MTH641A CLASSICAL GROUPS AND THEIR LIE ALGEBRAS

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1. Lecture 1: Classical groups and their Lie algebras

1.1. Let k be a field and let V be a finite dimensional vector space over k. The group of invertible k-linear ransformations is denoted by $\mathrm{GL}_k(V)$. For any bilinear form $B : V \times V \to k$, we denote by $\mathrm{O}(V, k)$, the subgroup of $\mathrm{GL}_k(V)$ which preserve the form B, i.e., $T \in \mathrm{GL}_k(V)$ such that

$$B(Tv, Tw) = B(v, w),\ v, w \in V.$$

The group $\mathrm{SO}_k(V)$ is the subgroup of $\mathrm{O}_k(V)$ defined as the kernel of the determinant map. Let K/k be a quadraic extension and let σ be a non-trivial element of $\mathrm{Gal}(K/k)$. Let W be a K vector space. A hermitian form $h : W \times W \to K$ on W is a k-bilinear and

$$h(\alpha v, \beta w) = \alpha\sigma(\beta)h(v, w),\ v, w \in W.$$

and

$$h(v, w) = \overline{h(w, v)},\ v, w \in W.$$

The group $\mathrm{U}(W, h)$ is the subgroup of $\mathrm{GL}_K(W)$ consisting of all K-linear transformations of W such that

$$h(T(v), T(w)) = h(v, w),\ v, w \in W.$$

When k is $\mathbb{C}$ or $\mathbb{R}$ the groups $\mathrm{GL}_k(V)$, $\mathrm{SL}_k(V)$, $\mathrm{O}(V, h)$, $\mathrm{SO}(V, B)$ and $\mathrm{U}(W, h)$ are some of the fundamental examples of *transformation groups.* These are the groups which underline various geometries. This couse will initially deal with the infinitisimal structure of these groups called the Lie groups. However, we will first strudy the structure of these groups when $k = \mathbb{R}$ or $k = \mathbb{C}$.

1.2. **Lie algebras.** . The Lie algebra is motivated by the neighbourhood of the group of linear transformation groups. Note that the groups defined in the above paragraph are solutions to some equations with coefficients in k, i.e.,

$$G = \{x \in k^{n^2} : f_1(x) = f_2(x) = \cdots f_k(x) = 0\}$$

for some $f_i \in k[x_{ij} : 1 \leq i, j \leq n]$ and $G(A)$ is a subgroup of $\mathrm{GL}_n(A)$ for all k-algebras A. Now consider the ring $k[\epsilon]$, where $\epsilon^2 = 0$. Let $G(k[\epsilon])$ be the set of $k[\epsilon]$ valued solutions to the equation $f_1, f_2, \ldots, f_k$. Since the multiplication law on G is a polynomial (with coefficients in k) in coordinates $\{x_{ij}\}$, we get that $G(k[\epsilon])$ is a group and moreover we have a group homomorphism

$$G(k[\epsilon]) \xrightarrow{\pi} G(k).$$

The kernel of π consists of matricies of the form $\mathrm{id} + \epsilon M$, where M varies over a vector k-vector space denoted by $\mathfrak{g}$. Note that the map

$$G(k[\epsilon_1]) \times G(k[\epsilon_2]) \to G(k[\epsilon_1, \epsilon_2]);\ (g_1, g_2) \mapsto g_1 g_2 g_1^{-1} g_2^{-1}$$

around id is given by

$$(1 + \epsilon_1 M_1)(1 + \epsilon_2 M_2)(1 - \epsilon_1 M_1)(1 - \epsilon_2 M_2) = 1 + \epsilon_1\epsilon_2(M_1M_2 - M_2M_1)$$

Note that $G(k[\epsilon_1\epsilon_2])$ is a subgroup of $G(k[\epsilon_1, \epsilon_2])$. Thus, we get that $M_1M_2 - M_2M_1$ belongs to $\mathfrak{g}$. Thus, the space $\mathfrak{g}$ is closed under the operation $(M_1, M_2) \mapsto M_1M_2 - M_2M_1$. The abstract definition of this structure is called a Lie algebra:

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Definition 1.1. A k-vector space $\mathfrak{g}$ equipped with an alternating bilinear form

$$[\,]:\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$$

is called a Lie algebra if it satisfies the condition

$$[X[YZ]]+[Z[XY]]]+[Y[ZX]]=0. \tag{1.1}$$

The definitions of a Lie subalgebra is likewise defined.

Exercise 1.2.1.
(1) Show that the Lie algebra associated with SL_n is given by trace-zero matricies (this is denoted by $\mathfrak{sl}_n$).
(2) Let B be a symmetric or an alternating bilinear form on a k-vector space V, and let $\mathfrak{g}$ be the space

$$\{T\in \mathrm{End}_k(V): B(Tv,w)+B(v,Tw)=0.$$

Show that $\mathfrak{g}$ under the operation $[T_1,T_2]=T_1T_2-T_2T_1$ is a Lie algebra.
(3) You may replace B by a hermitian form on W and $\mathfrak{g}$ consists of K-linear transformations with

$$h(Tv,w)+h(v,Tw)=0.$$

The algebra $\mathfrak{g}$ is a Lie subalgebra of $\mathrm{End}_k(V)$.
(4) Let $(\mathfrak{g},[\,])$ be the pair $(\mathbb{R}^3,\times)$. Show that $\mathfrak{g}$ is a Lie algebra. Can you get an isomorphism of this Lie algebra with the types studied in the previous examples.
(5) Show that any two dimensional Lie algebra is abelian, i.e., the bracket operation is the zero operation.

We will later prove that a finite dimensional Lie algebra over an algebraically closed field k is a Lie subalgebra of $\mathfrak{gl}_k(V)$ and that any Lie subalgebra of $\mathfrak{gl}_k(V)$ arrises as a Lie algebra of an algebraic group. The simplest groups are abelian groups, then we study nilpotent followed by solvable and then simple groups. Lie algebras being associated with groups there are analogues of these notions in Lie algebras as well.

Definition 1.2. *A subalgebra $\mathfrak{g}_0$ of $\mathfrak{g}$ is called an ideal of $\mathfrak{g}$ if for all $X\in\mathfrak{g}$ and $Y\in\mathfrak{g}_0$ we have $\mathfrak{g}_0$.*

Naturally kernerls of Lie algebra homomorphisms is an ideal and vice-versa, the first isomorphism theorem.

Exercise 1.2.2. *What are the ideals of $\mathfrak{sl}_n$ and the ideals of Lie algebra associated with a non-degenerate symmetric or alternating bilinear form B?*

Exercise 1.2.3. A derivation of a Lie algebra is a map $\delta:\mathfrak{g}\to\mathfrak{g}$ such that $\delta([X,Y])=[\delta(X)\ Y]+[X,\delta(Y)]$. Show that the space of derivations of a Lie algebra $\mathfrak{g}$ is a Lie algebra under composition. Let $\mathrm{Der}(\mathfrak{g})$ be the space of Lie derivations of $\mathfrak{g}$. Let $\delta_X:\mathfrak{g}\to\mathfrak{g}$ be the map $Y\mapsto[X,Y]$. Show that the map $\delta:\mathfrak{g}\to\mathrm{Der}(\mathfrak{g})$ given by $X\mapsto\delta_X$ is a map of Lie algebras. The kernel of the map δ is called the centre of the Lie algerba $\mathfrak{g}$. The image of the map δ is called the space of inner derivations. Given example where δ is neither surjective nor injective.

1.3. **Solvable and Nilpotent Lie algebras.** Let $\mathfrak{g}$ be a Lie algebra and let $D_k(\mathfrak{G})$ be a decreasing sequence of characteristic ideals of $\mathfrak{g}$ defined by setting $D_k(\mathfrak{g})$ by setting $D^1(\mathfrak{g})=\mathfrak{g}$ and $D^{k+1}(\mathfrak{g})=[D^k(\mathfrak{g}),D^k(\mathfrak{g})]$. This decreasing sequence will be called the derived central series. If this series is eventually zero then we call $\mathfrak{g}$ is *solvable*. Similarly, we may define $C_k(\mathfrak{g})$ by setting $C^1(\mathfrak{g})=\mathfrak{g}$ and $C^{k+1}(\mathfrak{g})=[\mathfrak{g},C^k(\mathfrak{g})]$. If this series decreases to zero then we call the Lie algebra to be *nilpotent Lie algebra*. Examples being the space of upper traiangular matrices $\mathfrak{b}_n(k)$ and $\mathfrak{u}_n(k)$, the upper triangular matricies with all diagonal entries as zero. Infact if $\mathfrak{g}$ is nilpotent then we can show that $\mathfrak{g}$ embeds as a subalgebra of $\mathfrak{u}_n(k)$ and if $\mathfrak{g}$ is solvable then $\mathfrak{g}$ is a subalgerba of $\mathfrak{b}_n(k)$, when k is algebraically closed.

Theorem 1.3 (Lie and Engel's theorem)**.** *Let $\mathfrak{g}$ be a nilpotent (resp. solvable) Lie subalgebra of $\mathfrak{gl}_k(V)$. Then $\mathfrak{g}$ is isomorhic to a subalgebra of $\mathfrak{u}_n(k)$ (resp. when k is algebraically closed, subalgebra of $\mathfrak{b}_n(k)$).*

Proof. We need to show that there exists a vector $v\in V$ such that $x.v=0$, for all $x\in\mathfrak{g}$. To do this we use induction of the dimension of $\mathfrak{g}$ and simultaneously for all vector spaces V over k. The proof uses the following trick: construct a co-dimension one ideal of $\mathfrak{g}$. From the definition, we know that $ad_x:\mathfrak{g}\to\mathfrak{g}$ is a nilpotent operator. Assume that $\mathfrak{g}_1$ is a Lie subalgebra of dimension $m<n$. Note that $ad_x:\mathfrak{g}/\mathfrak{g}_1\to\mathfrak{g}/\mathfrak{g}_1$ is nilptoent for all $x\in\mathfrak{g}_1$. Thus, there exists an element $y\in\mathfrak{g}\backslash\mathfrak{g}_1$ such that $ad_x(y)=0$ for all $x\in\mathfrak{g}_1$. Thus, we get that $\mathfrak{g}_1$ is an ideal of $\mathfrak{g}_1+\langle y\rangle$. Using induction we get that $\mathfrak{g}$ has a codimension one ideal.

The space U annihilated by $\mathfrak{g}_1$ is non-zero by induction hypotesis and $x \in \mathfrak{g} \backslash \mathfrak{g}_1$ fixes U. Thus, we get that there exists a non-zero vector $u \in U$ such that $xu = 0$. Hence, we get the theorem. Now using a similar modification of this result, when k is algebraically closed, one can prove Lie's theorem on solvable subalgebras of $\mathfrak{gl}_k(V)$. For $X \in \mathfrak{g}$, the map $X \mapsto [X, Y]$ is denoted by ad_X.

Exercise 1.3.1.
1. *Show that* $\mathfrak{g}$ *is nilpotent is equivalent to saying that there exists a decresing sequence ideals* $\mathfrak{g}_i$ *such that* $[\mathfrak{g}, \mathfrak{g}_i] \subseteq \mathfrak{g}_{i+1}$.
2. $\mathfrak{g}$ *is nilpotent then the map* ad_X *is nilpotent.*
3. *Show that the centre of a nilpotent Lie algebra is non-zero.*

□

2. Introduction to manifolds

2.1. A manifold is the data $(X, \mathcal{F})$, where X is a second countable locally compact Housdorff topological space and $\mathcal{F}$ is a sheaf with the following property: for all $p \in X$, there exists an open set U_p of X containing p and an open set $\mathcal{U}_p$ of $\mathbb{R}^n$ such that

$$(U_p, \mathrm{res}_{U_p} \mathcal{F}) \simeq (\mathcal{U}_p, C^\infty_{\mathcal{U}_p}).$$

Here $C^\infty_{\mathcal{U}_p}$ is the sheaf of C^∞ functions on $\mathcal{U}_p$. We denote by $\mathcal{F}$ with the notation C^∞_X and the space $C^\infty_X(X)$ with $C^\infty(X)$. This is the definition of C^∞-manifold and we may similarly define analytic and holomorphic manifolds. We leave it for the reader to write such definitions. This is not the usual definition of manifolds we find in books so some work is needed to relate this to the well known definitions and I leave them in the following set of exercises:

Exercise 2.1.1. (Usual definition of a manifold) Let $(X, \mathcal{F})$ be a manifold. Show that there exists a cover $\{U_\alpha : \alpha \in I\}$ of X and $u_\alpha : U_\alpha \to \mathbb{R}^n$ homeomorphisms on to their respective images such that whenever $U_\alpha \cap U_\beta$ is non-empty the map $u_\alpha \circ u_\beta^{-1}$ is a diffeomorphism.

The elements of $C^\infty(X)$ are identified with the set of functions on X.

Exercise 2.1.2 (Fact)**.** *Note that our definition of a manifold $(X, \mathcal{F})$ implicitly assumes the paracompactness of X and hence the normality of X. Let $\{U_\alpha\}_{\alpha \in I}$ be a locally finite open cover of X. Show that there exists a family of differentiable functions $\{\psi_\alpha\}$ such that the support of ψ_α is contained in U_α and $\psi_\alpha \geq 1$ and $\sum_\alpha \psi_\alpha = 1$.*

Exercise 2.1.3. *Let U be any paracompact open subset of X show that $\mathcal{F}(X) \to \mathcal{F}(U)$ is surjective. Let K be any closed subset of X. Show that the restriction map $\mathcal{F}(X) \to \mathcal{F}(K)$ is surjective.*

Given the definition it is our job to actually give examples. The following exercise are helpful to get an introduction to the idea of sheaves in a quick way.

Exercise 2.1.4.
(1) Let $\{U_\alpha\}$ be an open cover of X and let $\mathcal{F}_\alpha$ be a family on sheaves on U_α. Assume that there exists a family of maps $u_{\alpha,\beta} : \mathrm{res}_{U_\alpha \cap U_\beta} \mathcal{F}_\alpha \simeq \mathrm{res}_{U_\alpha \cap U_\beta} \mathcal{F}_\beta$ such that $u_{\alpha\beta} u_{\beta\gamma} = u_{\alpha\gamma}$, then there exists a unique sheaf $\mathcal{F}$ such that $\mathrm{res}_{U_\alpha} \mathcal{F} \simeq \mathcal{F}_\alpha$.
(2)