilde{\mathfrak{s}}$ is contained in A^{G_0}.

Let $(y_1, \ldots, y_d)$ be the tautological point of $\mathfrak{c}(B)$, where $B = k[\mathfrak{c}^d]$. Note that $B = k[\mathfrak{c}^d]$ is a polynomial algebra in the variables $b_j(y_i)$, $1 \le i \le d$ and $1 \le j \le n$. Let

$$\beta : TS^n(k[X_1, X_2, \ldots, X_d]) \to k[\mathfrak{c}^d]$$

be the isomorphism given by $\beta(X_{i,j}) = b_j(y_i)$, where $X_{i,j}$ is the variable X_i in the j-th copy, for $i \in [d]$ and $j \in [n]$. Hence, we get a map $\mathfrak{s} : k[\mathfrak{c}^d] \to A^{G_0}$ such that

$$\mathfrak{s} \circ \beta(\theta^{\otimes n}) = N_+ \circ p(\theta), \tag{5.7}$$

for all $\theta \in k[X_1, \ldots, X_d]$. Thus we obtain a map of schemes $\mathfrak{s} : \mathfrak{C}^d(\mathfrak{g}_1)//G_0 \to \mathfrak{c}^d//W_{\mathfrak{c}}$. We show that the map $\mathfrak{s} : \mathfrak{C}^d(\mathfrak{g}_1)//G_0 \to \mathfrak{c}^d//W_{\mathfrak{c}}$ is the inverse of the Chevalley restriction map $\mathfrak{i} : \mathfrak{c}^d//W_{\mathfrak{c}} \to \mathfrak{C}^d(\mathfrak{g}_1)//G_0$.

For the diagonal action of the symplectic group G_0 on the d-copies of $\mathfrak{gl}(V)$, the ring of invariants $k[\mathfrak{gl}(V)^d]^{G_0}$ is generated by the elements $Tr(M)$, where M is a monomial in X_j, X_j^*, $j = 1, 2, \cdots, d$ of degree less than or equal to $2^n - 1$ (see Theorem 10.1 of [Pro76]). Recall that the space $\mathfrak{g}_1$ is the -1 eigen space of the map $\theta(X) = -X^*$. So the elements in $\mathfrak{g}_1$ satisfy $X^* = X$. Since $\mathfrak{C}^d(\mathfrak{g}_1)$ is a closed subscheme of $\mathfrak{gl}(V)^d$ and G_0 is (linearly) reductive, the restriction map $k[\mathfrak{gl}(V)^d] \to k[\mathfrak{C}^d(\mathfrak{g}_1)]$ induces a surjective map $k[\mathfrak{gl}(V)^d]^{G_0} \to k[\mathfrak{C}^d(\mathfrak{g}_1)]^{G_0}$. So the restrictions of the functions $Tr(M)$, where M is a monomial in X_j, $j = 1, 2, \cdots, d$ form a generating set for $k[\mathfrak{C}^d(\mathfrak{g}_1)]^{G_0}$. Let $(x_1, x_2, \cdots, x_d)$ be the tautological point of the commuting scheme $\mathfrak{C}^d(\mathfrak{g}_1)$. Then the set $\{\phi_{\underline{a}} = \mathrm{Tr}(x_1^{a_1} x_2^{a_2} \ldots x_d^{a_d}) : \underline{a} = (a_1, \ldots, a_d) \in \mathbb{Z}^d_{\ge 0}\}$ is a generating set for $k[\mathfrak{C}^d(\mathfrak{g}_1)]^{G_0}$.

Note that

$$\mathfrak{i}(\phi_{\underline{a}}) = 2 \sum_{j=1}^{n} \prod_{i} b_j(y_i)^{a_i}.$$

From the above equation, we get that $\mathfrak{i}(\phi_{\underline{a}})$ generate the k-algebra $k[\mathfrak{c}^d]^{W_{\mathfrak{c}}}$. We set $\psi_{\underline{a}} = \mathfrak{i}(\phi_{\underline{a}})$, for all n-tuples $\underline{a} \in \mathbb{Z}^n_{\ge 0}$. In order to show that $\mathfrak{c}$ and $\mathfrak{s}$ are inverses of each other we need to show that $\mathfrak{s}(\psi_{\underline{a}}) = \phi_{\underline{a}}$, for all n-tuples $\underline{a} \in \mathbb{Z}^n_{\ge 0}$.

Let $\theta \in k[X_1, X_2, \cdots, X_d] \otimes k[t]$ be the polynomial $t - X_1^{a_1} X_2^{a_2} \ldots X_n^{a_n}$. Then $p(\theta) = tI - x_1^{a_1} x_2^{a_2} \ldots x_d^{a_d} \in \mathfrak{g}_1(A \otimes k[t])$, where I is the identity matrix. Since $\det(x) = N_+(x)^2$ for any $x \in \mathfrak{g}_1(R)$, where R is a k-algebra, we have

$$\det(p(\theta)) = N_+(p(\theta))^2.$$

Using (5.7) we also have

$$N_+(p(\theta)) = \mathfrak{s}(\beta(\theta^{\otimes n})).$$

By the definition of the map β we have $\beta(\theta_a^{\otimes n}) = \prod_{j=1}^{n} (t - \prod_{i=1}^{d} b_j(y_i)^{a_i})$.

Hence, we get that

$$\det(tI - x_1^{a_1}x_2^{a_2}\dots x_d^{a_d}) = \mathfrak{s}(\prod_{j=1}^{n}(t - \prod_{i=1}^{d} b_j(y_i)^{a_i}))^2.$$

Note that $\det(tI - x_1^{a_1}x_2^{a_2}\dots x_d^{a_d})$ is the characteristic polynomial of the matrix $x_1^{a_1}x_2^{a_2}\dots x_d^{a_d}$ and hence $\det(tI - x_1^{a_1}x_2^{a_2}\dots x_d^{a_d}) = t^{2n} - Tr(x_1^{a_1}x_2^{a_2}\dots x_d^{a_d})t^{2n-1} +$ lower degree terms in t. On the other hand

$$\mathfrak{s}(\prod_{j=1}^{n}(t - \prod_{i=1}^{d} b_j(y_i)^{a_i}))^2 = t^{2n} + 2\sum_{j=1}^{n}\prod_{i} b_j(y_i)^{a_i} t^{2n+1} + \text{ lower degree terms in } t.$$

Comparing the coefficients of t^{2n-1} we get that $\mathfrak{s}(\mathfrak{i}(\phi_a)) = \phi_a$.

Hence, we get that the Chevalley restriction map $\mathfrak{i} : \mathfrak{c}^d//W_{\mathfrak{c}} \to \mathfrak{C}^d(\mathfrak{g}_1)//G_0$ is an isomorphism.

5.3. **The symmetric pair** $AIII$**.** Let m, n be positive integers such that $m \geq n$. Let $\mathfrak{g}$ be the Lie algebra $\mathfrak{gl}_{n+m}(k)$ and let $\mathfrak{g}_0$ be the Lie algebra

$$\left\{\begin{pmatrix} A & 0 \\ 0 & B\end{pmatrix} : A \in \mathfrak{gl}_n(k), B \in \mathfrak{gl}_m(k)\right\},$$

and let $\mathfrak{g}_1$ be the space

$$\left\{\begin{pmatrix} 0 & X \\ Y & 0\end{pmatrix} : X, Y^t \in \mathrm{M}_{n\times m}(k)\right\}.$$

The pair $(\mathfrak{g}, \mathfrak{g}_0)$ is a symmetric pair for an involution with $\mathfrak{g}_0$, the space of invariants and $\mathfrak{g}_1$, the -1 eigenspace. The adjoint action of the group $G_0 = \mathrm{GL}_n(k) \times \mathrm{GL}_m(k)$ on the space $\mathfrak{g}_1$ is given by

$$(g_1, g_2)(X, Y) = (g_1 X g_2^{-1}, g_2 Y g_1^{-1}).$$

A Cartan subspace, denoted by $\mathfrak{c}$, of the symmetric pair $(\mathfrak{g}, \mathfrak{g}_0)$ is given by the space consisting of matrices of the form

$$\begin{pmatrix} 0 & X \\ X^t & 0\end{pmatrix},$$

where $X = [\mathrm{diag}(b_1, b_2, \dots, b_n), 0_{n\times m-n}]$ for some $b_i \in k$, $i \in [n]$. The little Weyl group $W_{\mathfrak{c}}$ in this case is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n$.

Let $\mathfrak{C}^d(\mathfrak{g}_1)$ be the commuting scheme associated to the pair $(\mathfrak{g}, \mathfrak{g}_0)$. Let $A = k[\mathfrak{C}^d(\mathfrak{g}_1)]$ be the coordinate ring of $\mathfrak{C}^d(\mathfrak{g}_1)$ and let $(x_1, \dots, x_d)$ be the universal point of $\mathfrak{C}^d(\mathfrak{g}_1)(A)$. We set

$$x_i = \begin{pmatrix} 0 & Q_i \\ R_i & 0\end{pmatrix}, Q_i, R_i^t \in \mathfrak{gl}_{n\times m}(A).$$

Let R be a k-algebra and let $p : R[X_1, \dots, X_d] \to \mathfrak{gl}_{n+m}(A\otimes R)$ be the map sending $X_i \mapsto x_i$ for all $i \in [d]$. Let $R[X_1, \dots, X_d]^+$ be the algebra spanned by even degree monomials. Note that the image of the map $p : R[X_1, \dots, X_d]^+ \to \mathfrak{gl}_{n+m}(A\otimes R)$ is contained in $\mathfrak{gl}_n(A\otimes R) \times \mathfrak{gl}_m(A\otimes R)$. Let q_1 be the first projection of $\mathfrak{gl}_n(A\otimes R) \times \mathfrak{gl}_m(A\otimes R)$. Note that the composite map

$$R[X_1, \dots, X_d]^+ \xrightarrow{p} \mathfrak{gl}_n(A\otimes R) \times \mathfrak{gl}_m(A\otimes R) \xrightarrow{q_1} \mathfrak{gl}_n(A\otimes R) \xrightarrow{\det} A\otimes R$$

has degree n. By Roby's theorem we get a map

$$\tilde{\mathfrak{s}} : TS^n(k[X_1, \dots, X_d]^+) \to A$$

such that $\tilde{\mathfrak{s}}(\theta^{\otimes n}) = \det\ \circ q_1\ \circ p(\theta)$, for all $\theta \in k[X_1, \dots, X_d]^+$. Note that the image of the map $\tilde{\mathfrak{s}}$ is contained in A^{G_0}.

The algebra $TS^n k[X_1, \dots, X_d]^+$ can be identified with the subalgebra of $T^n k[X_1, \dots, X_d]$ of fixed points under $W_{\mathfrak{c}} = S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$. Now, we give an isomorphism of $TS^n k[X_1, \dots, X_d]^+$ with $k[\mathfrak{c}^d]^{W_{\mathfrak{c}}}$.

Let $(y_1, \ldots, y_d)$ be the tautological point of $\mathfrak{c}(k[\mathfrak{c}^d])$. Note that $k[\mathfrak{c}^d]$ is a polynomial algebra in the variables $b_j(y_i)$, $1 \le i \le d$ and $1 \le j \le n$. Let

$$\beta : T^n k[X_1, \ldots, X_d] \to k[\mathfrak{c}^d]$$

be the isomorphism of algebras defined by $\beta(X_{ij}) = b_j(y_i)$, for $i \in [d]$ and $j \in [n]$. Here, X_{ij} is the i-th variable in the j-th copy of $k[X_1, \ldots, X_d]$. By restriction we have an isomorphism of algebras

$$\beta : TS^n k[X_1, \ldots, X_d]^+ \to k[\mathfrak{c}^d]^{W_\mathfrak{c}}.$$

Then it follows that we have an algebra map

$$\mathfrak{s} : k[\mathfrak{c}^d]^{W_\mathfrak{c}} \to A^{G_0}$$

such that

$$\mathfrak{s}\beta(\theta^{\otimes n}) = \det \circ q_1 \circ p(\theta), \text{ for all } \theta \in k[X_1, \ldots, X_d]^+.$$

Thus we obtain a map of schemes $\mathfrak{s} : \mathfrak{C}^d(\mathfrak{g}_1)//G_0 \to \mathfrak{c}^d//W_\mathfrak{c}$.

Let $\mathfrak{i} : k[\mathfrak{C}^d(\mathfrak{g}_1)]^{G_0} \to k[\mathfrak{c}^d]^{W_\mathfrak{c}}$ be the restriction map. We will show that $\mathfrak{s}$ and $\mathfrak{i}$ are inverses of each other.

The ring of invariants $k[\mathfrak{C}^d(\mathfrak{g}_1)]^{G_0}$ is generated by the images of the following polynomials, via the restriction map $k[\mathfrak{g}_1^d]^{G_0} \to k[\mathfrak{C}^d(\mathfrak{g}_1)]^{G_0}$:

$$\mathrm{Tr}(M_1 M_2 \ldots M_k) \tag{5.8}$$

where $M_i = Q_{n_i} R_{m_i}$ for some $n_i, m_i \in [d]$; and we denote the polynomial in (5.8) by P (see Section 4.1). We then have

$$\mathfrak{i}(P) = \sum_{j=1}^{n} \prod_{(n_i,m_i)} b_j(y_{n_i}) b_j(y_{m_i}).$$

From the above equation, we get that $\mathfrak{i}(\phi_{\underline{a}})$ generate the k-algebra $k[\mathfrak{c}^d]^{W_\mathfrak{c}}$. Let R be the algebra $k[t]$, and consider the element $\theta = t - \prod_{(n_i,m_i)} X_{n_i} X_{m_i}$ in $R[X_1, X_2, \ldots, X_d]^+$. We then have

$$\beta(\theta^{\otimes n}) = \prod_{j=1}^{n} (t - \prod_{(n_i,m_i)} b_j(y_{n_i}) b_j(y_{m_i})).$$

We then observe that $\det \circ q_1 \circ p(\theta)$ is equal to

$$\det(t - \prod_{(n_i,m_i)} b_j(y_{n_i}) b_j(y_{m_i})).$$

Since $\mathfrak{s} \circ \beta(\theta^{\otimes n}) = \det \circ q_1 \circ p(\theta)$, we get that

$$t^n - \mathfrak{s}\left(\sum_{j=1}^{n} \prod_{(n_i,m_i)} b_j(y_{n_i}) b_j(y_{m_i}) \right) t^{n-1} + \cdots = t^n - \mathrm{Tr}(\prod_{(n_i,m_i)} Q_{n_i} R_{m_i}) t^{n-1} + \cdots$$

Comparing the coefficients of t^{n-1} we get that $\mathfrak{s} \circ \mathfrak{i}(P) = P$. Thus, we get that $\mathfrak{s}$ is a section of the map $\mathfrak{i}$ and hence $\mathfrak{i}$ is an isomorphism.

5.4. **The symmetric pair** BDI. Let m, n be positive integers such that $m \ge n$ and let $\mathfrak{g}$ be the Lie algebra $\mathfrak{so}_{n+m}(k)$. Let $\mathfrak{g}_0$ be the Lie algebra

$$\left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} : A \in \mathfrak{so}_n(k), B \in \mathfrak{so}_m(k) \right\},$$

and let $\mathfrak{g}_1$ be the space

$$\left\{ \begin{pmatrix} 0 & X \\ -X^t & 0 \end{pmatrix} : X \in \mathrm{M}_{n\times m}(k) \right\}.$$

The pair $(\mathfrak{g}, \mathfrak{g}_0)$ is a symmetric pair for an involution with $\mathfrak{g}_0$, the space of invariants and $\mathfrak{g}_1$, the -1 eigenspace. The adjoint action of the group $G_0 = \mathrm{SO}_n(k) \times \mathrm{SO}_m(k)$ on the space $\mathfrak{g}_1$ is given by

$$(g_1, g_2)(X, Y) = (g_1 X g_2^{-1}, g_2 Y g_1^{-1}).$$

A Cartan subspace, denoted by $\mathfrak{c}$, of the symmetric pair $(\mathfrak{g}, \mathfrak{g}_0)$ is given by the space consisting of matrices of the form

$$\begin{pmatrix} 0 & X \\ -X^t & 0 \end{pmatrix},$$

where X is a matrix of the form $[A, 0_{n\times(m-n)}]$, with $A = \mathrm{diag}(b_1, b_2, \ldots, b_n)$ for some $b_i \in k$, $i \in [n]\}$. The little Weyl group $W_{\mathfrak{c}}$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}^n \rtimes S_n$. This case is similar to the case of $AIII$, however since the space $\mathfrak{g}_1$ differs from its counter part in $AIII$, we briefly describe the section for the Chevalley restriction map.

Let $\mathfrak{C}^d(\mathfrak{g}_1)$ be the commuting scheme associated to the pair $(\mathfrak{g}, \mathfrak{g}_0)$. Let $A = k[\mathfrak{C}^d(\mathfrak{g}_1)]$, and let $(x_1, \ldots, x_d)$ be the universal point in $\mathfrak{C}^d(\mathfrak{g}_1)(A)$. We set

$$x_i = \begin{pmatrix} 0 & Q_i \\ -Q_i^t & 0 \end{pmatrix}, Q_i \in \mathfrak{gl}_{n\times m}(A).$$

Let R be a k-algebra and let $p : R[X_1, \ldots, X_d] \to \mathfrak{gl}_{n+m}(A \otimes R)$ be the map sending $X_i \mapsto x_i$ for all $i \in [d]$. Let $R[X_1, \ldots, X_d]^+$ be the subalgebra of $R[X_1, \ldots, X_d]$ spanned by the even degree monomials. Note that the image of the map $p : R[X_1, \ldots, X_d]^+ \to \mathfrak{gl}_{n+m}(A \otimes R)$ is contained in $\mathfrak{gl}_n(A \otimes R) \times \mathfrak{gl}_m(A \otimes R)$. Let q_1 be the first projection of $\mathfrak{gl}_n(A \otimes R) \times \mathfrak{gl}_m(A \otimes R)$. Note that the composite map

$$R[X_1, \ldots, X_d]^+ \xrightarrow{p} \mathfrak{gl}_n(A \otimes R) \times \mathfrak{gl}_m(A \otimes R) \xrightarrow{q_1} \mathfrak{gl}_n(A \otimes R) \xrightarrow{\det} A \otimes R$$

is multiplicative and has degree n. So by Roby's theorem we get a map

$$\tilde{\mathfrak{s}} : TS^n(k[X_1, \ldots, X_d]^+) \to A$$

such that $\tilde{\mathfrak{s}}(\theta^{\otimes n}) = \det \circ q_1 \circ p(\theta)$, for all $\theta \in k[X_1, \ldots, X_d]^+$. We note that the image of $\tilde{\mathfrak{s}}$ belongs to A^{G_0}.

Let $(y_1, \ldots y_d)$ be a tautological point of $k[\mathfrak{c}^d](B)$, where $B = k[\mathfrak{c}^d]$. Let

$$\beta : TS^n(k[X_1, \ldots, X_d]^+) \to k[\mathfrak{c}^d]$$

be the map $\beta(X_{ij}) = b_j(y_i)$, for all $i \in [d]$ and $j \in [n]$. Here, X_{ij} be the variable X_j in the i-th copy in $TS^n(k[X_1, \ldots, X_d])$. As in the case of $AIII$, β restricts to an isomorphism of algebras

$$\beta : TS^n k[X_1, \ldots, X_d]^+ \to k[\mathfrak{c}^d]^{W_{\mathfrak{c}}}.$$

Then it follows that we have an algebra map

$$\mathfrak{s} : k[\mathfrak{c}^d]^{W_{\mathfrak{c}}} \to A^{G_0}$$

such that

$$\mathfrak{s}\beta(\theta^{\otimes n}) = \det \circ q_1 \circ p(\theta), \text{ for all } \theta \in k[X_1, \ldots, X_d]^+.$$

Thus we obtain a map of schemes $\mathfrak{s} : \mathfrak{C}^d(\mathfrak{g}_1)//G_0 \to \mathfrak{c}^d//W_{\mathfrak{c}}$.

We assume that m is odd. The ring of invariants $k[\mathfrak{C}^d(\mathfrak{g}_1)]^{G_0}$ is generated by the images of the following polynomials, via the restriction map $k[\mathfrak{g}_1^d]^{G_0} \to k[\mathfrak{C}^d(\mathfrak{g}_1)]^{G_0}$:

$$\mathrm{Tr}(M_1 M_2 \ldots M_k) \tag{5.9}$$

where $M_i = Q_{n_i} Q_{m_i}^t$ for some $n_i \in [d]$; and we denote the polynomial in (5.9) by P (see Subsection 4.2). We then have

$$\mathfrak{i}(P) = \sum_{j=1}^{n} \prod_{n_i} b_j(y_{n_i}) b_i(y_{m_i}).$$

From the above equation, we get that $\mathfrak{i}(\phi_{\underline{a}})$ generate the k-algebra $k[\mathfrak{c}^d]^{W_\mathfrak{c}}$. Let R be the algebra $k[t]$, and set $\theta = t - \prod_{n_i} X_{n_i}X_{m_i}$ in the algebra $R[X_1, X_2, \dots, X_d]^+$. We then have

$$\beta(\theta^{\otimes n}) = \prod_{j=1}^{n}(t - \prod_{(n_i,m_i)} b_j(y_{n_i})b_j(y_{m_i})).$$

We then observe that $\det \circ q_1 \circ p(\theta)$ is equal to

$$\det(t - \prod_{(n_i,m_i)} b_j(y_{n_i})b_j(y_{m_i})).$$

Since $\mathfrak{s} \circ \beta(\theta^{\otimes n}) = \det \circ q_1 \circ p(\theta)$, we get that

$$t^n - \mathfrak{s}\left(\sum_{j=1}^{n}\prod_{(n_i,m_i)} b_j(y_{n_i})b_j(y_{m_i})\right)t^{n-1} + \cdots = t^n - \mathrm{Tr}(\prod_{(n_i,m_i)} Q_{n_i}Q_{m_i}^t)t^{n-1} + \cdots$$

Comparing the coefficients of t^{n-1} we get that $\mathfrak{s} \circ \mathfrak{i}(P) = P$. Thus, we get that $\mathfrak{s}$ is a section of the map $\mathfrak{i}$.

Remark 5.1. When $m = n$ is even, the ring of invariants $k[\mathfrak{C}_d(\mathfrak{g}_1)]^{G_0}$ has additional invariants of the form

$$\det(T_1 \otimes A_1 + T_2 \otimes A_2 + \cdots + T_d \otimes A_d) \tag{5.10}$$

where $T = (T_1, \dots, T_d)$ is an element of $M^d_{r\times r}$ and $r \geq 1$. We denote by ψ_T the invariant in (5.10). Consider the element

$$\theta = \det(T_1 \otimes X_1 I_{n\times n} + T_2 \otimes X_2 I_{n\times n} + \cdots + T_d \otimes X_d I_{n\times n})$$

in the ring $R[X_1, \dots, X_d]^+$, where $R = k[M^d_{r\times r}]$. The identity $\mathfrak{s} \circ \beta(\theta^{\otimes n}) = \det \circ q_1 \circ p(\theta)$ implies that $\mathfrak{s} \circ \mathfrak{i}(\psi_T) = \psi_T$. Now, under the assumption that the invariants in (5.10) and (5.9) together generate the algebra $k[\mathfrak{C}_d(\mathfrak{g}_1)]^{G_0}$, we get that the map $\mathfrak{s}$ is the section of the Chevalley restriction map.

5.5. **The symmetric pair CI.** Let V be a $2n$-dimensional k-vector space and let ω be a non-degenerate skew symmetric bilinear form on V. Let $(v_1, v_2, \dots v_n, v_{-1}, \dots, v_{-n})$ be a Witt basis for the pair (V, ω). The matrix of the form ω is equal to

$$J = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix},$$

where I_n is the $n \times n$ identity matrix. Let V_+ and V_- be the spaces $\langle v_1, \dots, v_n\rangle$ and $\langle v_{-1}, \dots, v_{-n}\rangle$ respectively. In this basis we identify $\mathfrak{g} = \mathfrak{sp}(V, \omega)$ as a Lie subalgebra of $\mathfrak{gl}_{2n}(k)$. Let $T \in GL(V)$ be such that $T|_{V_+} = id$ and $T|_{V_-} = -id$. Then the conjugation by T defines an involution of $\mathfrak{g}$ such that the Lie algebra $\mathfrak{g}_0$ is identified with the Lie algebra

$$\{\begin{pmatrix} A & 0 \\ 0 & -A^t \end{pmatrix} : A \in \mathfrak{gl}_n(k)\},$$

and $\mathfrak{g}_1$ is the space

$$\{\begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix} : X, Y \in \mathfrak{gl}_n(k), X^t = X, Y^t = Y\}.$$

Let G_0 be the connected subgroup of $\mathrm{Sp}(V, \omega)$ with $\mathfrak{g}_0$ as its Lie algebra. We identify G_0 with $\mathrm{GL}_n(k)$. As a G_0 module $\mathfrak{g}_1$ is isomorphic to $\mathrm{Sym}^2(k^n) \oplus (\mathrm{Sym}^2(k^n))^*$. We choose the Cartan subspace $\mathfrak{c}$ to be the subspace

$$\{\begin{pmatrix} 0 & X \\ X & 0 \end{pmatrix} : X = \mathrm{diag}(b_1, \dots, b_n), b_i \in k, i \in [n]\}$$

of $\mathfrak{g}_1$. Then the little Weyl group, denoted by $W_{\mathfrak{c}}$, is isomorphic to $S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$.

Let $\mathfrak{C}^d(\mathfrak{g}_1)$ be the d-fold commuting scheme associated with the symmetric pair $(\mathfrak{g}, \mathfrak{g}_0)$ and let $A = k[\mathfrak{C}^d(\mathfrak{g}_1)]$ be the coordinate ring. Let $(x_1, \dots, x_d)$ be the universal point in $\mathfrak{C}^d(\mathfrak{g}_1)(A)$. Let Q_i and R_i be matrices with values in A such that

$$x_i = \begin{pmatrix} 0 & Q_i \\ R_i & 0 \end{pmatrix}, i \in [d]. \tag{5.11}$$

Note that $Q_i^t = Q_i$ and $R_i^t = R_i$, for all $i \in [d]$. Let R be a k algebra and let $p : R[X_1, \dots, X_d] \to \mathfrak{gl}_{2n}(A \otimes R)$ be the map defined by $X_i \mapsto x_i$. Let $R[X_1, \dots, X_d]^+$ be the R-subalgebra generated by even degree monomials. Note that the image of the map p is contained in $\mathfrak{gl}_n(A \otimes R) \times \mathfrak{gl}_n(A \otimes R)$. Let q_1 and q_2 be the first and second projections of $\mathfrak{gl}_n(A \otimes R) \times \mathfrak{gl}_n(A \otimes R)$ respectively. We have

$$R[X_1, \dots, X_d]^+ \xrightarrow{p} \mathfrak{gl}_n(A \otimes R) \times \mathfrak{gl}_n(A \otimes R) \xrightarrow{det \circ q_1} A \otimes R.$$

Since the map det is multiplicative, the composition $\det \circ q_1 \circ p$ is multiplicative and homogeneous of degree n. Thus by Roby's theorem we get a homomorphism of k-algebras:

$$\tilde{\mathfrak{s}} : TS^n(k[X_1, \dots, X_d]^+) \to A$$

such that

$$\tilde{\mathfrak{s}}(\theta^{\otimes n}) = det \circ q_1(p(\theta)), \text{ for all } \theta \in k[X_1, \dots, X_d]. \tag{5.12}$$

Note that the image of the map $\tilde{\mathfrak{s}}$ is contained in A^{G_0}.

The subalgebra $TS^n k[X_1, \dots, X_d]^+$ can be identified with the subalgebra of $T^n k[X_1, \dots, X_d]$ of fixed points under $S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$. Now, we give an isomorphism of $TS^n k[X_1, \dots, X_d]^+$ with $k[\mathfrak{c}^d]^{W_{\mathfrak{c}}}$. Let $(y_1, \dots, y_d)$ be the tautological point of $\mathfrak{c}(k[\mathfrak{c}^d])$. Note that $k[\mathfrak{c}^d]$ is a polynomial algebra in the variables $b_j(y_i)$, $1 \le i \le d$ and $1 \le j \le n$. Let

$$\beta : T^n k[X_1, \dots, X_d] \to k[\mathfrak{c}^d]$$

be the isomorphism of algebras defined by $\beta(X_{ij}) = b_j(y_i)$, for $i \in [d]$ and $j \in [n]$. Here, X_{ij} is the i-th variable in the j-th copy of $k[X_1, \dots, X_d]$. By restriction we have an isomorphism of algebras

$$\beta : TS^n k[X_1, \dots, X_d]^+ \to k[\mathfrak{c}^d]^{W_{\mathfrak{c}}}.$$

Then it follows that we have an algebra map

$$\mathfrak{s} : k[\mathfrak{c}^d]^{W_{\mathfrak{c}}} \to A^{G_0}$$

such that

$$\mathfrak{s}\beta(\theta^{\otimes n}) = \det \circ q_1 \circ p(\theta), \text{ for all } \theta \in k[X_1, \dots, X_d]^+.$$

Thus we obtain a map of schemes $\mathfrak{s} : \mathfrak{C}^d(\mathfrak{g}_1)//G_0 \to \mathfrak{c}^d//W_{\mathfrak{c}}$.

Let $\mathfrak{i} : k[\mathfrak{C}^d(\mathfrak{g}_1)]^{G_0} \to k[\mathfrak{c}^d]^{W_{\mathfrak{c}}}$ be the Chevalley restriction map. We will show that $\mathfrak{s}$ and $\mathfrak{i}$ are inverses of each other.

Recall that $\mathfrak{g}_1$ as a G_0-module is isomorphic to $\mathrm{Sym}^2(k^n) \oplus (\mathrm{Sym}^2(k^n))^*$. The ring of invariants $k[\mathfrak{C}^d(\mathfrak{g}_1)]^{G_0}$ is generated by the images of the following polynomials, via the restriction map $k[\mathfrak{g}_1^d]^{G_0} \to k[\mathfrak{C}^d(\mathfrak{g}_1)]^{G_0}$:

$$\mathrm{Tr}(M_1 M_2 \dots M_k) \tag{5.13}$$

where $M_i = Q_{n_i} R_{m_i}$ for some $n_i, m_i \in [d]$; and we denote the polynomial in (5.13) by P (see Subsection 4.3). Here Q_{n_i} and R_{n_i} are as defined in (5.11). We then have

$$\mathfrak{i}(P) = \sum_{j=1}^{n} \prod_{(n_i, m_i)} b_j(y_{n_i}) b_j(y_{m_i}).$$

From the above equation, we get that $\mathfrak{i}(\phi_{\underline{a}})$ generate the k-algebra $k[\mathfrak{c}^d]^{W_\mathfrak{c}}$. Let R be the algebra $k[t]$, and consider the element $\theta = t - \prod_{(n_i,m_i)} X_{n_i}X_{m_i}$ in $R[X_1, X_2, \ldots, X_d]^+$. We then have

$$\beta(\theta^{\otimes n}) = \prod_{j=1}^{n}(t - \prod_{(n_i,m_i)} b_j(y_{n_i})b_j(y_{m_i})).$$

We then observe that $\det \circ q_1 \circ p(\theta)$ is equal to

$$\det(t - \prod_{(n_i,m_i)} Q_{n_i}R_{m_i}).$$

Since $\mathfrak{s} \circ \beta(\theta^{\otimes n}) = \det \circ q_1 \circ p(\theta)$, we get that

$$t^n - \mathfrak{s}\left(\sum_{j=1}^{n}\prod_{(n_i,m_i)} b_j(y_{n_i})b_j(y_{m_i})\right)t^{n-1} + \cdots = t^n - \mathrm{Tr}(\prod_{(n_i,m_i)} Q_{n_i}R_{m_i})t^{n-1} + \cdots$$

Comparing the coefficients of t^{n-1} we get that $\mathfrak{s} \circ \mathfrak{i}(P) = P$. Thus, we get that $\mathfrak{s}$ is a section of the map $\mathfrak{i}$ and hence $\mathfrak{i}$ is an isomorphism.

6. The symmetric pair of type CII

Let n be a positive integer and let $n = q + r$ for some positive integers q and r with $r \leq q$. Let V be a $2n$ dimensional k-vector space and let ω be a non-degenerate symplectic form on V. Let

$$(w_1, w_{-1}, w_2, w_{-2}, \ldots, w_n, w_{-n})$$

be a Witt-basis for V such that $\omega(w_i, w_j) = 1$, for $i + j = 0$, $i > 0$ and $\omega(w_i, w_j) = 0$, for all i, j such that $i + j \neq 0$. Let V_1 and V_2 be the subspaces of V spanned by $\{w_{\pm 1}, w_{\pm 2}, \ldots, w_{\pm r}\}$ and $\{w_{\pm(r+1)}, w_{\pm(r+2)}, \ldots, w_{\pm(q+r)}\}$ respectively. Let T_s be the $2s \times 2s$ matrix given by $\mathrm{diag}(\mu, \mu, \ldots, \mu)$, where μ is the matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Let $\mathfrak{g}$ be the Lie algebra defined by

$$\{X \in \mathrm{End}_k(V) : \omega(Xv, w) + \omega(v, Xw) = 0\}.$$

Using the above Witt-basis, we identify $\mathfrak{g}$ with the Lie algebra $\mathfrak{sp}_{2n}(k)$. Let $\mathfrak{g}_0$ be the Lie subalgebra of $\mathfrak{g}$ consisting of matrices

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

where A and B belong to $\mathfrak{sp}_{2r}(k)$ and $\mathfrak{sp}_{2q}(k)$ respectively. Let $\mathfrak{g}_1$ be the subspace of $\mathfrak{g}$ consisting of matrices of the form

$$\begin{pmatrix} 0 & X \\ T_r X^t T_q & 0 \end{pmatrix}.$$

Note that $(\mathfrak{g}, \mathfrak{g}_0)$ is a symmetric pair of type CII. The Cartan subspace $\mathfrak{c}$ of $(\mathfrak{g}, \mathfrak{g}_0)$ is given by the set of matrices of the form

$$\begin{pmatrix} 0 & X \\ T_r X^t T_q & 0 \end{pmatrix}.$$

where X is of the form $\begin{pmatrix} BT_r \\ 0_{q-r} \end{pmatrix}$ and B is the diagonal matrix $[b_1, b_1, \ldots, b_r, b_r]$. The little Weyl group $W_\mathfrak{c}$ is equal to $(\mathbb{Z}/2\mathbb{Z})^r \rtimes S_r$.

Let $\mathfrak{C}^d(\mathfrak{g}_1)$ be the d-fold commuting scheme attached with the pair $(\mathfrak{g}, \mathfrak{g}_0)$. Let A be the coordinate ring of the affine scheme $\mathfrak{C}^d(\mathfrak{g}_1)$, and let $(x_1, x_2, \dots, x_d)$ be the universal point of the scheme $\mathfrak{C}^d(\mathfrak{g}_1)$. We then have

$$x_i = \begin{pmatrix} 0 & X_i \\ T_r X_i^t T_q & 0 \end{pmatrix},$$

for $i \in [d]$. Let R be a k-algebra, and let $p : R[X_1, \dots, X_d] \to \mathfrak{gl}_{2n}(A \otimes R)$ be the map $X_i \mapsto x_i$, for some $i \in [d]$. Let $R[X_1, X_2, \dots, X_d]^+$ be the subalgebra consisting of even degree polynomials in $R[X_1, X_2, \dots, X_d]$. We note that the image of p restricted to $R[X_1, X_2, \dots, X_d]^+$ lands in $\mathfrak{gl}_{2r}(A \otimes R) \times \mathfrak{gl}_{2q}(A \otimes R)$. Moreover, from the relations

$$X_i T_r X_j^t = X_j T_r X_i^t$$

and

$$X_i^t T_q X_j = X_j^t T_q X_i$$

we get that the image of the composite map

$$R[X_1, \dots, X_d]^+ \xrightarrow{p} \mathfrak{gl}_{2r}(A \otimes R) \times \mathfrak{gl}_{2q}(A \otimes R)$$

is contained in $\mathfrak{g}_r^+(A \otimes R) \times \mathfrak{g}_q^+(A \otimes R)$. Here, $\mathfrak{g}_m^+(A \otimes R)$ is given by

$$\{X \in M_{2m \times 2m}(A \otimes R) : T_m X - X^t T_m = 0\}.$$

Consider the following composite map

$$R[X_1, X_2, \dots, X_d]^+ \xrightarrow{p} \mathfrak{g}_r^+(A \otimes R) \times \mathfrak{g}_q^+(A \otimes R) \xrightarrow{q_1} \mathfrak{g}_r^+(A \otimes R) \xrightarrow{N_+} A \otimes R$$

Here, N_+ is the Pfaffian norm map as defined in 5.4 and q_1 is the first projection. Note that the map $N_+ \circ q_1 \circ p$ is a degree r map and hence Roby's theorem implies that there exists an k-algebra homomorphism

$$\tilde{s} : TS^r R[X_1, \dots, X_d]^+ \to A^{G_0}$$

such that $\tilde{s}(\theta^{\otimes r}) = N_+ \circ q_1 \circ p(\theta)$ for all $\theta \in R[X_1, \dots, X_d]^+$.

Let $(y_1, \dots, y_d)$ be the tautological point of $\mathfrak{c}(k[\mathfrak{c}^d])$. Note that $k[\mathfrak{c}^d]$ is a polynomial algebra in the variables $b_j(y_i)$, $1 \le i \le d$ and $1 \le j \le r$. Let

$$\beta : T^r k[X_1, \dots, X_d] \to k[\mathfrak{c}^d]$$

be the isomorphism of algebras defined by $\beta(X_{ij}) = b_j(y_i)$, for $i \in [d]$ and $j \in [r]$. Here, X_{ij} is the i-th variable in the j-th copy of $k[X_1, \dots, X_d]$. By restriction we have an isomorphism of algebras

$$\beta : TS^r k[X_1, \dots, X_d]^+ \to k[\mathfrak{c}^d]^{W_\mathfrak{c}}.$$

Then it follows that we have an algebra map

$$\mathfrak{s} : k[\mathfrak{c}^d]^{W_\mathfrak{c}} \to A^{G_0}$$

such that

$$\mathfrak{s}\beta(\theta^{\otimes r}) = N_+ \circ q_1 \circ p(\theta), \text{ for all } \theta \in k[X_1, \dots, X_d]^+.$$

Note that $\mathfrak{g}_1$ as a G_0 module is isomorphic to $M_{2r \times 2q}(k)$. Since $\mathfrak{C}^d(\mathfrak{g}_1)$ is a closed subscheme of $\mathfrak{g}_1^d$, the ring of invariants $k[\mathfrak{C}^d(\mathfrak{g}_1)]^{G_0}$ is generated by polynomials of the form

$$\mathrm{Tr}(M_{n_1 m_1} M_{n_2 m_2} \dots M_{n_l m_l})$$

where $M_{ij} = M_i^t T_r M_j T_q$ and $n_i, m_i \in [d]$ and $M_i \in M_{2r \times 2q}$ is a $2r \times 2q$ matrix (see subsection 4.4). We denote the above polynomial by P. Let $\mathfrak{i} : k[\mathfrak{C}^d(\mathfrak{g}_1)]^{G_0} \to k[\mathfrak{c}^d]^{W_\mathfrak{c}}$ be the Chevalley's restriction map. Note that $\mathfrak{i}(P)$ is given by

$$\mathfrak{i}(P) = 2 \sum_{j=1}^{p} \prod_{(n_i, m_i)} b_j(y_{n_i}) b_j(y_{m_i}).$$

Hence, by the explicit nature of the polynomials $\mathfrak{i}(P)$, we get that $k[\mathfrak{c}^d]^{W_\mathfrak{c}}$ is generated by polynomials of the form $\mathfrak{i}(P)$. Let R be the algebra $k[t]$, and consider the element $\theta = t - \prod_{(n_i,m_i)} X_{n_i}X_{m_i}$ in $R[X_1, X_2, \ldots, X_d]^+$. We then have

$$\beta(\theta^{\otimes r}) = \prod_{j=1} \left(t - \prod_{(n_i,m_i)} b_j(y_{n_i})b_j(y_{m_i}) \right) \tag{6.1}$$

Note that $N_+ \circ q_1 \circ p(\theta)$ is equal to

$$N_+ \left(t - \prod_{(n_i,m_i)} X_{n_i}T_rX^t_{m_i}T_q \right) \tag{6.2}$$

Taking square of (6.1) and (6.2) and using $\mathfrak{s} \circ \beta(\theta^{\otimes r}) = N_+ \circ q_1 \circ p(\theta)$ we get that

$$\det \left(t - \prod_{(n_i,m_i)} X_{n_i}T_rX^t_{m_i}T_q \right) = \prod_{j=1} \left(t - \prod_{(n_i,m_i)} b_j(y_{n_i})b_j(y_{m_i}) \right)^2 .$$

Comparing the coefficients of t^{2r-1}, we get that $\mathfrak{s} \circ \mathfrak{i}(P) = P$, for all $P \in k[\mathfrak{C}^d(\mathfrak{g}_1)]^{G_0}$. Hence $\mathfrak{s}$ is a section of $\mathfrak{i}$.

This concludes the proof of Theorem 1.1.

Remark 6.1. The above techniques do not work in the case where $(\mathfrak{g}, \mathfrak{g}_0)$ is equal to $(\mathfrak{so}_{2n}(k), \mathfrak{gl}_n(k))$. Note that the rank of the above symmetric pair is $[n/2]$ and the little Weyl group is isomorphic to the Weyl group of type B_n. Let $(x_1, x_2, \ldots, x_d)$ be the universal point of $\mathfrak{C}_d(\mathfrak{g}_1)(A)$, where $A = k[\mathfrak{C}_d(\mathfrak{g}_1)]$. For any k-algebra R, let $p : R[X_1, X_2, \ldots, X_d]^+ \to \mathfrak{gl}_{2n}(A \otimes R)$ be the map $X_i \mapsto x_i$. The image of the above map lands in $\mathfrak{g}_0(A \otimes R) = \mathfrak{gl}_n(A \otimes R)$. Since the determinant has degree n, we do not know any multiplicative map on the image of p of degree $[n/2]$.

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