On the duality involution for p-adic General Spin Groups

Amiya Kumar Mondal and Santosh Nadimpalli

December 30, 2022

Abstract

Let Gspin_m be the quasi-split general spin group over a p-adic field F for a positive integer m. For a certain class of quasi-split reductive groups, including the general spin groups, a specific involution ι, called the duality involution, is constructed in [Pra19]–generalising the MVW involution on the classical groups. In this article, for any irreducible admissible representation (π, V) of $\mathrm{Gspin}_m(F)$, we show that $\pi^\vee \simeq \pi^\iota$. Here $\pi^\vee$ is the contragredient of π and π^ι is the composite of π with ι. We also prove the analogue of this result for the split general spin groups over $\mathbb{F}_q((t))$.

1 Introduction

Let F be a non-Archimedean local field with the ring of integers $\mathfrak{o}_F$. Let $\mathfrak{p}_F$ be the maximal ideal of $\mathfrak{o}_F$, and let k_F be the residue field of F. Let $\mathbf{H}$ be a connected reductive algebraic group defined over F, and let H be the group of F-rational points of $\mathbf{H}$. Let (π, V) be an irreducible admissible complex representation of H with $(\pi^\vee, V^\vee)$, the contragredient (or smooth dual) of (π, V).

Though the contragredient representation $\pi^\vee$ of an irreducible admissible representation π is defined on the space $V^\vee$, the space of smooth linear functionals, one can often realize this representation on the space V itself. This explicit realisation of the contragredient representations goes back to the work of Gelfand and Kazhdan [GK75]. We recall their result for general linear groups. For $g \in \mathrm{GL}_n(F)$, let ${}^tg$ denote the transpose of g. Define an involution $g \mapsto g^\delta$ on $\mathrm{GL}_n(F)$ where $g^\delta := {}^tg^{-1}$ (here δ is merely a symbol to keep consistency with the later discussion). Let (π, V) be an irreducible admissible representation of $\mathrm{GL}_n(F)$. Define the representation (π^δ, V) (on the same space as of π) by

$$\pi^\delta(g) := \pi(g^\delta).$$

Gelfand and Kazhdan proved that the representations π^δ and $\pi^\vee$ are isomorphic. This theorem is referred to as the duality theorem for general linear groups.

An extension of the duality theorem for other groups depends on a suitable choice of the involution $g \mapsto g^\delta$. The duality theorems have been proved for classical groups and metaplectic groups by Moeglin-Vignéras-Waldspurger in [MgVW87]. They proved that the involution $g \mapsto g^\delta$ is given by conjugation of g by an element δ from outside the given group, i.e., it is an outer involution. The involution $g \mapsto g^\delta$ is referred to as MVW involution. The duality theorems have been proved for similitude classical groups by Roche-Vinroot [RV18, Theorem B]. It is worth mentioning in passing that the existence of MVW involutions have also been proved for real quaternionic classical groups in [LST14].

In this article, we focus on the general spin groups $\mathbf{G}' := \mathrm{GSpin}_m$ where m is a positive integer. These groups are certain algebraic covers of special orthogonal groups $\mathbf{G} := \mathrm{SO}_m$. The notations $\mathbf{G}$ and $\mathbf{G}'$ will be fixed throughout the paper. Before we state the main theorem of this paper, let us recall the

2010 Mathematics Subject Classification. Primary 22E50; Secondary 11F70
Keywords: Representation theory of p-adic groups, general spin groups, MVW involution.

following. For a certain class of quasi-split reductive groups $\mathbf{H}$ defined over F (including the general spin groups), an involution $\iota_{\mathbf{H}}$ (called the duality involution) has been constructed in [Pra19] as an element of $\mathrm{Aut}(\mathbf{H})(F)$ which is given by the composite of two involutions one fixing a pinning $(\mathbf{H}, \mathbf{B}, \mathbf{S}, \{X_\alpha\})$ and the other automorphism which takes each X_α to $-X_\alpha$ for all X_α occurring in the pinning. Note that $\mathbf{S}$ is a maximal torus of $\mathbf{H}$ contained in $\mathbf{B}$. The construction of this involution $\iota_{\mathbf{H}}$ is recalled in Section 2.2. For an irreducible admissible representation (π, V) of the group $G' := \mathrm{GSpin}_m(F)$, we define the representation $(\pi^{\iota_{\mathbf{G}'}}, V)$ of G' by

$$\pi^{\iota_{\mathbf{G}'}}(g) := \pi(\iota_{\mathbf{G}'}(g)) \; \forall \, g \in G'.$$

We prove the following theorem on the contragredients of irreducible admissible representations of a quasi-split general spin groups over p-adic fields:

Theorem 1.1. *Let F be a p-adic field and let $\mathbf{G}' := \mathrm{GSpin}_m$ be a quasi-split general spin group defined over F. Let (π, V) be an irreducible admissible representation of the quasi-split general spin group G'. The involution $\iota_{\mathbf{G}'}$ (called the duality involution) constructed by [Pra19] has the property that $\pi^{\iota_{\mathbf{G}'}} \cong \pi^\vee$.*

Since general spin groups are algebraic covers of special orthogonal groups, we make use of the MVW involution on special orthogonal groups (which is known from [MgVW87]) to prove our theorem. Using Theorem 1.1 and the 'close field' arguments (see [Gan15, §2.3]), we prove the duality theorem for split general spin groups over function fields also.

Theorem 1.2. *Let F be a finite extension of the field of Laurent series $\mathbb{F}_q((t))$ and let $\mathbf{G}' := \mathrm{GSpin}_m$ be a split general spin group defined over F. Let (π, V) be an irreducible admissible representation of G'. The involution $\iota_{\mathbf{G}'}$ (called the duality involution) constructed by [Pra19] has the property that $\pi^{\iota_{\mathbf{G}'}} \cong \pi^\vee$.*

For any p-adic general spin group G', it is possible to extend the MVW-involution δ on G to an involution θ on G' such that the dual of an irreducible admissible representation (π, V) of G' is isomorphic to the twisted representation π^θ (see Remark 2). Let us briefly describe the contents of this paper. In Section 2, we describe the general spin groups and the basics of duality involution. In Section 3, we recall the construction of duality involution, following [Pra19], of $\mathrm{GSpin}_m(F)$ and show that this is compatible with the MVW involution of $\mathrm{SO}_m(F)$. Finally in Section 4, we prove Theorems 1.1 and 1.2. In particular, Theorem 1.1 is proved in Section 4.1 by using the duality involution constructed in Section 3. Theorem 1.2 is proved in Section 4.2.

2 Preliminaries

Let F be a non-Archimedean local field with its norm $| \, |_\nu$. For any affine algebraic variety V defined over F, the set of F-rational points $V(F)$ is equipped with the natural topology induced from the norm $| \, |_\nu$ on F, and we refer to this topology as the ν-adic topology on $V(F)$ (see [PR83, Section 3.1]). For any linear algebraic group $\mathbf{H}$ over F, the ν-adic topology on $\mathbf{H}(F)$ makes it an l-group in the sense of [BZ76].

2.1 The General spin group

Let W be an m dimensional vector space over F with $q : W \longrightarrow F$, a quadratic form on W. The pair (W, q) is called a quadratic space. Let $T(W)$ be the tensor algebra of W and let $I(W, q)$ be the 2-sided ideal of $T(W)$ generated by the set of elements

$$\{w \otimes w - q(w) : \, w \in W\}.$$

Definition 2.1. *The Clifford algebra $C(W,q)$ (or $C(q)$ if W is clear from the context) associated to the quadratic space (W,q) is the F-algebra*

$$C(W,q) := T(W)/I(W,q).$$

Note that if w is replaced by $-w$, the definiton of the Clifford algebra $C(W,q)$ does not change (since $I(W,q)$ does not change). Let $\alpha : C(W,q) \longrightarrow C(W,q)$ denote the corresponding automorphism induced by the negative automorphism of the quadratic space (W,q).

Definition 2.2. *The Clifford group $\Gamma(W,q)$ is the group of units $u \in C(W,q)^\times$ such that $\alpha(u)Wu^{-1} = W$.*

Definition 2.2 says that $\Gamma(W,q)$ has a natural representation ρ on W given by

$$\rho(u)w = \alpha(u)wu^{-1} \text{ for } u \in \Gamma(W,q) \text{ and } w \in W.$$

Theorem 2.3. *[Knu88, Proposition 6, p. 63] The image of $\Gamma(W,q)$ under ρ is $\mathrm{O}_m(F)$ and the kernel of ρ is equal to the central subgroup $F^\times$. Hence $\Gamma(W,q)$ fits into the following short exact sequence:*

$$1 \longrightarrow \mathbb{G}_m \xrightarrow{\iota} \Gamma(W,q) \xrightarrow{p} \mathrm{O}_m \longrightarrow 1\ . \tag{2.1}$$

Now we define the main object of the paper. The *general spin group* $\mathrm{GSpin}_m(F)$ is defined by

$$\mathrm{GSpin}_m(F) := \Gamma(W,q) \cap C(W,q)_0,$$

where $C(W,q)_0$ is the even part of the Clifford algebra $C(W,q)$ (i.e., the image of the even part $T(W)_0$ of $T(W)$ in $C(W,q)$). Note that the same is called *special Clifford group* in [Knu88, Chapter 6, p. 62] and [Che97].

Proposition 2.4. *[Knu88, Proposition 6, p. 63] As an algebraic group, GSpin_m fits into the following short exact sequence:*

$$1 \longrightarrow \mathbb{G}_m \xrightarrow{\iota} \mathrm{GSpin}_m \xrightarrow{p} \mathrm{SO}_m \longrightarrow 1\ . \tag{2.2}$$

We recall in passing the following alternative description of the group GSpin_m. Let Spin_m denote the spin group which is an algebraic double cover of SO_m, i.e., we have a short exact sequence

$$1 \longrightarrow \mathbb{Z}_2 \xrightarrow{\iota} \mathrm{Spin}_m \xrightarrow{p} \mathrm{SO}_m \longrightarrow 1\ . \tag{2.3}$$

Following [Asg02, Definition 2.3] we define for $m \geq 3$,

$$\mathrm{GSpin}_m := \frac{\mathbb{G}_m \times \mathrm{Spin}_m}{\{(1,1),(-1,c)\}}, \tag{2.4}$$

where $\{1,c\}$ is the kernel of ι in (2.3). It can be shown that (2.2) follows from this definition of GSpin_m as well. We refer to [HS16, Chapter 4] and [Asg02, §2] for details on the structure of general spin groups.

2.2 The duality involution

Let $\mathbf{H}$ be a quasi-split reductive algebraic group defined over F and let $\mathbf{B}$ be a Borel subgroup of $\mathbf{H}$, defined over F, containing a maximal torus $\mathbf{S}$ also defined over F. Let $\Phi(\mathbf{H},\mathbf{S})$ be the set of roots of $\mathbf{H}$

with respect to $\mathbf{S}$. We denote by $\Delta := \Delta(\mathbf{H}, \mathbf{S})$, the set of simple roots of $\Phi(\mathbf{H}, \mathbf{S})$ with respect to $\mathbf{B}$. The tuple

$$\mathcal{P} := (\mathbf{H}, \mathbf{B}, \mathbf{S}, \{X_\alpha\}),$$

with $\{X_\alpha : \alpha \in \Delta(\mathbf{H}, \mathbf{S})\}$, is called a pinning on $\mathbf{H}$. Let $w_{\mathbf{H}}$ be a representative in the normaliser $N_{\mathbf{H}}(\mathbf{S})$ of the longest element of the Weyl group $N_{\mathbf{H}}(\mathbf{S})/\mathbf{S}$ of $\mathbf{H}$ which takes $\mathbf{B}$ to the opposite Borel $\mathbf{B}^-$. A *Chevalley involution* $c_{\mathbf{H},\mathcal{P}}$ associated with the pinning $\mathcal{P}$ is the unique element of $\mathrm{Aut}(\mathcal{P})$ such that $c_{\mathbf{H},\mathcal{P}}(s) = w_{\mathbf{H}} s^{-1} w_{\mathbf{H}}^{-1}$, for all $s \in \mathbf{S}$. The Chevalley involution belongs to the centre of the group $\mathrm{Aut}(\mathcal{P})$. Hence the Chevalley involution commutes with the F-rational structure on $\mathbf{H}$.

The *duality involution* $\iota_{\mathbf{H},\mathcal{P}}$ on $\mathbf{H}$ associated to the pinning $\mathcal{P} = (\mathbf{H}, \mathbf{B}, \mathbf{S}, \{X_\alpha\})$ is of the form

$$\iota_{\mathbf{H},\mathcal{P}} := \mathrm{Ad}(\iota_-) \cdot c_{\mathbf{H},\mathcal{P}},$$

where $\iota_- \in \mathbf{S}^{\mathrm{ad}}(F)$ is the unique element such that $\alpha(\iota_-) = -1$ for all $\alpha \in \Delta(\mathbf{H}, \mathbf{S})$. We refer to [Pra19, §3] for details.

The group $\mathbf{G}$ stands for the special orthogonal group associated with the pair (W, q) consisting of a vector space W of dimension m and q is a non-degenerate quadratic form on W. We also assume that $\mathbf{G}$ is quasi-split unless mentioned otherwise. Let $\mathcal{P} = (\mathbf{G}, \mathbf{B}, \mathbf{T}, \{X_\alpha\})$ be a pinning on $\mathbf{G}$ with the associated duality involution $\iota_{\mathbf{G},\mathcal{P}} = \mathrm{Ad}(\iota_-) \cdot c_{\mathbf{G},\mathcal{P}}$. By [Pra19, Proposition 1 and Example 2], up to inner conjugation by an element of G, the duality involution $\iota_{\mathbf{G},\mathcal{P}}$ is independent of the choice of the pinning $\mathcal{P}$. Hence from now onward, we denote the duality involution on $\mathbf{G}$ by $\iota_{\mathbf{G}}$. By [Pra19, Example 1, p. 5], we know that the duality involution $\iota_{\mathbf{G}}$ is the MVW involution for special orthogonal groups [MgVW87, Theorem II.I, p. 91].

3 Construction of the duality involution of general spin groups

Recall that we use the notation $\mathbf{G}'$ for the general spin group GSpin_m. In this section, we will construct a duality involution $\iota_{\mathbf{G}'}$ of $\mathbf{G}'$ which commutes with $\iota_{\mathbf{G}}$ of $\mathbf{G}$. Though the duality involution $\iota_{\mathbf{G}}$ is independent of the chosen pinning $\mathcal{P}$ on $\mathbf{G}$, one needs to fix a pinning $\mathcal{P}$ of $\mathbf{G}$ to construct the duality involution $\iota_{\mathbf{G},\mathcal{P}}$ of $\mathbf{G}$. We will construct a pinning $\mathcal{P}'$ of $\mathbf{G}'$ from the chosen pinning $\mathcal{P}$ of $\mathbf{G}$ to construct the duality involution $\iota_{\mathbf{G}',\mathcal{P}'}$ of $\mathbf{G}'$. Finally, we will prove that the duality involution $\iota_{\mathbf{G}',\mathcal{P}'}$ is independent of the pinning $\mathcal{P}'$, and hence the duality involution of $\mathbf{G}'$ will be denoted by $\iota_{\mathbf{G}'}$.

We first describe the Chevalley involution $c_{\mathbf{G}',\mathcal{P}'}$. Let us start with the pinning $\mathcal{P} = (\mathbf{G}, \mathbf{B}, \mathbf{T}, \{X_\alpha\})$ on $\mathbf{G}$. Choose a maximal torus $\mathbf{T}'$ in $\mathbf{G}'$ which surjects on to $\mathbf{T}$ under the short exact sequence (2.2). Let $\mathbf{B}'$ be a Borel subgroup in $\mathbf{G}'$ containing $\mathbf{T}'$. For a simple root $\alpha \in \Phi(\mathbf{G}, \mathbf{T})$, the root $\alpha' := \alpha \circ p$ is a simple root in $\Phi(\mathbf{G}', \mathbf{T}')$. The set of simple roots in $\mathbf{G}'$ is $\Delta' = \{\alpha' := \alpha \circ p : \alpha \in \Delta\}$. We get a homomorphism

$$p^* : \Phi(\mathbf{G}, \mathbf{T}) \longrightarrow \Phi(\mathbf{G}', \mathbf{T}')$$

given by $\alpha \mapsto \alpha'$. Note that the root subgroups corresponding to α and α' are isomorphic to $\mathbb{G}_a$ via the map p. Let $X_{\alpha'}$ be an element in the Lie algebra of G' which maps to X_α under the map induced by p on the Lie algebras of G' and G respectively. Hence, we have obtained a pinning $\mathcal{P}' = (\mathbf{G}', \mathbf{B}', \mathbf{T}', \{X_{\alpha'}\})$ on $\mathbf{G}'$. We now consider the unique Chevalley involution (up to conjugation) $c_{\mathbf{G}',\mathcal{P}'}$ associated to the pinning $\mathcal{P}'$ on $\mathbf{G}'$. We refer to [Pra19, §3] for details.

Let $\mathbf{Z} := Z(\mathbf{G})$ (resp. $\mathbf{Z}' := Z(\mathbf{G}')$) denote the centre of $\mathbf{G}$ (resp. $\mathbf{G}'$). From the definitions $\mathbf{T}^{\mathrm{ad}} := \mathbf{T}/\mathbf{Z}$ and $(\mathbf{T}')^{\mathrm{ad}} := \mathbf{T}'/\mathbf{Z}'$, we observe that the map p induces an isomorphism $(\mathbf{T}')^{\mathrm{ad}} \cong \mathbf{T}^{\mathrm{ad}}$. Let $\iota'_- \in (\mathbf{T}')^{\mathrm{ad}}(F)$ be the element which corresponds to $\iota_- \in \mathbf{T}^{\mathrm{ad}}(F)$ under the above isomorphism. Note that ι'_- is the unique element of $(\mathbf{T}')^{\mathrm{ad}}(F)$ such that $\alpha'(\iota'_-) = -1$ for all simple roots $\alpha' \in \Phi(\mathbf{G}', \mathbf{T}')$.

Finally we define the duality involution $\iota_{\mathbf{G}',\mathcal{P}'}$ as $\iota_{\mathbf{G}',\mathcal{P}'} := \mathrm{Ad}(\iota'_-) \cdot c_{\mathbf{G}',\mathcal{P}'}$ on $\mathbf{G}'$. Note that our construction of the duality involution $\iota_{\mathbf{G}'}$ apriori depends on the pinning $\mathcal{P}'$. In particular, it depends on the chosen maximal torus $\mathbf{T}'$ in $\mathbf{G}'$ to begin with. The following lemma shows that $\iota_{\mathbf{G}',\mathcal{P}'}$ is independent of the choice of the pinning $\mathcal{P}'$ on $\mathbf{G}'$.

Lemma 3.1. *The duality involution $\iota_{\mathbf{G}',\mathcal{P}'}$, up to conjugation with an element of G', is independent of the choice of the pinning $\mathcal{P}'$ on $\mathbf{G}'$.*

Proof of Lemma 3.1. Let $\mathbf{Z}' := Z(\mathbf{G}')$ denote the centre of $\mathbf{G}'$ with $Z' := \mathbf{Z}'(F)$, the group of F-rational points of $\mathbf{Z}'$. Let $G_F := \mathrm{Gal}(\overline{F}/F)$ denote the absolute Galois group of $\overline{F}/F$. We have a short exact sequence

$$1 \longrightarrow \mathbf{Z}' \xrightarrow{\ \iota\ } \mathbf{T}' \xrightarrow{\ p\ } (\mathbf{T}')^{\mathrm{ad}} \longrightarrow 1\ , \tag{3.1}$$

which gives the following long exact sequence

$$1 \longrightarrow Z' \xrightarrow{\ \iota\ } T' \xrightarrow{\ p\ } (T')^{\mathrm{ad}} \longrightarrow H^1(G_F, \mathbf{Z}') \xrightarrow{H^1(\iota)} H^1(G_F, \mathbf{T}')\ . \tag{3.2}$$

By [Pra19, Proposition 1, p. 6], it is enough to show that $2\mathrm{Ker}\,\{H^1(G_F, \mathbf{Z}') \longrightarrow H^1(G_F, \mathbf{T}')\} = 0$. We know the structure of $\mathbf{Z}'$ and $\mathbf{T}'$ from [AS06, Proposition 2.3, p. 143].

When m is odd, we know that $\mathbf{Z}' \cong \mathbb{G}_m$ and hence $H^1(G_F, \mathbf{Z}') = 0$ from Hilbert's Theorem 90 [Ser79, Proposition 2, p. 150]. When m is even note that $\mathbf{Z}'$ contains $\mathbb{G}_m$ with quotient isomorphic to $\mathbb{Z}_2$. Thus, we observe that $2H^1(G_F, \mathbf{Z}') = 0$ in this case. Hence the required condition is satisfied and the lemma is proved in this case.

□

Hence, from now onward, we denote the duality involution on $\mathbf{G}'$ by $\iota_{\mathbf{G}'}$ instead of $\iota_{\mathbf{G}',\mathcal{P}'}$. In the next lemma we prove that $\iota_{\mathbf{G}'}$ commutes with $\iota_{\mathbf{G}}$.

Lemma 3.2. *The duality involution $\iota_{\mathbf{G}'}$ of $\mathbf{G}'$ commutes with the duality involution $\iota_{\mathbf{G}}$ of $\mathbf{G}$, i.e., we have the following commutative diagram:*

$$\begin{array}{ccc} \mathbf{G}' & \xrightarrow{\ p\ } & \mathbf{G} \\ \downarrow{\scriptstyle \iota_{\mathbf{G}'}} & & \downarrow{\scriptstyle \iota_{\mathbf{G}}} \\ \mathbf{G}' & \xrightarrow[\ p\ ]{} & \mathbf{G}. \end{array} \tag{3.3}$$

Proof of Lemma 3.2. We look at the following short exact sequence defining the group $\mathbf{G}'$:

$$1 \longrightarrow \mathbb{G}_m \xrightarrow{\ \iota\ } \mathbf{G}' \xrightarrow{\ p\ } \mathbf{G} \longrightarrow 1\ .$$

Note that, it is enough to show that the maps $p \circ \iota_{\mathbf{G}'}$ and $\iota_{\mathbf{G}} \circ p$ factor through $\mathbb{G}_m$. Since $\iota_{\mathbf{G}'} = \mathrm{Ad}(\iota'_-) \cdot c_{\mathbf{G}',\mathcal{P}'}$, it is enough to prove that both the automorphisms $\mathrm{Ad}(\iota'_-)$ and $c_{\mathbf{G}',\mathcal{P}'}$ factor through $\mathbb{G}_m$. Since $\mathbb{G}_m \subseteq \mathbf{Z}'$ and $\mathrm{Ad}(\iota'_-)$ is an inner automorphism by $\iota'_- \in (\mathbf{T}')^{\mathrm{ad}}(F)$, it is clear that $\mathbb{G}_m$ is invariant under ι'_-.

Let $w_{\mathbf{G}'}$ be a representative in $N_{\mathbf{G}'}(\mathbf{T}')$ of the longest element in the Weyl group of $\mathbf{G}'$ (taking $\mathbf{B}'$ to the opposite Borel $(\mathbf{B}')^-$). The explicit construction of $c_{\mathbf{G}',\mathcal{P}'}$ as in [Pra19, Section 3], says that

$$c_{\mathbf{G}',\mathcal{P}'}(t) = w_{\mathbf{G}'} t^{-1} w_{\mathbf{G}'}^{-1}\ \forall t \in \mathbf{T}'.$$

Since $w_{\mathbf{G}'}$ acts trivially on $\mathbf{Z}'$, we have $c_{\mathbf{G}',\mathcal{P}'}(z) = z^{-1}\ \forall\, z \in \mathbf{Z}'$. Hence $\mathbb{G}_m$ is invariant under $c_{\mathbf{G}',\mathcal{P}'}$ as well. This proves the lemma. □

The next lemma is the central technical result in this paper. Before stating the lemma, let us recall the following. Let $\mathbf{H}$ be a connected reductive group defined over F. An element h of $\mathbf{H}$ is said to be *regular* if $\dim_F C_{\mathbf{H}}(h) = \operatorname{rank}\mathbf{H}$, where $C_{\mathbf{H}}(h)$ denotes the centralizer of h in $\mathbf{H}$. An element h of $\mathbf{H}$ is called *strongly regular* if $C_{\mathbf{H}}(h)$ is a maximal torus in $\mathbf{H}$. Such an element is regular and semisimple by definition. Let $\mathbf{H}_{sr}$ be the set of strongly regular elements of $\mathbf{H}$. Note that $\mathbf{H}_{sr}$ is a Zariski-open subset of $\mathbf{H}$ [Ste65, §2.15, p. 54]. Although the proof in [Ste65, §2.15, p. 54] is written for semisimple groups, the same proof works for the case of reductive groups. For any irreducible smooth variety V defined over F, and for any Zariski-open F-subvariety W of V, the set $W(F)$ is dense in $V(F)$ for the ν-adic topology on $V(F)$ induced from that of F (see [PR83, Lemma 3.2, p. 114]). Thus, we get that $\mathbf{H}_{sr}(F)$ is dense in $\mathbf{H}(F)$ for the ν-adic topology on $\mathbf{H}(F)$ induced from F.

We have the following lemma:

Lemma 3.3. *Let g be a strongly regular element in G'. Then $g^{\iota_{\mathbf{G}'}}$ and g^{-1} are conjugate in G'.*

Proof of Lemma 3.3. We first prove that g^{-1} and $g^{\iota_{\mathbf{G}'}}$ are conjugate in $\mathbf{G}'$ and then show that the conjugation actually takes place in G'.

Since g is a strongly regular semisimple element in G', the $\mathbf{G}'$-centralizer $C_{\mathbf{G}'}(g)$ of g is a maximal torus in $\mathbf{G}'$. Since any two maximal tori are conjugate in $\mathbf{G}'$ [Spr09, Theorem 6.4.1, p. 108], there exists $h \in \mathbf{G}'$ such that $hC_{\mathbf{G}'}(g)h^{-1} = \mathbf{T}'$.

Since $hgh^{-1} \in \mathbf{T}'$, we have $\iota_{\mathbf{G}'}(hgh^{-1}) = w_{\mathbf{G}'}(hg^{-1}h^{-1})w_{\mathbf{G}'}^{-1}$. On the other hand, $\iota_{\mathbf{G}'}(hgh^{-1}) = \iota_{\mathbf{G}'}(h)\iota_{\mathbf{G}'}(g)\iota_{\mathbf{G}'}(h)^{-1}$. This implies that

$$\iota_{\mathbf{G}'}(g) = \gamma g^{-1}\gamma^{-1},$$

where γ is equal to $\iota_{\mathbf{G}'}(h)^{-1}w_{\mathbf{G}'}h$.

It remains to show that the conjugation takes place in G' itself. Note that if γ conjugates g^{-1} and $g^{\iota_{\mathbf{G}'}}$, then so does γ^{σ} for $\sigma \in G_F := \mathrm{Gal}(\overline{\mathrm{F}}/\mathrm{F})$. This can be seen easily as g and $g^{\iota_{\mathbf{G}'}}$ are invariant under G_F. Let $\mathbf{T}_1'$ be the group $C_{\mathbf{G}'}(g)$ and let $\mathbf{T}_1$ be the image of $\mathbf{T}_1'$ under the map p. Note that $\gamma^{-1}\gamma^{\sigma} \in \mathbf{T}_1' \subseteq \mathbf{G}'$, and the cocycle map $f_\gamma : G_F \longrightarrow \mathbf{T}_1'$ given by $\sigma \mapsto \gamma^{-1}\gamma^{\sigma}$ gives a cohomology class in $H^1(G_F, \mathbf{T}_1')$. It is enough to prove that

$$f_\gamma \equiv 1 \text{ in } H^1(G_F, \mathbf{T}_1').$$

The short exact sequence (2.2) gives rise to the following long exact sequence:

$$\begin{aligned} 1 \longrightarrow F^* \xrightarrow{\ \iota\ } T_1' \xrightarrow{\ p\ } T_1 \xrightarrow{\ \delta^0\ } H^1(G_F, \mathbb{G}_m) \\ \xrightarrow{H^1(\iota)} H^1(G_F, \mathbf{T}_1') \xrightarrow{H^1(p)} H^1(G_F, \mathbf{T}_1)\cdots. \end{aligned} \tag{3.4}$$

We first note that

$$H^1(p) : H^1(G_F, \mathbf{T}_1') \longrightarrow H^1(G_F, \mathbf{T}_1) \tag{3.5}$$

is an embedding. This follows from the fact that $H^1(G_F, \mathbb{G}_m) = \{1\}$ by Hilbert's Theorem 90.

Now we observe that

$$\begin{aligned}\gamma g^{-1}\gamma^{-1} =& g^{\iota_{\mathbf{G}'}}\\ p(\gamma)p(g)^{-1}p(\gamma)^{-1} =& p(g^{\iota_{\mathbf{G}'}})\\ p(\gamma)p(g)^{-1}p(\gamma)^{-1} =& [p(g)]^{\iota_{\mathbf{G}}} \text{ (Lemma 3.3).}\end{aligned}$$

Hence $p(\gamma)$ conjugates $p(g)^{-1}$ and $[p(g)]^{\iota_{\mathbf{G}}}$ in $\mathbf{G}$. Note that $p(g)$ is a strongly regular element of G. Hence as before, it follows that $f_{p(\gamma)} \in H^1(G_F, \mathbf{T}_1)$. Since $\iota_{\mathbf{G}}$ is the MVW involution for $\mathbf{G}$, it follows that $f_{p(\gamma)} \equiv 1$ in $H^1(G_F, \mathbf{T}_1)$. Since the image of $f_\gamma \in H^1(G_F, \mathbf{T}_1')$ in $H^1(G_F, \mathbf{T}_1)$ is $f_{p(\gamma)}$ and $H^1(p): H^1(G_F, \mathbf{T}_1') \longrightarrow H^1(G_F, \mathbf{T}_1)$ is an embedding, we conclude that $f_\gamma \equiv 1$ in $H^1(G_F, \mathbf{T}_1')$. Thus, there exists a $t \in \mathbf{T}_1'$ such that $f_\gamma(\sigma) = t^\sigma t^{-1}$, for all $\sigma \in G_F$. Note that $\gamma t^{-1} = \gamma^\sigma (t^{-1})^\sigma$, for all $\sigma \in G_F$. Hence $\gamma t^{-1} \in G'$ and $\gamma t^{-1} g^{-1} t \gamma^{-1} = \iota_{\mathbf{G}'}(g)$. □

Remark 1. *Since $\iota_{\mathbf{G}'}(t) = t^{-1}$, for all $t \in Z(\mathbf{G}')$, the duality involution $\iota_{\mathbf{G}'}$ is a non-trivial automorphism of $\mathbf{G}'$ even if $\iota_{\mathbf{G}}$ is trivial. For instance when $\mathbf{G}$ is the group SO_{2n+1}, the involution $\iota_{\mathbf{G}}$ is the trivial automorphism.*

4 Duality involution and the contragredient

We prove Theorem 1.1 and Theorem 1.2 in this section. We prove Theorem 1.1 in Section 4.1. In Section 4.2, we supply the proof of Theorem 1.2.

4.1 Proof of Theorem 1.1

Let (π, V) be an irreducible admissible representation of the quasi-split general spin group $G' := \mathrm{GSpin}_m(F)$ for a p-adic field F. For the duality involution $\iota_{\mathbf{G}'}$ constructed in Section 3, define the representation $\pi^{\iota_{\mathbf{G}'}}$ of G' by

$$\pi^{\iota_{\mathbf{G}'}}(g) := \pi(g^{\iota_{\mathbf{G}'}})\ \forall\, g \in G',\quad g^{\iota_{\mathbf{G}'}} := \iota_{\mathbf{G}'}(g).$$

Let $(\pi^\vee, V^\vee)$ denote the contragredient representation of (π, V).

Let χ_π denote the Harish-Chandra trace character of π (see [HC70, Theorem 5, p. 99]). For strongly regular semisimple elements $g \in G'$, using Lemma 3.3 we get

$$\chi_{\pi^{\iota_{\mathbf{G}'}}}(g) = \chi_\pi(g^{\iota_{\mathbf{G}'}}) = \chi_\pi(g^{-1}) = \chi_{\pi^\vee}(g).$$

Let the distribution Θ_π on G' defined by $\Theta_\pi(f) := \mathrm{tr}(\pi(f))$ for $f \in C_c^\infty(G')$, denote the character of π (see [BZ76, §2.17, p. 21]). Since the set of all strongly regular elements of G' is dense in G', we get that

$$\Theta_{\pi^{\iota_{\mathbf{G}'}}}(f) = \Theta_{\pi^\vee}(f),\ \forall f \in C_c^\infty(G').$$

Hence the theorem is proved in this case by [BZ76, Corollary 2.20, p. 21].

Remark 2. *Let $\widetilde{\delta}$ be the involution on Spin_m which extends the MVW involution δ on $\mathbf{G} = \mathrm{SO}_m$. The automorphism $(t, g) \mapsto (t^{-1}, \widetilde{\delta}(g))$ of the group $\mathbb{G}_m \times \mathrm{Spin}_m$ induces an automorphism θ on $\mathbf{G}' = \mathrm{GSpin}_m$ (see (2.4)). As suggested in [Pra19, Question 1, p. 7], we can prove, even when $\mathbf{G}'$ is not quasi-split, that $\pi^\theta \simeq \pi^\vee$, for all irreducible smooth representations π of G'. Towards this direction, for $\mathbf{G}' := \mathrm{GSpin}_m$ (need not be quasi-split), note that Lemmas 3.2 and 3.3 hold for the automorphism θ. Hence as in Theorem 1.1, we see that $\pi^\theta \cong \pi^\vee$, for all irreducible smooth representations π of G'.*

4.2 Proof of Theorem 1.2

Now let F be a finite extension of the field of Laurent series over finite fields $\mathbb{F}_q((t))$. We prove Theorem 1.2 in this case for split general spin groups. In fact, we prove a more general statement. Let $\mathbf{H}$ be any split reductive group defined over $\mathbb{Z}$. We prove that if Theorem 1.1 holds for $\mathbf{H}$ defined over p-adic fields, then it holds for $\mathbf{H}$ defined over function fields using 'close field' arguments (see [Gan15, §2.3]). In particular the theorem then holds for split general spin groups.

Let F' be a finite extension of $\mathbb{Q}_p$ such that F and F' are m-close, i.e., there exists an isomorphism of rings:

$$\Lambda : \mathfrak{o}_F/\mathfrak{p}_F^m \simeq \mathfrak{o}_{F'}/\mathfrak{p}_{F'}^m.$$

Let $\mathbf{H}$ be a split reductive group defined over $\mathbb{Z}$. We choose a pinning $(\mathbf{H}, \mathbf{B}, \mathbf{S}, \{X_\alpha\})$ for the group $\mathbf{H}$. Let K_m and K'_m be the m-th congruence subgroups of $\mathbf{H}(\mathfrak{o}_F)$ and $\mathbf{H}(\mathfrak{o}_{F'})$ respectively. There exists an isomorphism of Hecke algebras

$$\phi : \mathcal{H}(\mathbf{H}(F), K_m) \to \mathcal{H}(\mathbf{H}(F'), K'_m)$$

(we will recall the precise nature of this isomorphism in the following paragraph). Assume that $\iota_{\mathbf{H}}$ and $\iota'_{\mathbf{H}}$ be the duality involutions on $\mathbf{H}/F$ and $\mathbf{H}/F'$ respectively, with respect to the chosen pinning. Note that $\iota_{\mathbf{H}}$ and $\iota'_{\mathbf{H}}$ preserve the groups K_m and K'_m respectively. Thus, we obtain involutions on the Hecke algebras

$$\iota_{\mathbf{H}} : \mathcal{H}(\mathbf{H}(F), K_m) \to \mathcal{H}(\mathbf{H}(F), K_m)$$

and

$$\iota'_{\mathbf{H}} : \mathcal{H}(\mathbf{H}(F'), K'_m) \to \mathcal{H}(\mathbf{H}(F'), K'_m).$$

Note that we use the same notations $\iota_{\mathbf{H}}$ and $\iota'_{\mathbf{H}}$ for the induced involutions on the Hecke algebras.

Let Φ be the set of roots of $\mathbf{H}$ with respect to the torus $\mathbf{S}$ and let $\mathbf{B}$ be a Borel subgroup containing $\mathbf{S}$. We denote by Δ the set of simple roots in Φ with respect to $\mathbf{B}$. Let $\lambda \in X_*(\mathbf{S})_-$ be the set of elements

$$\{\lambda \in X_*(T) : \langle \lambda, \alpha \rangle \leq 0, \ \alpha \in \Delta\}.$$

Let π_F be a uniformizer of F and let $\pi_{F'}$ be a uniformizer of F' such that

$$\Lambda(\pi_F + \mathfrak{p}_F^m) = \pi_{F'} + \mathfrak{p}_{F'}^m.$$

Let $\pi_\lambda \in \mathbf{S}(F)$ (resp. π'_λ) be the element $\lambda(\pi_F)$ (resp. $\lambda(\pi_{F'})$). Note that

$$\mathbf{H}(F) = \coprod_{\lambda \in X_*(\mathbf{S})_-} \mathbf{H}(\mathfrak{o}_F)\pi_\lambda \mathbf{H}(\mathfrak{o}_F).$$

The set

$$C_\lambda := \{K_m h K_m : K_m h K_m \subseteq \mathbf{H}(\mathfrak{o}_F)\pi_\lambda \mathbf{H}(\mathfrak{o}_F)\} \tag{4.1}$$

is a homogenous space under

$$\mathbf{H}(\mathfrak{o}_F/\mathfrak{p}_F^m) \times \mathbf{H}(\mathfrak{o}_F/\mathfrak{p}_F^m)$$

and let Γ_λ be the stabilizer of $K_m \pi_\lambda K_m$ under the action of the above group. Let Γ'_λ and C'_λ be the corresponding objects associated with F'. Kazhdan showed that the map Λ induces an isomorphism of Γ_λ and Γ'_λ. Thus, we get a correspondence $K \mapsto K'$ between elements of C_λ and C'_λ. For any subset K of $\mathbf{H}(F)$, we denote by t_K the characteristic function of K and $t_{K'}$ to be the characteristic function of the corresponding K' in $\mathbf{H}(F')$. The algebra $\mathcal{H}(\mathbf{H}(F), K_m)$ is spanned by the set of elements $\{t_K : K \in C_\lambda, \lambda \in X_*(\mathbf{S})_-\}$. The Kazhdan homomorphism ϕ sends the element t_K to the element $t_{K'}$. Thus, we get that Kazhdan isomorphism commutes with the duality involutions $\iota_{\mathbf{H}}$ and $\iota'_{\mathbf{H}}$ constructed by [Pra19], in the sense that:

$$\phi \circ \iota_{\mathbf{H}} = \iota'_{\mathbf{H}} \circ \phi. \tag{4.2}$$

Let (π, V) be an irreducible smooth representation of $\mathbf{H}(F)$ such that $V^{K_m} \neq 0$. Using the Kazhdan isomorphism we can define a $\mathcal{H}(\mathbf{H}(F'), K'_m)$-module structure on V^{K_m}, to be denoted by $\phi_*(V^{K_m})$ by setting

$$\phi_*(f)v = \pi(\phi^{-1}(f))v, v \in V^{K_m}, f \in \mathcal{H}(\mathbf{H}(F'), K'_m).$$

We note that

$$[\phi_*(V^{K_m})]^\vee \simeq [\phi_*(V^{K_m})]^{\iota'_{\mathbf{H}}}.$$

Note that $[\phi_*(V^{K_m})]^\vee$ is isomorphic to $\phi_*((V^\vee)^{K_m})$ and $[\phi_*(V^{K_m})]^{\iota'_{\mathbf{H}}}$ is isomorphic to $\phi_*((V^{K_m})^{\iota_{\mathbf{H}}})$. Thus, we get that

$$(V^{K_m})^{\iota_{\mathbf{H}}} \simeq (V^\vee)^{K_m}.$$

This shows that the representations $\pi^\vee$ and $\pi^{\iota_{\mathbf{H}}}$ are isomorphic as $\mathbf{H}(F)$-smooth representations. Taking $\mathbf{G}' = \mathbf{H}$, we complete the proof of Theorem 1.2.

Remark 3. *The duality theorem for quasi-split (and non-split) general spin groups over function fields is yet to be explored. One can ask for the duality theorem for general spin groups over finite fields as well. These questions need possibly more attention and we hope to address these questions in future.*

Acknowledgement

The authors would like to thank Prof. Eyal Kaplan for suggesting the problem. The first named author was supported by the ISRAEL SCIENCE FOUNDATION grant No. 421/17 (Mondal). The authors would like to thank the anonymous referee for the comments in the earlier versions of the paper, in particular for Remark 2.

References

[AS06] Mahdi Asgari and Freydoon Shahidi, *Generic transfer from* GSp(4) *to* GL(4), Compos. Math. **142** (2006), no. 3, 541–550. MR 2231191

[Asg02] Mahdi Asgari, *Local L-functions for split spinor groups*, Canad. J. Math. **54** (2002), no. 4, 673–693. MR 1913914

[BZ76] I. N. Bernšteĭn and A. V. Zelevinskiĭ, *Representations of the group $GL(n, F)$, where F is a local non-Archimedean field*, Uspehi Mat. Nauk **31** (1976), no. 3(189), 5–70. MR 0425030

[Che97] Claude Chevalley, *The algebraic theory of spinors and Clifford algebras*, Springer-Verlag, Berlin, 1997, Collected works. Vol. 2, Edited and with a foreword by Pierre Cartier and Catherine Chevalley, With a postface by J.-P. Bourguignon. MR 1636473

[Gan15] Radhika Ganapathy, *The local Langlands correspondence for* GSp_4 *over local function fields*, Amer. J. Math. **137** (2015), no. 6, 1441–1534. MR 3432266

[GK75] I. M. Gelfand and D. A. Kajdan, *Representations of the group* $\mathrm{GL}(n, K)$ *where* K *is a local field*, Lie groups and their representations (Proc. Summer School, Bolyai János Math. Soc., Budapest, 1971), 1975, pp. 95–118. MR 0404534

[HC70] Harish-Chandra, *Harmonic analysis on reductive p-adic groups*, Lecture Notes in Mathematics, Vol. 162, Springer-Verlag, Berlin-New York, 1970, Notes by G. van Dijk. MR 0414797

[HS16] Joseph Hundley and Eitan Sayag, *Descent construction for GSpin groups*, Mem. Amer. Math. Soc. **243** (2016), no. 1148, v+124. MR 3517154

[Knu88] Max-Albert Knus, *Quadratic forms, Clifford algebras and spinors*, Seminários de Matemática [Seminars in Mathematics], vol. 1, Universidade Estadual de Campinas, Instituto de Matemática, Estatística e Ciência da Computação, Campinas, 1988. MR 1099376

[LST14] Yanan Lin, Binyong Sun, and Shaobin Tan, *MVW-extensions of quaternionic classical groups*, Math. Z. **277** (2014), no. 1-2, 81–89. MR 3205764

[MgVW87] Colette Mœ glin, Marie-France Vignéras, and Jean-Loup Waldspurger, *Correspondances de Howe sur un corps p-adique*, Lecture Notes in Mathematics, vol. 1291, Springer-Verlag, Berlin, 1987. MR 1041060

[PR83] V. P. Platonov and S. A. Rapinchuk, *Algebraic groups*, Algebra. Topology. Geometry, Vol. 21, Itogi Nauki i Tekhniki, Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1983, pp. 80–134. MR 724615

[Pra19] Dipendra Prasad, *Generalizing the MVW involution, and the contragredient*, Trans. Amer. Math. Soc. **372** (2019), no. 1, 615–633. MR 3968781

[RV18] Alan Roche and C. Ryan Vinroot, *A factorization result for classical and similitude groups*, Canad. Math. Bull. **61** (2018), no. 1, 174–190. MR 3746483

[Ser79] Jean-Pierre Serre, *Local fields*, Graduate Texts in Mathematics, vol. 67, Springer-Verlag, New York-Berlin, 1979, Translated from the French by Marvin Jay Greenberg. MR 554237

[Spr09] T. A. Springer, *Linear algebraic groups*, second ed., Modern Birkhäuser Classics, Birkhäuser Boston, Inc., Boston, MA, 2009. MR 2458469

[Ste65] Robert Steinberg, *Regular elements of semisimple algebraic groups*, Inst. Hautes Études Sci. Publ. Math. (1965), no. 25, 49–80. MR 0180554

[Ste74] ———, *Conjugacy classes in algebraic groups*, Lecture Notes in Mathematics, Vol. 366, Springer-Verlag, Berlin-New York, 1974, Notes by Vinay V. Deodhar. MR 0352279