MTH204 2022-FEB; MIDSEM EXAM

INSTRUCTOR: SANTOSH NADIMPALLI

Instructions:

(1) You can only use concepts and results, introduced and proved respectively, in this course to answer the following questions.
(2) Partial scores will be given only for a complete and correct statement, if they are relevant to the solution of the final problem.
(3) Please do not discuss these questions with other students or on the internet forums like math stackexhange.
(4) I reserve the right to give zero score if two candidates produce the exact same answers. I reserve the right to give zero score if the solution is found exactly the same solution on an internet forum.
(5) Please remember that there are two pages in this question paper.
(6) You are suggested to restrict your answers to one side of a page for each question.
(7) Note that the set $\{1, 2, \ldots, n\}$ is denoted by $[n]$.
(8) Answer any choice of questions with total 40 points. **You can score a maximum of 40 points in this exam.**

1. **15 points** Let n be a positive integer and let R_n be the vector space $\mathbb{C}[X_1, \ldots, X_n]$ consisting of polynomials in n-variables $(X_1, \ldots, X_n)$. Let S_n be the group of bijections between $[n]$ to $[n]$. For $f \in C[X_1, \ldots, X_n]$, we denote by $S_f \subseteq S_n$ the subgroup defined by

$$S_f = \{\sigma \in S_n : f(X_1, X_2, \ldots, X_n) = f(X_{\sigma(1)}, X_{\sigma(1)}, \ldots, X_{\sigma(n)})\}.$$

Does there exists an $f \in R_4$ such that S_f is isomorphic to S_3, A_4 (permutations in S_4 with sign 1) , Q_8 (quaternions), $(\mathbb{Z}/2\mathbb{Z})^2$, $(\mathbb{Z}/2\mathbb{Z})$, D_4(dihedral group of cardinality 8) and $\mathbb{Z}/3\mathbb{Z}$.
(Not a part of the exam: Given any subgroup H of S_n, show that there exists an $f \in R_n$ such that $S_f = H$.)

2. **15 points** For any two groups G and H, we denote by $\mathrm{Hom}(G, H)$, the set of group homomorphisms from G to H.

(1) **(4 points)** Compute the cardinality of the set $\mathrm{Hom}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z})$, where n a nd m are arbitrary positive integers.
(2) **(5 points)** Compute the cardinality of the set $\mathrm{Hom}(\mathbb{Z}^2, S_6)$. (Here, $\mathbb{Z}^k$ is the group $\mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z}$, the k fold direct product of $\mathbb{Z}$.)
(3) **(6 points)** Show that for any finite group G, $|G|$ divides $|\mathrm{Hom}(\mathbb{Z}^3, G)|$.

It is interesting to generalise this question. One may think about computing (or form a generating series of) the cardinality of $\mathrm{Hom}(\mathbb{Z}^k, S_n)$, for arbitrary k and n. (Although not a part of the exam).

3. **(5 points)** Let p be a prime number. Show that there exists a matrix $g \in \mathrm{GL}_2(\mathbb{F}_p)$ such that g does not have any eigenvalues in $\mathbb{F}_p$, i.e., there exists no $\lambda \in \mathbb{F}_p$ such that $g(v) = \lambda v$, for some non-zero $v \in \mathbb{F}_p^2$. Show that the set $K = \{aI_{2\times 2} + bg : a, b \in \mathbb{F}_p\}$ is a field.

4. **(5 points)** Every group G with exactly 5 subgroups is abelian.

5. **(10 points)** Give very short answers.

5.1. **(2 point)** Give an example of a non-abelian group with all its subgroups normal.

5.2. **(2 point)** What are the subgroups of $\mathbb{Z} \rtimes \mathrm{Aut}(\mathbb{Z})$.

5.3. **(3 point)** Prove or disprove the following statement: Let p be the least prime dividing $|G|$. If N is a subgroup of G such that $|G/N| = p$. Then N is normal.

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5.4. **(3 points)** Show that any element of A_n can be written as $xyx^{-1}y^{-1}$ for some $x, y \in A_n$