MTH201 SUMMER TERM 2021; ENDSEM

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1. **3 points+7 points**
Let F be a field, and let n, m be two positive integers. We define

$$r_F(n,m) = \max\{\text{rank}(A) : A \in M_{n\times m}(F) : A^t A = 0\}.$$

Here, A^t is the transpose of the matrix A. Determine the functions $r_{\mathbb{R}}(m,n)$ and $r_{\mathbb{C}}(m,n)$.

2. **10 points**
Let n be a positive integer, and let $A, B \in \mathrm{M}_{n\times n}(\mathbb{C})$ be non-zero matrices such that $AB - BA = A + B$, then show that there exists an invertible matrix P such that PAP^{-1} and PBP^{-1} are both upper triangular matrices.

3. **4 points+3 points**
Let $A \in M_{n\times n}(F)$, and let $T_A : M_{n\times n}(F) \to M_{n\times n}(F)$ be the linear transformation $X \mapsto AXA$, for $X \in M_{n\times n}(F)$. What is the rank of T_A? Show that there exists a matrix B such that $ABA = A$.

4. **7 points +8 points**
Given any finite subset S of a finite dimensional $\mathbb{Q}$-vector space V, show that there exists a linear functional $l : V \to \mathbb{Q}$ such that $l(x) \neq 0$, for any $x \in S$. Now use this to show the following: Let n be a positive integer and, let $X = \{x_1, x_2, \ldots, x_{2n-1}\}$ be a subset of $\mathbb{R}\backslash\mathbb{Q}$ with $|X| = 2n-1$. Prove that there exists $Y = \{y_1, y_2, \ldots, y_n\}$ a subset of X with $|Y| = n$ such that for all vectors $(a_1, a_2, \ldots, a_n) \in \mathbb{Q}^n$ with $a_1 \geq 0, a_2 \geq 0, \ldots, a_n \geq 0$ and $\sum_{i=1}^n a_i > 0$, the number $\sum_{i=1}^n a_n y_n \in \mathbb{R}\backslash\mathbb{Q}$.

5. **5 points+5 points**
Let $V_n \subset \mathbb{C}[X,Y]$ be the vector space of polynomials in two variables X, Y of degree less than or equal to n. Let $T : V_n \to V_n$ be the operator $f \mapsto Y\dfrac{\partial f}{\partial X}$. What are the invariant subspaces of T. What are the invariant subspaces of $S : V_n \to V_n$ given by $f \mapsto X\dfrac{\partial f}{\partial Y} - Y\dfrac{\partial f}{\partial X}$

6. **8 points**
Let $B : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a positive definite bilinear form. Let $S \subseteq \mathbb{R}^n$ be a set of vectors such that $B(v,w) < 0$, for any $v, w \in S$ with $v \neq w$. Is S finite? and if S is a finite set determine the maximum possible size of S.

or

Let $p(t) \in \mathbb{R}[t]$ be a monic polynomial in one variable t. Does there exists a matrix $M \in M_{n\times n}(\mathbb{R})$ such that $\det(M - t\,\mathrm{id}) = p(t)$