MTH201: LINEAR ALGEBRA

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1. Lecture 1

1.1. In physics we study vector and scalar like quantities, for instance, an electric field behaves as a vector, i.e., has a direction and a magnitude. Where as charge is a scalar quantity. In quantum mechanics, we use operators on Hilbert spaces which take a ket or bra vector as an input and produce new ket or bra vector in some order, for example positition, momentum operators. In problems related to circuits, we have to deal with a large number of linear equations in several variables. The mathematical abstraction to understand these situations is the concept of a vector space, which consists of vector like objects on which a set of scalars, to be called as fields, act via certain scalar multiplication. In this lecture we will introduce these basic objects i.e., set of scalars, to be called as fields, and sets of vectors on which elements of these fields scale the magnitude called the vector space.

1.2. **Fields.** The set of scalars quantities will be called a field. We will come to the precise definition of a field. A *field* is a tupe $(F, +, .)$ consisting of a set F, equipped with two operations, to be denoted by $+$ and $.$. The set F must contain two distinct elements 0 and 1 such that the following properties hold:

$$0 + x = x + 0 = 0, x \in F,$$

$$x + (y + z) = (x + y) + z, x, y, z \in F,$$

for any x, there exists an element $-x$ (which turns out to be unique) such that $x + (-x) = 0$, and

$$x + y = y + x.$$

Such a structure $(F, +)$ is called an *abelian group*. As far as the second operation and its compatibility with the first is concerned we assume that

$$1.x = x.1 = x, x \in F,$$

$$x.y = y.x, x, y \in F, x.(y.z) = (x.y).z,$$

$$x.(y + z) = x.y + x.z, x, y, z \in F,$$

and for any $y \in F$ such that $y \neq 0$, there exists a $z \in F$ such that $yz = 1$. If the last condition is not included then the the tuple $(F, +, .)$ is called a *ring*. It is now important to give some examples. The set of integers $\mathbb{Z}$ with usual addition and multiplication is a ring but not a field. Examples of fields include the set of rational numbers $\mathbb{Q}$, the set of real numbers $\mathbb{R}$, and the set of complex numbers $\mathbb{C}$, all with their usual addition and multiplication operations.

Note that the set $\{0, 1\}$ along with $+$ and multiplication $.$, which satisfy the relations:

$$0.0 = 0, 0.1 = 1, 1.1 = 1 \text{ and } 1 + 1 = 0$$

is a field. This field is denoted by $\mathbb{F}_2$. This is widely used in computer science, for information is stored as bits of zeros and ones. Of course, its a nightmare in mathematics that scalars consists of just 2-elements. Fix a prime number p, and let $\mathbb{F}_p$ be the set $\{0, 1, 2, \ldots p - 1\}$. We define $x + y$ as an element $z \in \mathbb{F}_p$ such that $x + y - z$ is divisible by p. Similarly we define $x.y$ in the field $\mathbb{F}_p$ to be the integer $z \in \mathbb{F}_p$ such that $xy - z$ is divisible by p. The tuple $(\mathbb{F}, +, .)$ is called a *prime field* of characteristic p.

After defining structures in mathematics, we need to define maps between these structures, a map of two fields $(F_1, +, .)$, and $(F_2, +, .)$, is a map $f : F_1 \to F_2$ such that

$$f(0) = 0, \text{and } f(1) = 1,$$

$$f(a +_1 b) = f(a) +_2 f(b)$$

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and

$$f(a._1b) = f(a).{}_2f(b).$$

Lemma 1.1. *If f is a field map between $(F_1, +_1, .1)$, and $(F_2, +_2, .2)$, then $f : F_1 \to F_2$ is an injection.*

Proof. Let S be the set $\{x \in F_1 : f(x) = 0\}$. If $S \neq 0$, then there exists an element $x \neq 0$ such that $f(x) = 0$. But

$$1 = f(x.{}_1x^{-1}) = f(x).{}_2f(x^{-1}) = 0$$

and this is a contradiction to the fact that $0 \neq 1$ (you must think about the above equality, this needs some justification). Hence $S = 0$. Now if $f(x) = f(y)$, then $f(x - y) = 0$ and thus, $x = y$. □

We can take an abstract field $(F, +, .)$ and consider the set

$$\{0, 1, 1 + 1, \ldots 1 + 1 + \cdots + 1, \ldots\}$$

If no two elements of the above set are equal, then we get that the set of natural numbers $\mathbb{N}$ is contained in the set F and hence and set of rational numbers $\mathbb{Q}$ is contained in F. Assume that there exists $r \neq s$ such that

$$1 + 1 + \ldots \ r \text{ times} = 1 + 1 + \ldots \ s \text{ times}.$$

Then there exists an integer $n > 0$ such that

$$1 + 1 + \ldots \ n \text{ times} = 0$$

We may assume that n is the least such integer. If n is not a prime number, we set $n = n_1n_2$ with $1 < n_i < n$. Then,

$$(1 + 1 + \ldots \ n_1 \text{ times})(1 + 1 + \ldots \ n_2 \text{ times}) = 0.$$

Thus we get a contradiction to the minimality of the definition of n. Thus, there exists a prime number p such that

$$1 + 1 + \ldots \ p \text{ times} = 0,$$

and all the elements $1 + 1 + \ldots \ r$ times $\neq 0$, for $0 \leq r < p$. Hence, the prime field $\mathbb{F}_p$ is contained in the field F. The above argument proves the following lemma:

Lemma 1.2. *Given any field $(F, +, .)$, then either $(\mathbb{F}_p, +, .)$ is contained in $(F, +, .)$, for some prime number p or else $(\mathbb{Q}, +, .)$ is contained in $(F, +, .)$.*

If $(F, +, .)$ contains $(\mathbb{F}_p, +, .)$, then the prime number p is called the characteristic of the field (show that the prime number p is unique). If $\mathbb{Q}$ is contained in F then the characteristic is said to be zero. The set of all polynomials with coefficients in a field F will be denoted by $F[X]$. A field F is called *algebraically closed* if for any polynomial $P \in F[X]$ has a root in F. It is common knowledge that $\mathbb{C}$ is algebraically closed. However, a proof of this fact is less common in common courses. If one is uncomfortable with general fields, you may assume that $(F, +, .)$ is the field of complex numbers or the field of real numbers.

1.3. **Vector spaces.** Now that we have defined fields, next abstract structure we want is the set of vectors with addition and a notion of multiplication by elements of a field. This leads us to the following definition. A *vector space* V over a field $(F, +_1, .1)$ is an abelian group $(V, +_2)$ and a map

$$* : F \times V \to V; (c, v) \mapsto c * v$$

such that

$$1.v = v, v \in V,$$
$$c * (v_1 + v_2) = c * v_1 +_2 c * v_2, v_1, v_2 \in V, c \in F,$$
$$(c_1 + c_2) * v = c_1 * v + c_2 * v, c_1, c_2, \in F, v \in V$$

and

$$c_1 * (c_2 * (v)) = (c_1.{}_1c_2) * v, c_1, c_2 \in F, v \in V.$$

Clearly the first example of a vector space is the set of scalars itself, i.e., the structure $(F, +)$ with $*$ operation given by . is a vector space over the field F. Let n be an integer and let F^n be the abelian group $\{(x_1, x_2, \ldots, x_n) : x_i \in F\}$. The set together with addition

$$(x_1, x_2, \ldots, x_n) +_1 (y_1, y_2, \ldots, y_n) = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$$

is an abelian group, and moreover, $(F^n, +_1)$ is an F vector space via the scalar multiplication map given by

$$c * (x_1, \ldots, x_n) = (c.x_1, c.x_2, \ldots, c.x_n), c, x_1, \ldots, x_n \in F.$$

A subset W of an abelian group $(V, +)$ is called a *subgroup* if $0 \in W$, $v + w \in W$ whenever $v, w \in W$ and $-v \in W$ for all $v \in W$. A subgroup W of a vector space V over a field $(F, +, .)$ is called a *vector subspace* if $c * w \in W$, for all $w \in W$ and $c \in F$.

Example 1.3. *If X is a set, and if F is a field then the set $\mathrm{Map}(X, F)$ consisting of functions from X to the field F is a vector space with pointwise addition and pointwise scalar multiplication, i.e., the following addition and sclar multiplication:*

$$(f_1 + f_2)(x) = f_1(x) + f_2(x); (c * f)(x) = c.f(x),$$

for all $f_1, f_2, f \in \mathrm{Map}(X, F)$, $x \in X$ and $c \in F$ makes $\mathrm{Map}(X, F)$ a vector space.

Example 1.4. *Let $x_1, x_2, \ldots, x_n$ be some variables and let A be the matrix (a_{ij}), i.e., the ij^{th}-entry of A is a_{ij}. The subset of tuples $(y_1, y_2, \ldots, y_n) \in F^n$ which satisfy the system of equations*

$$\sum_{j=1}^{n} a_{ij} x_j = 0, 1 \leq i \leq m,$$

is a vector space.

Example 1.5. *The space of $n \times m$-matrices with entries in a field $(F, +, .)$, to be denoted by $\mathrm{M}_{n \times m}(F)$, is a vector space. The addition is given by entrywise addition and scalar multiplication is given by scalar multiplication of all entries.*

Example 1.6. *Consider the linear differential operator, i.e.,*

$$D = p_n(x)\frac{d^n}{dx^n} + p_{n-1}(x)\frac{d^{n-1}}{dx^{n-1}} + \ldots p_0(x).$$

The set of all solutions for the operator D in the space $C^n(R)$ is a vector space.

Example 1.7. *If $(F_1, +, .)$ is a subfield of $(F_2, +, .)$, then the field F_2 is a vector space over F_1, where F_1 scalar multiplication on F_2 is given by multiplication in F_2. In this sense the field $\mathbb{C}$ is a vector space over $\mathbb{Q}$.*

Example 1.8. *The set of polynomials in n-variables with coefficients in the field F, denoted by $F[X_1, X_2, \ldots, X_n]$ is a vector space over F. Here the addition and scalar multiplication is the usual one.*

Example 1.9. *The abelian group of complex numbers $\mathbb{C}$, together with the scalar multiplication*

$$z * v = \overline{z}v, z \in \mathbb{C}, v \in \mathbb{C}$$

makes $\mathbb{C}$ a vector space over the field $\mathbb{C}$.

2. Lecture 2

2.1. **Linear transformations.** Now, that we have many examples for vector spaces, we need to define maps between vector spaces which preserve the vector space structure. This will help us in defining when two vector spaces are the "same". Let $(F, +, .)$ be a field and let $(V_1, *_1)$ and $(V_2, *_2)$ be two $(F, +, .)$ vector spaces. A *linear transformation* or a *linear operator* $T : V_1 \to V_2$ is a map of abelian groups between V_1 and V_2, i.e.,

$$T(0_{V_1}) = 0_{V_2},$$

and

$$T(v_1 +_{V_1} v_2) = T(v_1)_{V_2} + T(v_2), v_1, v_2, \in V_1,$$

such that $f(c *_1 v) = c *_2 f(v)$ for all $c \in F$ and $v \in V_1$. Linear transformation $T : V_1 \to V_2$ is said to be an isomorphism if there exists a $S : V_2 \to V_1$ such that $T \circ S = \mathrm{id}_{V_2}$ and $S \circ T = \mathrm{id}_{V_1}$. **From now we drop specifying the operators of scalar multiplication;** $c * v$ **will be just denoted by** cv **whenever the situation is clear**. Let us define some obvious linear transformations.

Example 2.1. *Let F be a field, and let V be an F-vector space. Let $S = \{v_1, v_2, \dots v_n\} \subset V$ be a set of n-vectors in V. Let $\Phi_S : F^n \to V$ be the linear transformation given by*

$$\Phi_S((x_1, x_2, \dots, x_n)) = x_1v_1 + x_2v_2 + \cdots + x_nv_n.$$

The above example is a fundamental example and it shows that all vector spaces can be studied using the standard space F^n, for some n. We will analyse this linear transformation in this lecture. Essentially, in this lecture, we want to study those vector spaces for which Φ_S is an isomorphism for some $S \subset V$.

Let V_1 and V_2 be two F-vector space and let $\mathrm{Hom}_F(V_1, V_2)$ be the set of linear transformations between V_1 and V_2. Note that $\mathrm{Hom}_F(V_1, V_2)$ is also an F-vector space. The addition on $\mathrm{Hom}_F(V_1, V_2)$ is given by

$$(T_1 + T_2)(v) = T_1(v) + T_2(v), T_1, T_2 \in \hom_F(V_1, V_2), v \in V_1.$$

The scalar multiplication is given by

$$(c * T)(v) = cT(v), c \in F, T \in \hom(V_1, V_2), v \in V_1.$$

When $V_2 = F$, the space $\mathrm{Hom}_F(V, F)$ is denoted by $V^\vee$. The space $V^\vee$ is called the dual space of V. An element of $V^\vee$ is called a *linear functional* on V. In the next lecture, we shall see that $V^\vee$ is very fundamental object. A linear transformation $T : V \to W$ is said to be an isomorphism if there exists a linear transformation $S : W \to V$ such that $ST = \mathrm{id}_V$ and $TS = \mathrm{id}_W$. Clearly, any isomorphism of vector spaces is injective and surjective. Now, if a linear map T is a bijection, then T^{-1} is also a linear transformation. Hence, a bijective linear transformation $T : V \to W$ is an isomorphism. The following lemma, illustrates the definitions in a concrete set up.

Lemma 2.2. *Let V_1 be the space F^n and V_2 be the space F^m for some positive integers n and m. The space of of linear transformations between V_1 and V_2 is isomorphic to the space of matrices $M_{n\times m}(F)$.*

Proof. We need to construct maps $\Phi : \mathrm{Hom}_F(V_1, V_2) \to M_{n\times m}(F)$ and $\Psi : M_{n\times m}(F) \to \mathrm{Hom}_F(V_1, V_2)$ such that $\Phi \circ \Phi$ and $\Phi \circ \Psi$ are identity maps on the respective domains. Let $T \in \mathrm{Hom}_F(V_1, V_2)$ and let $e_i = (0, \dots, 1, \dots, 0)$ where 1 is in the i-th position. Let $T(e_i) = (a_{i1}, a_{i2}, \dots a_{im})$, and we define $\Phi(T)$ to be the matrix $A = (a_{ij})$. Given a matrix $A = (a_{ij})$, we define $\Psi(A) : F^n \to F^m$ by setting

$$\Psi(A)(x_1, x_2, \dots x_n) = (\sum_{ij} a_{i1}x_1, \dots, \sum_{ij} a_{in}x_n).$$

We can check that $\Psi \circ \Phi$ and $\Phi \circ \Psi$ are identity maps. □

Before going any further, let introduce an important subspace which will be useful in understanding the infectivity of a linear transformation.

Definition 2.3. *Let V_1 and V_2 be two F-vector spaces and let $T : V_1 \to V_2$ be a linear transformation. The kernel of T, to be denoted by* $\ker(T)$*, is the following subset of V:*

$$\ker(T) = \{v \in V_1 : T(v) = 0\}.$$

Note that $\ker(T)$ *is a linear subspace of V_1.*

Definition 2.4. *Let $S = \{v_1, \ldots, v_n\}$ be a subset of V. The image of the map $\Phi_S : F^n \to V$ is called the linear span of S. It is customary to denote by $\langle v_1, v_2, \ldots, v_n \rangle$ for the image of the map Φ_S.*

Lemma 2.5. *A linear transformation $T : V_1 \to V_2$ between two F-vector spaces V_1 and V_2 is injective if and only if $\ker(T)$ the zero subspace of V_1, i.e., $\ker(T) = \{0\}$.*

Proof. Let T be an injective tranformation. If $T(v) = 0 = T(0)$, for some $v \in V$, then $v = 0$. This shows that $\ker(T) = \{0\}$. Let v_1 and v_2 be two vectors in V such that $f(v_1) = f(v_2)$ then we get that $f(v_1 - v_2) = 0$ and hence, $v_1 - v_2 \in \ker(T)$. Since, $\ker(T) = \{0\}$, we get that $v_1 = v_2$. This shows that if $\ker(T) = \{0\}$, then T is injective. □

Definition 2.6. *The vector space consisting of a single element namely zero is called a trivial vector space or a zero vector space.*

Example 2.7. *We will see the relation between linear equations and kernels of linear transformations. Assume that $A = (a_{ij})$ is an $n \times m$ matrix with entries in F. The linear system*

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = 0, 1 \leq i \leq m$$

has a non-zero solution if and only if $\ker(\Psi(A))$, where $\Psi(A) : F^n \to F^m$ is a linear transformation defined in Lemma 2.2 is a non-trivial space. In general finding solutions of differential equation or a linear system can be formulated using the ker of the respective linear transformations.

3. Lecture 3

3.1. Let V be a vector space over a field F and let S be a subset of vectors in V. The *linear span* of S is the set of all vectors of the form

$$x_1v_1 + x_2v_2 + \cdots + x_nv_n,$$

where $x_i \in F$ and $v_i \in S$. Note that the linear span of S is a subspace of V. The linear span of the set S is denoted by $\langle S\rangle$. The linear span of S is the minimal subspace of V containing the set S, i.e., if W' is any subspace of V containing S such that $W' \subseteq \langle S\rangle$, then $W' = \langle S\rangle$. A set of vectors in V, denoted by S, are said to be *linearly independent*, whenever you have a linear relation

$$x_1v_1 + x_2v_2 + \cdots + x_nv_n = 0,$$

for $v_1, v_2 \ldots, v_n \in S$ then $x_i = 0$ for all $1 \leq i \leq n$. A linearly independent subset S of V is called a *basis* if and only if $\langle S\rangle = V$. A vector space V is called a finite dimensional vector space if and only if there exists a finite set S of V such that $\langle S\rangle = V$.

3.2. Given a finite spanning set S of V, the next lemma shows that we can always find a linearly independent subset of V.

Lemma 3.1. *Let V be a vector space over a field F and let S be any finite subset of V consisting of non-zero vectors of V. There exists a subset S' of S which is linearly independent and $\langle S\rangle = \langle S'\rangle$.*

Proof. We prove the lemma using the cardinality of the set S. If $|S| = 1$, then $v \in S$ is non-zero and hence linearly independent. Assume that the lemma is proved for all S with $|S| \leq N$. Now, consider the case where $|S| = N + 1$. If the elements of S are linearly independent then we may take $S' = S$. Otherwise, there exists a non-trivial linear relation

$$x_1v_1 + x_2v_2 + \cdots + x_nv_n = 0, x_i \in F, v_i \in S$$

such that $(x_1, x_2, \ldots, x_n) \neq 0$. Without loss of generality we may assume that $x_1 \neq 0$. Then note that

$$v_1 = -\frac{x_2}{x_1}v_2 + \cdots - \frac{x_n}{x_1}v_n.$$

Note that $\langle S\rangle = \langle S'\rangle$, where $S' = S\backslash\{v\}$. Using induction hypothesis, we get $S'' \subseteq S'$ such that $\langle S''\rangle = \langle S'\rangle$ and S'' is linearly independent. Now, the lemma follows using the principle of induction. □

Thus, any finite dimensional vector space V has a basis. The next result is to prove that the cardinality of any basis is the same. We will first need the following lemma.

Lemma 3.2 (Exchange)**.** *Let V be a vector space over a field F and let I be a subset of V. Let $v \in \langle I\rangle$ and let $w \in I$ such that $v \notin \langle I\backslash\{w\}\rangle$, then the span of the set $J = (I\backslash\{w\}) \cup \{v\}$ is equal to $\langle I\rangle$.*

Proof. Clearly $\langle J\rangle$ is contained in $\langle I\rangle$. We have to show the converse. Since $v \in \langle I\rangle$, and $v \notin \langle I\backslash\{w\}\rangle$ there exists a linear relation of the form

$$v = x_0w + x_1w_1 + x_2w_2 + \cdots + x_nw_n, x_i \in F, w_i \in I\backslash\{w\},$$

where $x_0 \neq 0$. We then get that

$$w = x_0^{-1}v - \frac{x_1}{x_0}w_1 - \frac{x_2}{x_0}w_2 - \cdots - \frac{x_n}{x_0}w_n.$$

This implies that $w \in \langle J\rangle$. Hence, $\langle I\rangle$ is a subspace of $\langle J\rangle$. □

Lemma 3.3. *Let S be a linearly independent subset of vectors in a vector space V over a field F such that $\langle S\rangle$ is a proper subspace of V. Then $S \cup \{v\}$ is a linearly independent subset of V for any $v \in V\backslash\langle S\rangle$.*

Proof. Trivial. □

Theorem 3.4. *Let S_1 and S_2 be two basis of a finite dimensional vector space V over a field F, then $|S_1| = |S_2| < \infty$.*

Proof. Let $\mathcal{S}_0 = \{w_1, w_2, \ldots, w_m\}$ be a basis of the vector space V over the field F. We may assume that $\mathcal{S}_0$ is finite using Lemma 3.1. Let $S_1 = \{v_1, v_2, \ldots, v_i, v_{i+1}, \ldots, v_n\}$, using induction on the integer $i \leq n$, we show that the set

$$\{v_1, v_2, \ldots, v_i, w_{i+1} \ldots, w_n\}$$

is a basis of V. Since v_{i+1} does not belong to the space $\langle v_1, v_2, \ldots, v_i\rangle$, there exists a w_j with $j > i+1$ such that $v_{i+1} \notin \langle v_1, v_2, \ldots v_i, w_{i+1}, \ldots, w_j\rangle$. Thus, we get that $|S_1|$ is finite and $|S_1| \leq |\mathcal{S}_0|$. We may interchange $\mathcal{S}_0$ with S_1 and get the inequality $|S_1| \leq |\mathcal{S}_0$. □

Definition 3.5. *Let V be a finite dimensional vector space over a field F. The cardinality of a basis of V is called the dimension of V.*

3.3. **Exercises.**

3.3.1. Let V be a finite dimensional vector space and let W be a subspace of V. Show that $\dim_F(W) \leq \dim_F(V)$ the equality holds if and only if $W = V$.

3.3.2. Let V be a finite dimensional vector space over a field F. Let $(w_1, w_2, \ldots, w_k)$ be a tuple of linearly independent vectors in V. There exists a basis of the form $(w_1, w_2, \ldots, w_k, w_{k+1}, w_{k+2}, \ldots, w_n)$.

3.3.3. Let V be a finite dimensional vector space over a field F, and let W be a vector space over F. Let $B = \{v_1, v_2, \ldots, v_n\}$ be a basis of V. For any choice of vectors $(w_1, w_2, \ldots, w_n)$, $w_i \in W$, show that there exists a unique linear transformation $T : V \to W$ such that $T(v_i) = w_i$.

3.3.4. Let W be a subspace of F^n defined by setting

$$W = \{(x_1, x_2, \ldots, x_n) \in F^n : \sum x_i = 0\}.$$

Find a basis of W.

3.3.5. Let $\mathfrak{g}_0$ and $\mathfrak{g}_1$ be the spaces $\{M \in M_{n\times n}(F) : M^T = M\}$ and $\{M \in M_{n\times n}(F) : M^T = -M\}$ respectively. Find a basis of $\mathfrak{g}_0$ and $\mathfrak{g}_1$.

3.3.6. Let V be a vector space and let $V^\vee$ be the space $\mathrm{Hom}_F(V, F)$. Given a basis $(v_1, v_2, \ldots, v_n)$ show that there exists there exists a unique element $v_i^\vee \in V^\vee$ such that

$$v_i^\vee(v_j) = \delta_{ij}.$$

for $i, j \in [n]$. Show that $(v_1^\vee, v_2^\vee, \ldots, v_n^\vee)$ is a basis of $V^\vee$.

3.3.7. Let $B_1 = (v_1, v_2, \ldots, v_n)$ and $B_2 = (w_1, w_2, \ldots, w_n)$ be two basis for the vector space V. Let M be the matrix transforming B_1 to B_2, i.e., $M = (m_{ij})$ with

$$w_i = \sum_{j\in[n]} m_{ij}v_j, i \in [n].$$

Show that $(M^T)^{-1}$ is the matrix which transforms the dual basis of B_1 to the dual basis of B_2.

3.3.8. Let F be a finite field of cardinality q, and let V be a vector space of dimension n over F. Show that number of distinct basis is equal to

$$(q^n - 1)(q^n - q) \ldots (q^n - q^{n-1}).$$

3.3.9. Let V be a finite dimensional $\mathbb{C}$-vector space. We can consider V as an $\mathbb{R}$-vector space by restricting the scalar multiplication map to the real numbers. Let us denote this real vector space by $res_{\mathbb{C}/\mathbb{R}}V$. Note that the underlying abelian group structure on $res_{\mathbb{C}/\mathbb{R}}V$ and V is just the same, however, in the first one we only consider scalar multiplication from real numbers. What is the dimension of the vector space $res_{\mathbb{C}/\mathbb{R}}V$ as a $\mathbb{R}$-vector space. Conversely, given a real vector space V and a linear operator J such that $J^2 = -\mathrm{id}$, there exists a unique complex vector space structure on V which extends the real scalar multiplication and $J(v) = iv$.

3.3.10. Let V be a $\mathbb{C}$-vector space. We define a new scalar multiplication $z * v$ for $z \in \mathbb{C}$ and $v \in V$ by setting

$$z * v = \bar{z}v.$$

Show that $(V, +, *)$ is a $\mathbb{C}$-vector space and does there exists a vector space V such that $(V, +, *)$ is isomorphic to the vector space $(V, +, .)$. If $(V, +, .)$ is a vector space of dimension n, then what is the dimension of the vector space $(V, +, *)$.

4. Lecture 4

4.1. **Linear transformations and Basis.** In this lecture, we will explore the relations between linear transformations between two finite dimensional vector spaces V and W and matrices of size $\dim(V)\times\dim(W)$. We already showed that the space $\mathrm{Hom}_F(V,W)$ is isomorphic to $M_{n\times m}(F)$. However, the map defined in the lemma is not the only one. There are plenty of such maps depending on a choice of basis in V and W. Let $B_1=\{v_1,v_2,\ldots,v_n\}$ be a basis of V and let $B_2=\{w_1,w_2,\ldots,w_m\}$ be a basis of W. Given any linear transformation $T:V\to W$, we can associate a matrix $M_T=(m_{ij})$ where the values m_{ij} are uniquely determined by the equalities:

$$T(v_i)=\sum_{j=1}^{m} m_{ji}w_j, i\in[n].$$

The map $T\mapsto M_T$, gives a linear transformation denoted by $\Phi(B_1,B_2):\mathrm{Hom}_F(V,W)\to M_{n\times m}(F)$. Conversely, given any matrix M, we can define T a linear operator which satisfies the above equality. If we denote the map sending M to T, by

$$\Psi(B_1,B_2):M_{n\times m}(F)\to\mathrm{Hom}_F(V,W),$$

then $\Psi(B_1,B_2)$ is the inverse of $\Phi(B_1,B_2)$.

Let $v\in V$ and there exists a unique tuple $X=(x_1,x_2,\ldots,x_n)\in F^n$ such that

$$v=x_1v_1+x_2v_2+\ldots x_nv_n.$$

The vector X is called *coordinates or coordinate vector* with respect to the basis $(v_1,v_2,\ldots,v_n)$, and x_i is called the i-th coordinate of the vector v. The coordinates of the vector $T(v)$ are given by MX^t, where M is the matrix of the linear transformation T.

4.2. **Linear transformations and matrix multiplication.** Let V_1, V_2 and V_3 be three vector spaces and let B_1, B_2 and B_3 be basis of V_1, V_2 and V_3 respectively. Assume that $T_1:V_1\to V_2$ and $T_2:V_2\to V_3$ are two linear transformations.

Lemma 4.1. *Let $M_1=\Phi(B_1,B_2)(T_1)$ and let M_2 be the matrix $\Phi(B_2,B_3)(M_2)$. Then, we have*

$$\Phi(B_1,B_3)(T_2T_1)=M_2M_1.$$

Proof. Let $B_1=(v_1,v_2,\ldots,v_k)$, $B_2=(v_1',v_2',\ldots,v_l')$ and $B_3=(v_1'',v_2'',\ldots,v_m'')$. We set

$$T_1(v_i)=\sum_{j=1}^{l} m_{ji}v_j',$$

$$T_2(v_i')=\sum_{j=1}^{m} m_{ji}'v_j''.$$

Now

$$T_2(T_1(v_i))=\sum_{j=1}^{l} m_{ji}T_2(v_j')=\sum_{j=1}^{l} m_{ji}(\sum_{k=1}^{m} m_{kj}'v_k'')=\sum_{k=1}^{m}(\sum_{j=1}^{l} m_{kj}'m_{ji})v_k''.$$

This proves the lemma. □

4.3. **Rank-nullity lemma.**

Lemma 4.2. *Let V be a finite dimensional vector space, and let W be any vector space over a field F. Let $T\in\mathrm{Hom}_F(V,W)$. Then, $\mathrm{img}(T)$ is a finite dimensional vector space over F and*

$$\dim(\ker(T))+\dim(\mathrm{img}(T))=\dim(V).$$

Proof. Let $(w_1,w_2,\ldots,w_k)$ be a basis of the subspace $\ker(T)$. Extend this basis to a basis

$$(w_1,w_2,\ldots,w_k,w_{k+1},\ldots,w_n)$$

of V. Note that $\mathrm{img}(T)$ is spanned by the vectors $\{T(w_{k+1}),\ldots,T(w_n)\}$. This shows that $\mathrm{img}(T)$ is a finite dimensional vector space. Assume that there is a linear relation

$$a_{k+1}T(w_{k+1})+\cdots+a_nT(w_n)=0.$$

We get that $a_{k+1}w_{k+1} + \cdots + a_nw_n = a_1w_1 + \cdots + a_kw_k$. Thus, we get that $a_i = 0$, for all $i \in [n]$. Hence $(T(w_{k+1}, \ldots, T(w_n))$ forms a basis of $\mathrm{img}(T)$. This proves the lemma. However, we used the fact that we can extend a linearly independent subset to a basis and a subspace of a finite dimensional vector space is finite dimensional. Which are exercises from previous Lecture. □

Corollary 4.3. *Let $T : V \to W$ be a linear transformation of finite dimensional vector spaces. There exists a basis $(v_1, v_2, \ldots, v_n)$ of V and $(w_1, w_2, \ldots, w_r)$ of the vector space $\mathrm{img}(T)$ such that*

$$T(v_1) = w_1, \ldots, T(v_r) = w_r, T(v_j) = 0, j \geq r + 1.$$

In terms of matrices, for every $M \in M_{n\times m}(F)$, there exists invertible matrices P and Q such that

$$PMQ = \begin{pmatrix} \mathrm{id}_r & 0 \\ 0 & 0 \end{pmatrix},$$

Where id_r is the identity matrix in $M_{r\times r}(F)$.

Proof. The proof is exactly the same as in the previous lemma. The basis $(v_1, v_2, \ldots, v_n)$ can be taken to S' and $(w_1, w_2, \ldots, w_r)$ can be taken to be B. □

Corollary 4.4. *Let $T : V \to W$ be a linear map of finite dimensional vector spaces with $\dim(V) = \dim(W)$. if T is injective then T is surjective and viceversa, if T is surjective, then T is injective. Thus isomorphism of a finite dimensional vector spaces of same dimension is equivalent to either infectivity or surjectivity.*

4.4. **Direct sum.** Let V and W be two vector spaces over a field F. The product spaces $V \times W = \{(v, w) : v \in V, w \in W\}$ can be made as a vector space by the following operations:

$$(v_1, w_1) + (v_2, w_2) = (v_1 + v_2) + (w_1 + w_2), v_1, v_2 \in V, w_1, w_2 \in W$$

and

$$c(v, w) = (cv, cw), c \in F, v \in V, w \in W.$$

The set $V \times W$ together with these operations is called the direct sum and is denoted by $V \oplus W$. If V and W are finite dimensional vector spaces with basis $(v_1, v_2, \ldots, v_n)$ and $(w_1, w_2, \ldots, w_m)$ for V and W respectively then

$$((v_1, 0), \ldots, (v_n, 0), (0, w_1), \ldots, (0, w_m)$$

is a basis of $V \oplus W$.

Lemma 4.5. *Let V be a finite dimensional vector space and let W_1 and W_2 be two subspaces of V. We then have*

$$\dim(W_1 + W_2) + \dim(W_1 \cup W_2) = \dim(W_1) + \dim(W_2).$$

Proof. Apply rank nullity lemma for the linear transformation $T : W_1 \oplus W_2 \to V$ defined by:

$$T((w_1, w_2)) = w_1 - w_2.$$

□

4.5. **Exercises.**

4.5.1. Let V be a finite dimensional vector space of dimension n. Let $T_1, \ldots, T_{n+1}$ be $n+1$ linear operators on V to V. Show that a non-zero linear combination of T_i with $i \in [n + 1]$ is not invertible.

4.5.2. Let V be a vector space and let $V^\vee$ be the space $\mathrm{Hom}_F(V, F)$. Let $T : V \to W$ be a linear transformation of two vector spaces. Show that there exists a unique linear operator $T^\vee : W^\vee \to V^\vee$ such that

$$[T^\vee(w^\vee)](v) = w^\vee(Tv), v \in V, w^\vee \in W^\vee.$$

Now assume that $V = W$ and let $B = (v_1, v_2, \ldots, v_n)$ be a basis of V and a dual basis $B^\vee = (v_1^\vee, v_2^\vee, \ldots, v_n^\vee)$ of $V^\vee$. Let M be the matrix corresponding to the Linear transformation T with respect to the basis B. Show that the matrix of the linear transformation $T^\vee$ with respect to the basis $B^\vee$ is the transpose of the matrix M. Generalise this to any Linear transformation $T : V \to W$, where V and W are two finite dimensional vector spaces.

4.5.3. For any finite dimensional vector space V, show that there exists an isomorphism $\phi_V : V \to (V^\vee)^\vee$ such that for any linear transformation $T : V \to W$, we have

$$\phi_W \circ T = (T^\vee)^\vee \circ \phi_V.$$

4.5.4. Let V and W be two vector spaces and let $T : V \to M$ be an injective linear transformation. Show that the linear transformation $T^\vee$ is surjective. In particular, the inclusion of W in V induces the restriction map $\mathrm{res}_{V/W}\, V^\vee \to W^\vee$, given by just restricting a linear functional of V on W. This exercise shows that $\mathrm{res}_{V/W}$ is surjective.

4.5.5. Let V be a finite dimensional vector space and let W be a vector subspace of V. Let $\mathrm{Ann}(W)$ be the following linear subspace of $V^\vee$:

$$\{v^\vee \in V^\vee : v^\vee(w) = 0, \text{ for all } w \in W\}.$$

Show that the dimension of $\mathrm{Ann}(W)$ is equal to $\dim_F(V) - \dim_F(W)$.

4.5.6. Let $T \in \mathrm{Hom}_F(V, V)$ be a linear operator. Show that there exists a polynomial $P \in F[X]$ such that $P(T) = 0$. Let $m_T(X)$ be the monic minimal polynomial such that $m_T(X) = 0$. Let $Q \in F[X]$ be any polynomial such that $Q(T) = 0$. Then show that m_T devides Q.

4.5.7. Let V be a finite dimensional vector space of dimension n. Let $T \in \mathrm{Hom}_F(V, V)$ be a linear operator such that $T^n = 0$ and $T^{n-1} \neq 0$. Compute the rank of following linear operators ϕ_1 and ϕ_2 defined on the vector space $\mathrm{Hom}_F(V, V)$:

$$\Phi_1(S) = TS, S \in \mathrm{Hom}_F(V, V),$$
$$\Phi_2(S) = TS - ST, S \in \mathrm{Hom}_F(V, V).$$

4.5.8. Let M be a matrix in $M_{n\times m}(F)$. Show that the dimension of the linear span of columns and the linear span of the rows is the same and it is equal to the rank of linear transformation defined by the matrix.

4.5.9. Let $A \in M_{n\times n}(F)$ and $B \in M_{m\times m}(F)$. Let $T : M_{n\times m}(F) \to M_{n\times m}(F)$ be the linear transformation sending $X \mapsto AXB$. Determine the rank of T as a function of rank of A and rank of B. Show that there exists a matrix C such that $ACA = A$.

4.5.10. Let $A \in M_{n\times m}(F)$. Let E be any field containing F. Show that the rank of $T_E : E^n \to E^m$ given by $X \mapsto AX^t$ is equal to the rank of A.

5. Lecture, Invariants of matrices I

In this lecture we will study the trace function.

5.1. **Trace.** Let V be a finite dimensional vector space and let $(v_1, v_2, \ldots, v_n)$ be a basis of V. Using this basis, we can identify $\mathrm{Hom}_F(V,V)$ via the map $\Phi(B,B)$ with $M_{n\times n}(F)$. Let

$$\mathrm{tr} : M_{n\times n}(F) \to F.$$

be the trace function given by the sum of diagonal entries of matrix. Recall that

$$\mathrm{tr}(AB) = \mathrm{tr}(BA), A, B \in M_{n\times n}(F).$$

Thus we get a linear functional on the vector space $\mathrm{Hom}_F(V,V)$ via the composition of the maps:

$$\mathrm{tr} : \mathrm{Hom}_F(V,V) \xrightarrow{\Phi(B,B)} M_{n\times n}(F) \xrightarrow{\mathrm{tr}} F.$$

Thus the map $\mathrm{tr} : \mathrm{Hom}_F(V,V) \to F$ has the property that $\mathrm{tr}(TS) = \mathrm{tr}(ST)$, for all S and T in the space $\mathrm{Hom}_F(V,V)$.

Now, one may ask what is so special about this trace function. Indeed, the above trace function can depend on the choice of a basis. Now, we ask whether we can construct any other linear functionals $f : \mathrm{Hom}_F(V,V) \to F$ such that $f(TS) = f(ST)$, for any $T, S \in \mathrm{Hom}_F(V,V)$? It turns out that one cannot get any new linear functional other than scaling the trace function and we prove this in the following theorem.

Theorem 5.1. *Let V be a finite dimensional vector space over a field F and let $f : \mathrm{Hom}_F(V,V) \to F$ be a linear functional such that*

$$f(gTg^{-1}) = f(T)$$

for all $g \in \mathrm{GL}_F(V)$ and $T \in \mathrm{Hom}_F(V,V)$. Then, there exists a constant $c \in F$ such that $f = c\mathrm{tr}$.

Before, we prove this theorem, we recall another interesting result due to Burnside.

5.2. **Burnside's theorem.** Before, you see the proof it is instructive to prove the theorem for 2×2 matrices.

Theorem 5.2 (Burnside)**.** *Let V be a finite dimensional vector space over a field F. The linear span of the group $\mathrm{GL}_F(V)$ in the space $\mathrm{Hom}_F(V,V)$ is the whole space.*

We prove this theorem with a sequence of lemmas.

Lemma 5.3. *Any linear operator $T : V \to V$ can be written as a sum of rank one operators*

Proof. Let $(v_1, v_2, \ldots, v_n)$ be a basis and let $w_i = T(v_i)$, for $i \in [n]$. Now, define $T_i : V \to W$ to be the linear transformation defined by setting $T(v_i) = w_i$ and $T(v_j) = 0$, for $i \neq j$. Clearly the Linear operators T_i, for $i \in [n]$ has dimension one and $T = \sum_{i=1}^{n} T_i$. □

Lemma 5.4. *The linear span of $\mathrm{GL}_F(V)$ contains a rank one operators*

Proof. Let $(v_1, v_2, \ldots, v_n)$ be a basis of V. Now consider the operator $T(v_1) = v_1 + v_2$ and $T(v_i) = 0$, for $1 < i \leq n$. Consider the operator $T_1(v_1) = v_1 + v_2$, and $T_1(v_i) = v_i$, for $1 < i \leq n$. Then we get that $T_1 - \mathrm{id} = T$. □

Proof of theorem 5.2. Let W be the space $\langle \mathrm{GL}_F(V) \rangle$. Using lemma , there exists a rank one operator $T \in W$. Since the space W is stable under the left and right action of $\mathrm{GL}_F(V)$, we can now show that W contains all rank one operators. Hence, $W = \mathrm{Hom}_F(V,V)$. □

Proof of 5.1. Let $f : \mathrm{Hom}_F(V,V) \to F$ be a function such that $f(gT) = f(Tg)$, for all $g \in \mathrm{GL}_F(V)$ and $T \in \mathrm{Hom}_F(V,V)$. Then, using 5.2, we get that $f(ST) = f(TS)$, for all $S, T \in \mathrm{Hom}_F(V,V)$. This shows that $\langle TS - ST : T, S \in \mathrm{Hom}_F(V,V) \rangle$ is contained in the kernel of f. Hence, we get that $f = c\mathrm{tr}$, for some constant $c \in F$. □

5.3. **Exercises.**

5.3.1. Let A be a matrix in $\mathrm{M}_{n\times n}(F)$ and let $T_A : \mathrm{M}_{n\times n}(F) \to \mathrm{M}_{n\times n}(F)$ be the operator defined by $M \mapsto AM$. What is the trace of the operator T_A.

5.3.2. Let V_n be the space of homogenous polynomials of degree n in 2-variables. Let D_n be the linear operator $x\dfrac{\partial}{\partial y} - y\dfrac{\partial}{\partial x}$ acting on the space V_n. Let $f(n) = \mathrm{tr}(D_n)$, for $n \in \mathbb{N}$. Compute the function $f(n)$. Replace the operator D_n by the operator $-y^2\left\{\dfrac{\partial^2}{\partial^2 x} + \dfrac{\partial^2}{\partial^2 y}\right\}$ and compute the function $f(n)$.

5.3.3. The properties of trace function show that the trace function is trivial on linear transformations of the form $AB - BA$, where $A, B \in \mathrm{Hom}_F(V, V)$. For instance the relation $AB - BA = \mathrm{id}$ is impossible for any $A, B \in \mathrm{Hom}_F(V, V)$. But this relation $AB - BA = \mathrm{id}$ is the first quantisation. Can you find examples of such linear operators A, B with $AB - BA = \mathrm{id}$, when $F = \mathbb{F}_p$ and $F = \mathbb{C}$. When $F = \mathbb{F}_p$ can you find A and B on a finite dimensional vector space?

5.3.4. Let V be a finite dimensional vector space and let $f : \mathrm{Hom}_F(V, V) \to F$ be a linear functional. Show that there exists a matrix $T \in \mathrm{Hom}_F(V, V)$ such that $f(S) = \mathrm{tr}(TS)$, for $S \in \mathrm{Hom}_F(V, V)$.

6. Lecture

6.1. **General linear group.** Let V be a finite dimensional vector space of dimension n. Given any invertible linear transformation T, and a basis $(v_1, v_2, \ldots, v_n)$ of V, the tuple $(T(v_1), T(v_2), \ldots, T(v_n))$ is another basis of V. Let $B_1 = (v_1, v_2, d \ldots, v_n)$ and $B_2 = (w_1, w_2, \ldots, w_n)$ be two basis of V, there exists a unique invertible linear transformation T such that $T(v_i) = w_i$, for $i \in [n]$. Given two linear transformations T and S, T sends a fixed basis B_1 to B_2 and say S sends a basis B_2 to B_3:

$$B_1 \xrightarrow{T} B_2 \xrightarrow{S} B_3; B_1 \xrightarrow{ST} B_3, B_3 \xrightarrow{TS} B_1.$$

Let $\mathcal{B}$ be a set of basis of V. Then the set $\mathcal{B}$ is in bijection with the set of invertible linear transformations of V, to be denoted by $\mathrm{GL}_F(V)$. To define this bijection, fix a basis $B = (v_1, \ldots, v_n) \in \mathcal{B}$ and set

$$\mathrm{GL}_F(V) \to \mathcal{B}; T \mapsto (T(v_1), \ldots, T(v_n)).$$

The set $\mathrm{GL}_F(V)$ is called the general linear group. If we fix a basis $B = (v_1, \ldots, v_n)$ of V, then we can identify $\mathrm{GL}_n(F)$ with set of invertible matrices in $M_{n\times n}(F)$. Note that $\mathrm{GL}_F(V)$ is a subset of $\mathrm{Hom}_F(V, V)$, and we have

$$\Phi(B, B) : \mathrm{Hom}_F(V) \to M_{n\times n}(F).$$

Clearly, we have $\Phi(B, B)(TS) = \Phi(B, B)(T)\Phi(B, B)(S)$. This shows that the linear transformation is invertible if and only if the matrix $\Phi(B, B)(T)$ is invertible.

6.2. **Rudimentary Group theory and Group actions.** Groups occur in nature as set of permutations of objects or as automorphisms of some structures. Like the set of invertible linear transformations are set of automorphisms of vector spaces. Here is the official definition

Definition 6.1. *A set G together an map (also called multiplication operation)$* : G \times G \to G$ ($*(g_1, g_2)$ is denoted by $g_1 * g_2$) is called a group if the following axioms are satisfied:*

1. *There exists an element $e \in G$ such that $e * g = g * e = g$,*
2. *$g_1 * (g_2 * g_3) = (g_1 * g_2) * g_3$,*
3. *For any $g \in G$, there exists a $g^{-1} \in G$ such that $gg^{-1} = g^{-1}g = e$.*

In this set of lectures, we will study many such groups which arise naturally in linear algebra. The first one being the group $\mathrm{GL}_F(V)$, where V is a vector space over F. Given two groups $(G_1, *_1)$ and $(G_2, *_2)$, a homomorphism between them is a map $\phi : G_1 \to G_2$ such that $\phi(g *_1 h) = \phi(g) *_2 \phi(h)$, $g, h \in G_1$. A homomorphism ϕ is an isomorphism if and only if there exists a $\psi : G_2 \to G_1$ such that the compositions $\phi\psi$ and $\psi\phi$ are both identity maps. Here the multiplication operation is given by composition of linear transformations or matrix multiplication.

Given any set X, the set of automorphisms of X, as a set, is also a group where multiplication operation is given by composition of automorphisms. The group of automorphisms of a set X is denoted by $\mathrm{Aut}(X)$. An action of a group G on a set X is a homomorphism $\rho : G \to \mathrm{Aut}(X)$, i.e., for any $g \in G$, there exists an automorphism $\rho(g)$ on the set X. When the situation is clear, we use gx for $\rho(g)(x)$, for $g \in G$ and $x \in X$. An action of a group G on a set X is said to be transitive if for any $x, y \in X$, there exists a $g \in G$ such that $gx = y$.

Example 6.2. Let X be the set $\{1, 2, 3 \ldots, n\}$. The group $\mathrm{Aut}(X)$ is the denoted by S_n. Clearly the cardinality of S_n is $n!$. Let $\{X_1, X_2, \ldots, X_n\}$ be a set of n-variables. We set

$$D(X_1, X_2, \ldots, X_n) = \prod_{i<j}(X_i - X_j).$$

The polynomial D is often called as the discriminant polynomial. Let $\sigma \in S_n$. Now

$$D(X_{\sigma(1)}, \ldots, X_{\sigma(n)}) = \mathrm{sign}(\sigma)D(X_1, X_2, \ldots, X_n),$$

where $\mathrm{sign}(\sigma) \in \{\pm 1\}$. The set $\{\pm 1\}$ is a group of 2-elements under multiplication, and 1 is the identity element. The map

$$S_n \to \{\pm 1\}; \sigma \mapsto \mathrm{sign}(\sigma)$$

is clearly a surjective homomorphism. This is called the *sign character*.

Example 6.3. *The group G acts on itself via left or right multiplication. A group G acts on itself by setting $g \mapsto \phi_g$, where $\phi_g(h) = ghg^{-1}$. This later action is called as conjugation action.*

Let X be a set and let G be any group acting on X. We can define an equivalence relation on X by setting $x_1 \simeq x_2$ if there exists an $g \in G$ such that $gx_1 = x_2$. Note that if G acts transitively on X then there is just one equivalence class. Otherwise the set X decomposes into disjoint union of sets

$$X = \coprod_{\alpha \in I} X_\alpha$$

The group G acts transitively on each of the sets X_α, for $\alpha \in I$.

These abstract notaions will help us in understanding classical results in a better way.

6.3. **Exercises.**

6.3.1. Let V be any finite dimensional vector space and W be any subspace of V. Show that there exists an operator $p : V \to V$ such that $p^2 = p$ and $p(V) = W$. The operator p is called the projection operator with image W. Let $\mathcal{P}_W$ be the space of projection operators corresponding to the subspace W. Show that the elements of $\mathcal{P}_W$ are in one to one correspondence with the set of subspaces W' such that $W' \cap W = \{0\}$ and $W' + W = V$.

6.3.2. Let V be a finite dimensional vector space of dimension n over a field F. Let $0 \leq k \leq n$ be an integer and let $\mathrm{Gr}(k, V)$ be the set of k-dimensional subspaces of V. Show that the action of the group $\mathrm{GL}_F(V)$ on the space V induces a transitive action of $\mathrm{GL}_F(V)$ on $\mathrm{Gr}(k, V)$.

7. Lecture

7.1. **Row and Column operations.** Let us begin with two questions which will ultimately be related.

Question 7.1. *Let V be an n-dimesional vector space over a field F. Let $\mathrm{Gr}(k,V)$ be the set of k-diemensional subspaces of V. Can we parametrise the set $\mathrm{Gr}(k,V)$?*

Question 7.2. *Given a linear transformation $T : F^n \to F^m$, can we produce a basis for the kernel of the linear transformation T.*

For any matrix $A \in M_{m\times n}(F)$, we denote by $T_A : F^m \to F^n$ the linear transformation $T_A(X) = AX^t$, where $X \in M_{1\times n}(F)$.

Lemma 7.3. *Any subspace W of F^n is the kernel of the linear transformation T_A, for some matrix $A \in \mathrm{M}_{n\times n}(F)$.*

Proof. Let $(w_1, w_2 \ldots, w_k)$ be a basis of W, and extend this basis to a basis $(w_1, w_2, \ldots, w_k, w_{k+1}, \ldots, w_n)$, of V. Now set $T : F^n \to F^n$ to be the linear transformation $T(w_i) = 0$ for $i \in [k]$ and $T(w_i) = w_i$ for $i \geq k+1$. Note that the kernel of T is equal to W. □

Lemma 7.4. *Let $T_1, T_2 : V \to W$ be two linear transformations such that $\ker(T_1) = \ker(T_2)$. Then there exists an invertible matrix $S : W \to W$ such that $T_1 = ST_2$. In terms of matricies the solutions to the system $A_1X = 0$ and $A_2X = 0$ are the same if and only if there exists a matrix $P \in \mathrm{GL}_m(F)$ such that $A_1 = PA_2$.*

Proof. Let $(w_1, w_2, \ldots, w_k)$ be a basis of $\ker(T_1) = \ker(T_2)$. Then extend this basis to a basis $(w_1, w_2, \ldots, w_n)$ of V. Note that

$$(T_1(w_{k+1}), \ldots, T_1(w_n)), \text{ and } (T_2(w_{k+1}), \ldots, T_2(w_n))$$

are two linearly independent subsets of W. Hence, there exists an invertible linear transformation $S : W \to W$ such that $T_1 = ST_2$. □

We say two lineat transformations $T_1, T_2 : V \to W$ are equivalent if and only if there exists a matrix $S \in \mathrm{GL}_F(W)$ such that $T_1 = ST_2$. Similarly two matricies $A_1, A_2 \in M_{m\times n}(F)$ are said to be similar if and only if there exists an invertible matrix $P \in \mathrm{GL}_m(F)$ such that $PA_1 = A_2$. We will now use some simple invertible matricies to simply a matrix A. We begin with recalling of elementary matricies:

(1) Let $i \neq j \in [m]$, and let $c \in F^\times$. We denote by $e_{ij}(c)$ by the matrix whoose only non-zero entry is in the ij-th position and is equal to c. Let $E_{ij}(c) = \mathrm{id} + e_{ij}(c)$. We denote by $R_i(A)$ the i-th row of the matrix A. Note that $R_i(E_{ij}(c)A) = R_i(A) + cR_j(A)$.
(2) For any $i \in [n]$, and let $c \in F^\times$. We denote by $e_{ii}(c)$ by the matrix whoose only non-zero entry is in the ii-th position and is equal to c. Let $E_{ij}(c) = \mathrm{id} + e_{ij}(c-1)$. Note that $R_i(E_{ii}(c)A) = cR_i(A)$.
(3) Let $i \neq j \in [m]$, and let $w_{i,j}$ be the permutation matrix M which interchages two rows R_i and R_j of matricies, i.e., $R_i(w_{i,j}A) = R_j(A)$ and $R_j(w_{i,j}A) = R_i(A)$ and $R_k(W_{ij}A) = R_k(A)$, for any $k \notin \{i, j\}$.

Any matrix of the above form is called an elementary matrix.

Definition 7.5. *A matrix $M \in M_{n\times m}(F)$ is said to be in row reduced echelon form if and only if*

(1) If the j-th row of a matrix M is zero then the k-th row of the matrix M is zero for all $k > j$.
(2) If the i-th row is non-zero then, the first non-zero entry (called pivot) in the i-th row is equal to 1.
(3) If the $i+1$-th row is non-zero, the pivot is to the right of the pivot of the i-th row.

Let M be a row reduced echelon matrix such that $R_k(A) = 0$ and $R_{k-1}(A) \neq 0$. Let $S = n_1 < n_2 < \cdots < n_k$ be the position of pivots in $R_i(A)$, for $i \in [k]$. Note the system $MX = 0$ is given by the system of equations

$$x_{n_i} + \sum_{\substack{j>n_i,\\ j\notin S}} a_{ij}x_j = 0,\ i \in [k].$$

Hence the null space of M is of the form

$$\{((x_i : i \notin S), (-\sum_{\substack{j>n_i,\\ j\notin S}} a_{ij}x_j) : i \in S)) : x_i \in F, \forall i \notin S\}. \tag{7.1}$$

Hence this is the graph of the linear transformation $T : F^{n-|S|} \to F^{|S|}$, whose matrix M' is given by $(a_{ij})_{i\in S, j\notin S}$. Hence if M_1 and M_2 are two matricies in row reduced echelon form and if $\ker(M_1) = \ker(M_2)$, then clearly the equality (7.1) shows that $M_1 = M_2$. This discussion gives us the following lemma.

Lemma 7.6. *There exists a sequence of elementary matricies $\{E_i\}_{i=1}^{l}$ such that $(\prod_{i=1}^{l} E_i)M$ is a row ewduced echelon matrix.*

The group $\mathrm{GL}_F(W)$ acts on the set $\mathrm{Hom}_F(V, W)$ via the left multiplication $\rho(g)(T)$ is the composition gT. Similarly the group $\mathrm{GL}_m(F)$ acts on $M_{m\times n}(F)$ via the equality $\rho(g)(M) = gM$, for all $g \in \mathrm{GL}_m(F)$ and $M \in \mathrm{M}_{m\times n}(F)$.

Lemma 7.7. *Every equivalence class for the action $\rho : \mathrm{GL}_m(F) \to \mathrm{Aut}(\mathrm{M}_{m\times n}(F))$ contains a unique matrix in the row reduced echelon form.*

8. Lecture: Bilinear forms I

8.1. Let V, W and N be vector spaces over a field F and let $B : V \times W \to N$ be a function. The function B is called *bilinear* if

$$B(av_1 + bv_2, w) = aB(v_1, w) + bB(v_2, w), v_1, v_2 \in V, w \in W \tag{8.1}$$

$$B(v, aw_1 + bw_2) = aB(v, w_1) + bB(v, w_2), w_1, w_2 \in W, v \in V. \tag{8.2}$$

Given any bilinear form B, we may define two maps

$$l_B : W \to \mathrm{Hom}_F(V, N); l_B(v) = B(\ , w), w \in V. \tag{8.3}$$

and

$$r_B : V \to \mathrm{Hom}_F(W, N); l_B(v) = B(v,\ ), w \in V. \tag{8.4}$$

The form B is called left (resp. right) *non-degenerate* or *non-singular* if and only if $\ker(l_B) = 0$ (resp. $\ker(r_B) = 0$). If $V = W$, $N = F$ and if V is finite dimensional, if B is left (resp. right) non-degenerate, then by rank nullity theorem, we get that l_B (resp. r_B) is an isomorphism.

Consider any vector space V and let $\mathrm{ev} : V \to (V^\vee)^\vee$ be the map evaluation map, i.e., $\mathrm{ev}(v)(l) = l(v)$, for any $l \in V^\vee$.

Lemma 8.1. *Given any non-zero vector $v \in V$, there exists a linear functional $l \in V^\vee$ such that $l(v) \neq 0$.*

Proof. This lemma depends on the proof of existence of a basis. Let $S \subset V$ be a basis for V. We may assume that $v \in S$. The required linear functional can be constructed as a unique linear functional which extends the function which takes v to 1 and $v' \in S$ and $v' \neq v$ to zero. □

The kernel of the map ev is the following subspace of V

$$\{v \in V : l(v) = 0, \forall l \in V^\vee\}.$$

From the above lemma, we get that the above space is the trivial vector space. Hence, *ev* is always an injective map. If V is finite dimensional vector space, then *ev* is an isomorphism.

Proposition 8.2. *Let V be a finite dimensional vector space and let $B : V \times V \to F$ be a bilinear form. The form B is left non-degenerate if and only if it is right non-degenerate.*

Lemma 8.3. *The linear transformation $r_B : V \to V^\vee$, is the composite of the maps*

$$V \xrightarrow{\mathrm{ev}} (V^\vee)^\vee \xrightarrow{l_B^\vee} V^\vee.$$

Proof. Note that $r_B(v) \in V^\vee$, for any $v \in V$. Hence for $w \in V$, we have $r_B(v)(w) = B(v, w)$. Note that $l_B^\vee(\mathrm{ev}(v)) \in V^\vee$, now take a vector $w \in V$. Then we have

$$l_B^\vee(\mathrm{ev}(v))(w) = \mathrm{ev}(v)(B(\ , w)) = B(v, w) = r_B(v)(w), v, w \in V.$$

□

Lemma 8.4. *Let V be a finite dimensional subspace and let $B : V \times V \to F$ be a bilinear form. Then we have* $\mathrm{rank}(l_B) = \mathrm{rank}(r_B)$.

Proof. Thus, we get that $\mathrm{rank}(r_B)$ is equal to $\mathrm{rank}(l_B^\vee \mathrm{ev})$, and since ev is an isomorphism for finite dimensional vector spaces, we get that rank of r_B and rank of $l_B^\vee$ is the same. Since rank of $T^\vee$ and $T^\vee$ is the same for any linear transformation $T : V \to W$, for two finite dimensional vector spaces V and W over the field F, we get that $\mathrm{rank}(r_B)$ and $\mathrm{rank}(l_B)$ are the same. □

Proof of proposition. The proof of the proposition is now a corollary of the above lemma. □

From now we will simply call a left or right non-degenerate bilinear forms as non-degenerate bilinear form.

8.2. **Bilinear forms and basis.** Let V be a finite dimensional vector space over a field F and let $B : V \times V \to F$ be a bilinear form on V. Let $B = (v_1, v_2, \ldots, v_n)$ be a basis of V. Let $X = (x_1, \ldots x_n)^t$ and $Y = (y_1, \ldots, y_n)$ be the coordinates of two vectors v and w respectively; we consider X and Y as column vectors. This means that

$$v = x_1v_1 + x_2v_2 + \cdots + x_nv_n,\ w = y_1v_1 + y_2v_2 + \cdots + y_nv_n.$$

Then we have

$$B(v, w) = X^tMY,$$

where $M \in M_{n\times n}(F)$ is the matrix $M = (B(v_i, v_j))$. Sometimes, if X and Y are coordinate vectors of v and w, then $B(X, Y)$ will be used instead of $B(v, w)$. The matrix M is called the matrix of the bilinear form with respect to the basis $(v_1, v_2, \ldots, v_n)$.

Let $(w_1, w_2, \ldots, w_n)$ be another basis of the vector space V. Let $P = (p_{ij})$ be the change of basis matrix, i.e.,

$$v_i = p_{i1}w_1 + p_{i2}w_2 + \cdots + p_{in}w_n,\ 1 \leq i \leq n.$$

Let X' and Y' be the coordinates of v and w in the new basis $(w_1, w_2, \ldots, w_n)$. Then we have $X' = P^tX$ and $Y' = P^tY$. Thus, we have $B(X', Y') = XPMP^tY$. Note that the matrix of the bilinear form B in the new basis $(w_1, w_2, \ldots, w_n)$ is given by PMP^t.

8.3. Let V be a finite dimensional vector space over a field F. A bilinear form $B : V \times V \to F$ is called symmetric if and only if $B(v, w) = B(w, v)$, for all $v, w \in V$. The form B is called anti-symmetric if and only if $B(v, v) = 0$, for all $v \in V$. If characteristic of F is not equal to 2, then B is anti-symmetric form if and only if $B(v, w) = -B(v, w)$, for all $v, w \in V$. Given any bilinear form $B : V \times V \to F$, we can write $B = B_s + B_a$, where B_s and B_a are symmetric and anti-symmetric bilinear forms on V. If characteristic of F is not equal to 2, we may set

$$B_s(v, w) = \frac{1}{2}\{B(v, w) + B(w, v)\}$$

and

$$B_s(v, w) = \frac{1}{2}\{B(v, w) - B(w, v)\}.$$

In the general case, we may define a anti-symmetric form from a symmetric form. To do this, first if $B : V \times V \to F$ is a bilinear form, then $B^t : V \times V \to F$ be the form $B^t(v, w) = B(w, v)$. We define $T^2(V)$, the space of bilinear forms on V. Let $\mathrm{Sym}^2(V) \subset T^2(V)$ be the vector space of symmetric bilinear forms on V, and let $\mathrm{Alt}^2(V) \subset T^2(V)$ be the space of anti-symmetric bilinear forms on V. We define the natural map

$$S : T^2(V) \to \mathrm{Alt}^2(V)$$

by setting $B \mapsto B - B^t$. Note that the kernel of the map S is the space $\mathrm{Sym}^2(V)$. The image of S is contained in the space of anti-symmetric forms on V.

Lemma 8.5. *The image of the map S is equal to $\mathrm{Alt}^2(V)$.*

Proof. If $B : V \times V \to F$ is an anti-symmetric bilinear form then $B(v, w) + B(w, v) = 0$. If we fix $(v_1, v_2, \ldots, v_n)$ as a basis for the vector space V, then if B is alternating form, then in this basis the matrix of the bilinear form, M, is given equal to $L - L^t$, where L is some upper triangular matrix with all its entries zero. Now if we set B' to be a bilinear form on V whose matrix with respect to the basis $(v_1, \ldots, v_n)$ is L, then we get that $B = B' - (B')^t$. Thus the image of the map S is equal to $\mathrm{Alt}^2(V)$. □

Now, if characteristic of F is equal to 2, then we have $S^2 = 0$, thus, $\mathrm{img}(S) \subseteq \ker(S)$. Thus the span of $ker(S) + \mathrm{img}(S)$ is equal to $\ker(S)$. Thus, we cannot write every form as a sum of alternating plus symmetric bilinear forms.

8.4. **Null spaces.** Let V be a finite dimensional vector space and let $B : V \times V \to F$ be a bilinear form on V. Let $\mathrm{Nil}_r(B)$ be the space $\{v \in V : B(v, V) = 0\}$. and let $\mathrm{Nil}_l(B)$ be the space $\{v \in V : B(V, v) = 0\}$. Note that $\mathrm{Nil}_l(B)$ and $\mathrm{Nil}_r(B)$ are the kernels of l_B and r_B respectively.

Lemma 8.6. *Let V be any finite dimensional vector space over a field F and let $B : V \times V \to F$ be a bilinear form. There exists a subspace W such that $V = \mathrm{Nil}_l(B) \oplus W$ (resp. $V = \mathrm{Nil}_r(B) \oplus W$) such that $B : W \times W \to F$ is left (resp. right) non-degenerate*

Proof. Let $(v_1, v_2, \ldots, v_k)$ be a basis for the space $\mathrm{Nil}_l(B)$. We can extend this basis to a basis

$$(v_1, v_2, \ldots, v_k, \ldots, v_n)$$

for the space V. Let W be the span of the vectors $\{v_{k+1}, \ldots, v_n\}$. Note that $V = \mathrm{Nil}_l \oplus W$. Assume that $w \in W$ such that $B(W, w) = 0$, then $B(V, w) = 0$. Thus, $w = 0$. This implies that W is left non-degenerate. The right non-degenerate case is similar. Please note that the space W is not necessarily unique. □

A *non-degenerate* subspace of V is a subspace W such that the form $B : W \times W \to F$ is non-degenerate.

8.5. **Exercises.** Let V be a finite dimensional vector space over a field F. Let $B : V \times V \to F$ be a bilinear form on V. Let $(v_1, v_2, \ldots, v_n)$ be a basis of V, and let $M \in M_{n\times n}(F)$ be the matrix of the bilinear form B with respect to this basis. Show that the matrix of the linear transformations $l_B : V \to V^\vee$ and $r_B : V \to V^\vee$ with respect to the basis $(v_1, v_2, \ldots, v_n)$ and $(v_1^\vee, v_2^\vee, \ldots, v_n^\vee)$ is equal to M and M^t respectively.

9. Lecture, Hyperbolic spaces and Witt-Decomposition

The pair (V, B) consisting of a finite dimensional vector space over a field F and a bilinear form $B : V \times V \to F$ will be called as *bilinear space*. For simplicity we assume that the characteristic of F is not equal to 2.

9.1. **Isometries.** Let (V_1, B_1) and (V_2, B_2) be two bilinear spaces over a field F, a map of bilinear spaces is a linear transformation $T : V_1 \to V_2$ such that $B_1(v, w) = B_2(T(v), T(w))$, for all $v, w \in V$. Two bilinear spaces (W_1, B_1) and (W_2, B_2) are isomorphic if there is an invertible map of bilinear spaces between them. An *orthogonal transformation* is a linear map from (V, B) to itself. An *isometry* is an invertible orthogonal linear transformation. Note that any isometry of a bilinear form B preserves the forms B_s and B_a, given by $1/2(B + B^t)$ and $1/2(B - B^t)$ respectively. Hence, to understand isometries, we may concentrate on either symmetric or anti-symmetric bilinear forms. From now, we assume that $B : V \times V \to F$ is an ϵ-bilinear form on V, i.e.,

$$B(v, w) = \epsilon B(w, v), v, w \in V.$$

for some $\epsilon \in \pm 1$.

If T is an isometry of (V, B), then for any $v \in \mathrm{Nil}(B)$, then $T(v) \in \mathrm{Nil}(B)$. Let W be a non-degenerate vector subspace of V such that $V = \mathrm{Nil}(B) \oplus W$. Then $O(V, B)$ is isomorphic to the group

$$\begin{pmatrix} \mathrm{GL}_F(\mathrm{Nil}(B)) & \mathrm{Hom}_F(W, \mathrm{Nil}(B)) \\ 0 & O(W, B)). \end{pmatrix}$$

In the sense, that any isometry $T : V \to V$ takes $\mathrm{Nil}(B)$ to $\mathrm{Nil}(B)$ and $\mathrm{res}_W T = T_1 + T_2$, where $T_2 : W \to W$ is an isometry and $T_1 : W \to \mathrm{Nil}(B)$ is any linear transformation. Note that

$$B(w_1, w_2) = B(T(w_1), T(w_2)) = B(T_1(w_1) + T_2(w_1), T_1(w_2) + T_2(w_2)) = B(T_2(w_1), T_2(w_2)) = B(w_1, w_2).$$

Thus, to understand the structure of isometries we need to understand the part $\mathrm{O}(W, B)$.

9.2. **Bilinear forms and direct sums.** Let (W_1, B_1) and (W_2, B_2) be two bilinear spaces. We may define a bilinear form on $W_1 \oplus W_2$, to be denoted by $B_1 \oplus B_2$ as follows:

$$(B_1 \oplus B_2)((w_1, w_1'), (w_2, w_2')) = B_1(w_1, w_2) + B_2(w_1', w_2').$$

Let $(w_1, w_2, \ldots, w_n)$ is a basis of W_1 and the matrix of the bilinear form B_1 in this basis is M_1. Let $(w_1', w_2', \ldots, w_m')$ is a basis of W_2 and the matrix of the bilinear form B_2 in this basis is M_2. The basis of the form $B_1 \oplus B_2$ in the basis $((w_1, 0), (w_2, 0), \ldots, (w_n, 0), (0, w_1'), \ldots, (0, w_m')$ is given by the block matrix

$$\begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix}.$$

If (W_1, B_1) and (W_2, B_2) are both non-degenerate bilinear forms then, the bilinear space $(W_1 \oplus W_2, B_1 \oplus B_2)$ is also non-degenerate.

9.3. **Hyperbolic spaces.** Let (V, B) be an ϵ-bilinear form. A non-zero vector $v \in V$ is called *isotropic* if $B(v, v) = 0$, and a vector v is called *anisotropic* if $B(v, v) \neq 0$. If (V, B) is a non-degenerate ϵ-bilinear space, then we first prove some basic facts.

Lemma 9.1. *Let (V, B) be a non-degenerate ϵ-bilinear form and let W be a non-degenerate subspace. Then $V = W \oplus W^\perp$*

Proof. Note that $\mathrm{Nil}(W, B) = W \cap W^\perp$. Since $\mathrm{Nil}(W, B) = 0$, we get that $W \cap W^\perp = \{0\}$. Thus, by considering dimensions of W and $W^\perp$, we get that $V = W \oplus W^\perp$. □

Lemma 9.2. *Let (V, B) be a non-degenerate ϵ-bilinear form and let $v \in V$ be an isotropic vector. There exists an isotropic vector $v' \in V$ such that $B(v, v') = 1$.*

Proof. There exists a vector $w \in V$ such that $B(v, w) \neq 0$. Let $v' = w + cv$. If $\epsilon = -1$, then we can take $v' = w/B(v, w)$. When $\epsilon = 1$, then set $v_c = w + cv$. Now, we have

$$B(v_c, v_c) = B(w, w) + 2cB(w, cv).$$

We may choose c such that $B(v_c, v_c) = 0$. Note that $B(v, v_c) = B(v, w) \neq 0$. Thus, we may take $v' = v_c/B(v, w)$. □

Note that v and v' are linearly independent and the matrix of the bilinear form B restricted to the space $\langle v, v' \rangle$, with respect to the basis (v, v') is given by

$$H_\epsilon = \begin{pmatrix} 0 & 1 \\ \epsilon & 0 \end{pmatrix}.$$

A two dimensional vector space F^2 with the bilinear form $(X, Y) \mapsto X^t H_\epsilon Y$ is called *hyperbolic plane*, and is denoted by $\mathbb{H}_\epsilon$. Note that H_ϵ is an invertible matrix and hence the bilinear space $\mathbb{H}_\epsilon$ is non-degenerate subspace. The usual map from F^2 to $\langle v, v' \rangle$ given by

$$(x, y) \mapsto xv + yv'$$

is an invertible map of bilinear spaces $\mathbb{H}_\epsilon$ and $\langle v, v' \rangle$.

An *isotropic* subspace of a bilinear space (V, B) is a subspace W such that $W \subseteq W^\perp$. A subspace W is called *anisotropic* if for any non-zero vector $v \in V$, we have $B(v, v) \neq 0$. If (V, B) is non-degenerate, then $\dim(W) \leq \dim(V) - \dim(W)$, thus we get that $\dim(W) \leq \dim(V)/2$. There may exist isotropic subspaces W such that $\dim(W) = \dim(V)/2$. For example, the subspaces $\{(x, 0), x \in F\}$ and $\{(0, y), x \in F\}$ are isotropic subspaces of $\mathbb{H}_\epsilon$.

Lemma 9.3. *Let (V, B) be a non-degenerate ϵ-bilinear form over a field F, then there exists an anisotropic subspace V_a of V such that (V, B) is isomorphic to*

$$\mathbb{H}_\epsilon \oplus \cdots \oplus \mathbb{H}_\epsilon \oplus (V_a, B).$$

Proof. We prove this lemma using induction on the dimension of V. Assume that we proved the result for all non-degenerate ϵ-bilinear spaces of dimension k. Let (V, B) be a non-degenerate bilinear space of dimension $k + 1$. If all non-zero vectors of V are anisotropic, i.e., then V is an anisotropic space. If there exists an isotropic vector $v \in V$ then we can find a hyperbolic plane $\langle v, v' \rangle$ in V. Let W be the space $\langle v, v' \rangle^\perp$. Using induction hypothesis, we get that

$$W \simeq \mathbb{H}_\epsilon \oplus \cdots \oplus \mathbb{H}_\epsilon \oplus (W_a, B)$$

where W_a is an anisotropic subspace of V. Since an anisotropic subspace of W is also anisotropic subspace of V, we get that

$$V \simeq \mathbb{H}_\epsilon \oplus \cdots \oplus \mathbb{H}_\epsilon \oplus (V_a, B).$$

Thus, we prove the lemma using induction on the integer n. □

Lemma 9.4. *If (V, B) is a non-degenerate ani-symmetric bilinear space on an n-dimensional subspace V, then n is even.*

Proof. □

Definition 9.5. *Let (V, B) be a non-degenerate ϵ-bilinear space. A tuple of of $k+1$ subspaces of V, denoted by $(W_1, W_2, \ldots, W_{k+1})$, gives a Witt-decomposition if $(W_i, B) \simeq \mathbb{H}_\epsilon$ for $1 \leq i \leq k$, the space (W_{k+1}, B) is anisotropic,*

$$W = W_1 \oplus W_2 \oplus \cdots \oplus W_{k+1}$$

and $B(v, w) = 0$, for any $v_i \in W_i$ and $w \in W_j$ with $i \neq j$.

The above lemma shows that a Witt-decomposition always exists. Note that a Witt-decomposition is not unique for a bilinear form for example consider $V = \mathbb{R}^4$ and let M be the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Let $B : \mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{R}$ be the bilinear form $(X, Y) \mapsto X^t M Y$. Let e_1, e_2, e_3, e_4 be standard basis. Note that

$$\mathbb{R}^4 = \langle e_1, e_2 \rangle \oplus \langle e_3, e_4 \rangle$$

and

$$\mathbb{R}^4 = \langle e_1, e_3\rangle \oplus \langle e_2, e_4\rangle$$

are two different Witt-decompositions of $\mathbb{R}^4$. However the following is a fundamental theorem due to Witt. However, this is not the place to prove the theorem

Theorem 9.6 (E. Witt)**.** *If $(W_1, W_2, \ldots, W_{k+1})$ and $(U_1, U_2, \ldots, U_{l+1})$ are two tupes of subspaces giving two Witt-decompositions of a non-degenerate ϵ-bilinear space (V, B), then $k = l$ and*

$$(W_{k+1}, B) \simeq (U_{l+1}, B).$$

9.4. **Exercises.**

9.4.1. Let (V, B) be a non-degenerate ϵ-bilinear space such that

$$(V, B) \simeq \mathbb{H}_\epsilon \perp \cdots \perp \mathbb{H}_\epsilon.$$

Let W be any finite dimensional vector space and let $W^\vee$ be its dual space. Let $B_W : W \oplus W^\vee \to W \oplus W^\vee$ be the bilinear form

$$B_W((w_1, l_1), (w_2, l_2)) = l_1(w_2) + \epsilon l_2(w_1).$$

Show that $(V, B) \simeq (W \oplus W^\vee, B_W)$, for some vector space W.

9.4.2.

10. Lecture

In the previous lecture, we have seen various generalities on bilinear forms on finite dimensional vector spaces over arbitrary fields. In this lecture, we concentrate on the field $F = \mathbb{R}$. The field of real numbers are ordered by positivity, and the lengths are positive integers. We introduce positive definite bilinear forms, signature of a bilinear form on finite dimensional real vector spaces.

10.1. **Orthogonal basis and Gram schmidt process.** Let (V, B) be a pair consisting of a vector space voer an arbitrary field F and let $B : V \times V \to F$ be a non-degenerate symmetric bilinear form on V. We call a basis $(v_1, V_2, \ldots, v_n)$ as orthogonal basis if

$$B(v_i, v_j) = 0, i \neq j \in [n].$$

Lemma 10.1. *Let V be a finite dimensional vector space over a field of characteristic not equal to 2. For any symmetric non-degenerate bilinear form $B : V \times V \to F$, there exists an orthogonal basis for V with respect to the form B.*

Proof. For any non-zero vector v, there exists a vector w such that $B(v, w) \neq 0$. If $B(v, v) \neq 0$ or $B(w, w) \neq 0$, then we find a vector v' such that $B(v', v') \neq 0$. If $B(v, v) = B(w, w) = 0$, then we may consider $B(v + w, v + w) = 2B(v, w) \neq 0$. Hence, taking all possible cases into consideration, there exists a vector v such that $B(v, v) \neq 0$. Now, $V = \langle v \rangle \perp W$, where $\dim(W) = n - 1$. Using induction on the integer n, we get an orthogonal basis for the vector space with respect to the form B. □

10.1.1. *Gram–Schmidt process.* Let us consider a non-degenerate symmetric bilinear form $B : V \times V \to F$ such that $B(v, v) \neq 0$, for any $v \in V \backslash \{0\}$. As an example, we can take $F = \mathbb{F}_p$ with $p = 4k + 3$ and $B : \mathbb{F}_p^2 \times \mathbb{F}_p^2 \to \mathbb{F}_p$ is the form

$$B(x_1, y_1), (x_2, y_2)) = x_1x_2 + y_1y_2.$$

Or, take $F = \mathbb{R}$ and $B : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ a bilinear form given by setting

$$B(x_1, y_1), (x_2, y_2)) = -x_1x_2 - y_1y_2.$$

We describe a procedure to obtain an orthogonal basis. Let us begin with an arbitrary basis $(v_1, v_2, \ldots, v_n)$ for the vector space V. Let us consider $v_2' = v_2 - kv_1$,and we want

$$B(v_1, v_2') = 0.$$

Thus, $B(v_1, v_2) - kB(v_1, v_1) = 0$, and $k = -B(v_1, v_2)/B(v_1, v_1)$. Now we have a new basis $(v_1, v_2', \ldots, v_n)$ with $B(v_1, v_2') = 0$. We consider $v_3' = v_3 - k_1v_2' - k_2v_1$. We want $B(v_3', v_1) = 0$ and $B(v_3', v_2') = 0$. We get

$$-k_2B(v_1, v_1) - B(v_3, v_1) = 0, B(v_3, v_2') - k_1B(v_2', v_2').$$

We thus get a unique k_1, k_2 for which

$$B(v_1, v_2) = 0, B(v_2', v_3') = 0, B(v_3', v_1) = 0.$$

Thus, we have obtained an orthogonal basis $(v_1, v_2', \ldots, v_n')$ beginning with an arbitrary basis $(v_1, v_2, \ldots, v_n)$. If the field $F = \mathbb{R}$ and $B(v, v) \geq 0$, then we can also obtain a basis $(v_1, v_2, \ldots, v_n)$ for the vector space V such that $B(v_i, v_j) = \delta_{ij}$, for $i, j \in [n]$; and such a basis is called an orthonormal basis for the vector space V with respect to the form B.

10.2. **Signature of real bilinear forms.** Let V be a finite dimensional real vector space and let $B : V \times V \to \mathbb{R}$ be a non-degenerate symmetric bilinear form on V. In this section, we will try to classify bilinear forms up to isometry. Let $(v_1, v_2, \ldots, v_n)$ be an orthogonal basis for the vector space V, and we may further assume that $B(v_i, v_i) = \pm 1$. We may assume that

$$B(v_i, v_i) = 1, 1 \leq i \leq p,\ B(v_i, v_i) = -1, p + 1 \leq i \leq p + q = n.$$

The basis $(v_1, v_2, \ldots, v_n)$ will be called *normalised orthogonal basis* (this name is invented for the purposes of these lectures). The pair of integers (p, q) will be called the signature of the normalised orthogonal basis of V Thus, using the basis $(v_1, v_2, \ldots, v_n)$ the bilinear form is given by the map

$$(X, Y) \mapsto XI_{p,q}Y^{\mathrm{T}},$$

where $I_{p,q}$ is a matrix with all its non-diagonal entries equal to zero, the first p entries of diagonal are 1 and the rest of the diagonal entries are -1.

Theorem 10.2 (Sylvester's "law of inertia"[1])**.** *The signature of any normalised orthogonal basis is the same.*

Proof. Let $(v_1, v_2, \ldots, v_n)$ and $(w_1, w_2, \ldots, w_n)$ be two normalised orthogonal basis for the vector space V with respect to the form B. Assume that there exists v_i and w_j such that $B(v_i, v_i) = 1$ and $B(w_j, w_j) = 1$. By permuting the bases if necessary we may assume that $i = j = 1$. We then have

$$V = \langle v_1 \rangle \oplus \langle v_1 \rangle^{\perp} = \langle w_1 \rangle \oplus \langle w_1 \rangle^{\perp}.$$

If there exists an isometry $T : V \to V$ such that $T(v_1) = w_1$, then $\langle v_1 \rangle$ and $\langle w_1 \rangle$ are isometric. Moreover, $(v_2, v_3, \ldots, v_n)$ and $(w_2, w_3, \ldots, w_n)$ forms a basis for $\langle v_1 \rangle$ and $\langle w_1 \rangle$ respectively. Thus, we may use induction to deduce the proof of the theorem.

Thus, we need to construct an isometry $T : V \to V$ such that $T(v_1) = w_1$. This is the most important trick in the theory of bilinear forms and very geometric. Let us prove a more general lemma.

Lemma 10.3 (Rhombus lemma)**.** *Let V be a finite dimensional vector space over a field of characteristic not equal to 2. Let $B : V \times V \to F$ be a non-degenerate bilinear form and let v, w be two vectors such that $B(v, v) = B(w, w) \neq 0$. There exists an isometry $\tau : V \to V$ such that $\tau(v) = w$ and $\tau^2 = \mathrm{id}$.*

Proof. Let $v \in V$ be any vector such that $B(v, v) \neq 0$. A *reflection* along v is the following linear transformation:

$$\tau_v(w) = w - 2\frac{B(v, w)}{B(v, v)}, w \in V.$$

Note that $\tau_v(w) = 0$ for any $w \in \langle w \rangle^{\perp}$ and $\tau_v(v) = -v$. Note that $B(v + w, v + w)$ and $B(v - w, v - w)$ cannot both be zero simultaneously. We may assume that $B(v + w, v + w) \neq 0$. Note that $\tau_{v+w}(v) = w$. If $B(v - w, v - w) \neq 0$, then $\tau_{v-w}(v) = w$. This proves the lemma. □

The proof of the lemma completes the proof of theorem. □

Let $\mathrm{SO}(p, q)$ be the isometry group of the form $XI_{p,q}Y^{\mathrm{T}}$, i.e.,

$$\mathrm{SO}(p, q) = \{g \in \mathrm{GL}_{p+q}(\mathbb{R}) : gI_{p,q}g^{\mathrm{T}} = I_{p,q}\}.$$

The group $\mathrm{SO}(3, 1)$ is called Lorent's group and it is the fundamental group of symmetries in theory of special relativity.

10.3. **Positive definite forms.** Let $B : V \times V \to \mathbb{R}$ be a symmetric bilinear form such that $B(v, v) > 0$ for all $v \neq 0$. The signature of a positive definite form is obviously $(n, 0)$. We begin with an arbitrary bilinear form on F^n given by XMY^{T} and we would like to give a necessary and sufficient criterion using the minors of the matrix M for the positive definiteness of the form XMY^{T}.

Proposition 10.4. *Let M_k be the matrix obtained by deleting first k-left columns and first k-top rows of M. The form XMY^{T} is positive definite if and only if $\det(M_k) > 0$.*

Proof. aaa... □

10.4. **Exercises.**

10.4.1. *Iwasawa decomposition.* Show that any invertible matrix $g \in \mathrm{GL}_n(\mathbb{R})$ can be written as kb, where $k \in \mathrm{SO}_n$ and b is an upper triangular matrix.

10.4.2. Let V be a finite dimensional vector space with a non-degenerate anti-symmetric bilinear form $B : V \times V \to F$. Let F be a field of characteristic different from 2. Show that there exists vectors w_1 and w_2 in V such that $B(w_i, w_i) = 0$ and $B(w_1, w_2) = 1$. Show that there exists a basis for V such that the matrix of the bilinear form B is the matrix

$$\begin{pmatrix} 0 & \mathrm{id}_n \\ -\mathrm{id}_n & 1 \end{pmatrix}$$

and such a basis is called a *Witt basis*. This shows that $\dim_F(V)$ is even.

[1]this has nothing to do with the term "inertia" as used (or rather misused) in physics. It is inspired by Newton's law of inertia, which is a tautology. But it refers to the basic philosophy of finding invariants in mathematical structures.

10.4.3. Let $B_1 = (w_1, w_2, \ldots, w_n)$ and $B_2 = (w'_1, \ldots, w'_n)$ be two Witt-basis of a vector space V with respect to an anti-symmetric non-degenerate bilinear form B. Show that there exists an isometry of B such that $T(w_i) = w'_i$, for $i \in [2n]$.

10.4.4. *Optional.* Let F be a finite field of cardinality q. Find the cardinality of the set of Witt-basis of V with respect to some fixed non-degenerate anti-symmetric bilinear form B on V.

10.4.5. Let F be any field and let V be a finite dimensional vector space over F. Let $B : \mathrm{Hom}_F(V, V) \times \mathrm{Hom}_F(V, V) \to F$ be a bilinear form defined by setting

$$B(T, S) = \mathrm{tr}(TS).$$

Show that B is non-degenerate. If $F = \mathbb{R}$, then what is the signature of the form B. Assume that $F = \mathbb{R}$ and consider $b : V \times V \to F$ be a positive definite bilinear form and let T^* be the adjoint of T with respect to b. If we replace $B(T, S)$ with $\mathrm{tr}(TS^*)$, what is the signature of b. Let $\mathcal{N}$ be the set of nilpotent elements of $M_{2\times 2}(\mathbb{R})$. Show that $O(V, B)$ stabilizes $\mathcal{N}$. Calculate the orbits for the group $O(V, B)$ on $\mathcal{N}$. Can you generalise these for arbitrary n.

11. Lecture: Determinants

Let V be a finite dimensional vector space over a field F. Let n be the dimension of V. A function

$$B : \underbrace{V \times V \times \cdots \times V}_{k-\text{times}} \to F$$

is called multilinear form in k-variables if the map

$$B(v_1, \ldots, v_{i-1}, \ \_\_ \ , v_{i+1}, \ldots, v_k) : V \to F$$

is a linear functional, for all $i \in [k]$ and $v_1, v_2, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n \in V$. Given any tuple of k-linear functionals $S := (l_1, l_2, \ldots, l_k)$, for $l_k \in V^\vee$, we set

$$B_S(v_1, v_2, \ldots, v_k) = l_1(v_1) l_2(v_2) \ldots, l_k(v_k).$$

Note that B_S is a multilinear form in k-variables. A multilinear form $B : \underbrace{V \times \cdots \times V}_{k-\text{times}} \to F$ in k-variables is said to be symmetric if

$$B(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_k) = B(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_k),$$

for all $\neq j \in [k]$ and for all $v_1, \ldots, v_n \in V$. If we have

$$B(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_k) = 0, \text{ if } v_i = v_j$$

for all $i \neq j \in [k]$ and for all $v_1, \ldots, v_n \in V$, then B is called anti-symmetric. In these set of lectures, we only concentrate on anti-symmetric multilinear forms. However, other kinds of multilinear forms are very useful tools in representation theory and other parts of geometry. These forms are the basic examples of so called "tensors" as used in other natural sciences. Which will be discussed at a later stage in this notes.

Note that the set of multilinear forms in k-variables, denoted by $T^k V^\vee$, has a natural F-vector space structure, the addition and scalar multiplication are defined as follows:

$$(B_1 + B_2)(v_1, v_2, \ldots, v_k) = B_1(v_1, v_2, \ldots, v_k) + B_2(v_1, v_2, \ldots, v_k),$$

for $B_1, B_2 \in T^k(V)$, and $v_1, \ldots, v_k \in V$.

$$(cB)(v_1, v_2, \ldots, v_k) = cB(v_1, \ldots, v_k).$$

Let $S^k V^\vee$ and $\wedge^k V^\vee$ be the sets of symmetric and anti-symmetric multi-linear forms in k-variables respectively. Note that $S^k V^\vee$ and $\wedge^k V^\vee$ are vector subspaces of $T^k V^\vee$.

11.1. In this subsection, we are interested in the dimension of anti-symmetric multilinear forms in k-variables.

Proposition 11.1. *Let V be an n-dimensional vector space over a field F. The dimension of $\wedge^k V^\vee$ is equal to the zero vector space for $k > n$ and has dimension $\binom{n}{k}$, for $k \leq n$.*

Proof. Assume that $k > n$. Given any tuple of vectors $(v_1, v_2, \ldots, v_k)$, there exists a non-zero linear relation

$$a_1 v_1 + a_2 v_2 + \cdots a_k v_k = 0, a_i \in F, i \in [k].$$

Assume that $a_1 \neq 0$, and hence $v_1 = b_2 v_2 + \cdots b_k v_k$. This implies that $B(v_1, v_2, \ldots, v_k) = 0$, for all $v_1, v_2, \ldots, v_k \in V$. Thus B is the constant function zero. Now assume that $k \leq n$. Let $(v_1, v_2, \ldots, v_k)$ be a basis for the vector space V. Let $I = \{i_1 < i_2 < \cdots < i_k\}$ be any subset of $[n]$. Given any $B \in \wedge^k V^\vee$, we set M_B to be the tuple $(B(v_{i_1}, \ldots, v_{i_k}) : I \subseteq [n], |I| = k)$. Let $\psi : \wedge^k V^\vee \to F^{\binom{n}{k}}$ be the map defined by $\psi(B) = M_B$. Let

$$M = (m_I : I \subset [n] : |I| = k) \in F^{\binom{n}{k}}$$

be any tuple indexed by subsets of $[n]$, there exists a unique multi-linear form in k-variables, denoted by B, such that

$$B(e_{i_1}, e_{i_2}, \ldots, e_{i_k}) = m_I, I = \{i_1 < i_2, \ldots, i_k\}.$$

Lemma 11.2. *Given any tuple $(m_I : I \subseteq [n], |n| = k)$, indexed by subsets of $[n]$ of cardinality k, there exists a unique multi-linear form in k-variables $B : \underbrace{V \times \cdots \times V}_{} \to F$ such that*

$$B(e_{i_1}, e_{i_2}, \ldots, e_{i_k}) = m_I, I = \{i_1 < i_2, \ldots, i_k\}.$$

Proof. Let $v_1, \ldots, v_k$ be some k-elements in V. Say $X_i = (x_{i1}, \ldots, x_{in})$ be the coordinates for v_i, for $i \in [k]$. We then have

$$B(v_1, v_2, \ldots, v_k) = \sum_{\substack{I \subseteq [n] \\ I = \{i_1 < i_2 < \cdots < i_k\}}} \sum_{\sigma \in S_k} \operatorname{sign}(\sigma) x_{\sigma(1)i_1} \ldots x_{\sigma(k)i_k} B(e_{i_1}, \ldots, e_{i_k}). \tag{11.1}$$

Given any arbitrary values for $B(e_{i_1}, \ldots, e_{i_k})$, we get the required multi-linear form in k-variables using the above equation. The uniqueness is clear from construction. □

□

11.2. Let V be an n-dimensional vector space, given any two elements $B, B' \in \wedge^n V^\vee$, there exists a constant $c \in F$ such that $B = cB'$. Let $T : V \to V$ be a linear transformation, and let $B \in \wedge^n V^\vee$. We set $T(B) \in \wedge^n V^\vee$ by setting

$$T(B)(v_1, v_2, \ldots, v_n) = B(T(v_1), \ldots, T(v_n)).$$

Now, $B' \in \wedge^n V^\vee$, hence there exists a constant c_T depending on T such that $T(B) = c_T B$. We then have

$$c_T c_S = c_{TS}, T, S \in \operatorname{Hom}_F(V, V). \tag{11.2}$$

Note that constant c_T does not depend on the choice of B. Thus, we get a multiplicative map

$$c : \operatorname{Hom}_F(V, V) \to F.$$

Which will be called as determinant of T and denoted by $\det(T)$. Let $(v_1, v_2, \ldots, v_n)$ be a basis of V and let $T \in \operatorname{Hom}_F(V, V)$. Let $M = (m_{ij})$ be the matrix of the linear transformation T with respect to this basis. Then $\det(M)$ is equal to

$$\det(T) = \sum_{\sigma \in S_n} \operatorname{sign}(\sigma) m_{1\sigma(1)} m_{2\sigma(2)} \cdots m_{n\sigma(n)}. \tag{11.3}$$

The above formula follows from expanding

$$B(m_{11}v_1 + \cdots + m_{n1}v_1, \ldots, m_{n1}v_1 + \cdots + m_{nn}v_n)$$

as in (11.1) when $k = n$. Note that from our definition of determinant does not depend on the choice of a basis of V.

11.3. In this subsection, we want to apply the concept of multilinear froms in n-variables to give a formula for the inverse of a matrix. Here, we define the adjoin of a matrix M. Let $M = (m_{ij})$ be a matrix and M_{ij} be a matrix of size $(n-1) \times (n-1)$ obtained by deleting i-th column and j-th row. Let $M^{ad} = ((-1)^{i+j} M_{ij})^T$. Then we want to show that

$$MM^{ad} = \det(M)\,\mathrm{id}\,. \tag{11.4}$$

Before we prove this above result, let us demonstrate the method to solve certain system of linear equations using determinants. Say we have the following system:

$$\begin{aligned}
&a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = y_1 \\
&a_{21}x_1 + a_{22}x_2 + \cdots + a_{1n}x_n = y_2 \\
&\vdots \qquad\qquad\qquad \vdots \\
&a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = y_3.
\end{aligned}$$

Using the linearity of the determinant in each column, we get that

$$\det(M)x_i = \det \begin{pmatrix} a_{11} & \ldots & a_{1(i-1)} & y_1 & a_{1i+1} & \ldots & a_{1n} \\ a_{21} & \ldots & a_{2(i-1)} & y_2 & a_{2i+1} & \ldots & a_{2n} \\ \vdots & & \vdots & \vdots & \vdots & \vdots & \\ a_{n1} & \ldots & a_{n(i-1)} & y_n & a_{ni+1} & \ldots & a_{nn} \end{pmatrix}.$$

If we want to find the inverse of the matrix, then we need to solve the following system of equations

$$MX_i = Y_i,$$

where Y_i is equal to $(0, 0, \ldots, 1, \ldots, 0)^t$, where 1 is in the i-th position. If $X_1, X_2, \ldots, X_n$ are solutions of the system $MX = y_i$ respectively, then M^{-1} is equal to $[X_1X_2, \ldots, X_n]$. Let us set $X_i = (x_{1i}, x_{2i}, \ldots, x_{ni})$. Then we have

$$\det(M)x_{ji} = \det \begin{pmatrix} a_{11} & \ldots & a_{1(j-1)} & 0 & a_{1j+1} & \ldots & a_{1n} \\ a_{21} & \ldots & a_{2(j-1)} & 0 & a_{2j+1} & \ldots & a_{2n} \\ \vdots & & \vdots & \vdots & \vdots & \vdots & \\ a_{i1} & \ldots & a_{i(j-1)} & 1 & a_{ij+1} & \ldots & a_{2n} \\ \vdots & & \vdots & \vdots & \vdots & \vdots & \\ a_{n1} & \ldots & a_{n(j-1)} & 0 & a_{nj+1} & \ldots & a_{nn} \end{pmatrix} = (-1)^{i+j} M_{ij}. \tag{11.5}$$

The above equality is the same as the identity (11.4).

11.4. **Exercises.**

11.4.1. Let $A \in M_{n\times n}(F)$ be an invertible skew symmetric matrix, i.e., $A^t = -A$. Show that the determinant is equal to one.

11.4.2. Let X be the set of subspaces $V \subseteq M_{n\times n}(F)$ such that every non-zero matrix of V is invertible. Now let $d_F(n) = \max\{\dim(V) : V \in X\}$. What are the functions $d_{\mathbb{C}}(n)$, $d_{\mathbb{R}}(n)$[2] and $d_{\mathbb{Q}}(n)$.

11.4.3. Let A and B be two matricies in $M_{n\times n}(\mathbb{R})$ such that there exists a $P \in \mathrm{GL}_n(\mathbb{C})$ such that $PAP^{-1} = B$. Show that there exists a $Q \in \mathrm{GL}_n(\mathbb{R})$ such that $PAP^{-1} = B$.

[2]This is a hard question.

12. Lecture: Spectral theory of linear operators

Let V be a finite dimensional vector space over a field F, and let $T : V \to V$ be a linear transformation. To understand the geometry of a linear transformation, it is fundamental to determine its invariant subspaces.

12.1. An *invariant subspace* of a linear transformation $T : V \to V$ (or a T-invariant subspace) is a subspace W of V such that $T(w) \in W$, for all $w \in W$. Given a linear transformation $T : V \to V$, there may not be a non-zero invariant subspace of V: as an example consider the rotation transformation

$$R_\theta : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, x, y \in \mathbb{R}.$$

The linear transformation R_θ does not have invariant subspaces in $\mathbb{R}^2$. Let us consider another linear transformation $T : \mathbb{C}^2 \to \mathbb{C}^2$ given by

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, x, y \in \mathbb{C}$$

There is a unique one dimensional T-invariant subspace W of $\mathbb{C}^2$. Let W_1 and W_2 be two T-invariant subspaces of V such that $V = W_1 \oplus W_2$. Let $(w_1, \ldots, w_r)$ be a basis of W_1 and $(w_{r+1}, \ldots, w_n)$ be a basis of W_2. Then the matrix of the linear transformation T with respect to the basis $(w_1, w_2, \ldots, w_n)$ is of the form

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, A \in M_{r\times r}(F), B \in M_{(n-r)\times(n-r)}(F).$$

If $W_1, W_2, \ldots, W_n$ are n T-invariant subspaces of V with $\dim_F(W_i) = 1$, such that

$$V = W_1 \oplus W_2 \oplus \cdots \oplus W_n,$$

in other words there exists a basis for $(w_1, w_2, \ldots w_n)$ of V such that $w_i \in W_i$, then with respect to this basis the matrix of the linear transformation T is a diagonal matrix. Moroever, note that $Tw_i = \lambda_i w_i$. Since the diagonal matrix is the simplest form of a matrix, it is desired to find such a basis $(w_1, w_2, \ldots, w_n)$. However, the above two examples show that such a basis may not always exist. This lecture we shall study some general results about how far we can decompose a vector space into direct sum of invariant subspaces, if not one dimensional once.

12.2. **Eigenvalues and Eigenvectors.** A non-zero vector $v \in V$ is an eigenvector of a linear transformation $T : V \to V$ is called an *eigenvector* if $T(v) = \lambda v$, for some $\lambda \in V$. In other words $\langle v \rangle$ is an invariant subspace of V. Let $T : V \to V$ be a linear transformation and let M be the matrix of this linear transformation with respect to some basis B of V. The polynomial $\det(M - t\,\mathrm{id})$ is called the *characteristic polynomial* of T. Clearly, this polynomial does not depend on the chosen basis B, i.e., if B' is any other basis of B, then the matrix of the linear transformation T with respect to B' is equal to PMP^{-1}, where P is an invertible matrix with entries in F. Thus, $\det(M - t\,\mathrm{id})$ is equal to $\det(M' - t\,\mathrm{id})$; and we denote the polynomial $\det(M - t\,\mathrm{id})$ as $P_T(t)$.

A field F is called an algebraically closed field if for any polynomial $P(X) \in F[X]$, there exists an $\alpha \in F$ such that $P(\alpha) = 0$. For instance, it is a well known statement that $\mathbb{C}$ is an algebraically closed field, not perhaps a proof of this statement. Given any field F there exists an algebraically closed field F' containing F. If F is an algebraically closed field, then the characteristic polynomial $P_T(t)$ has a root say $\lambda \in F$. Then clearly $T - \lambda\,\mathrm{id}$ is not invertible linear transformation and hence there exists a non-zero vector v such that $T(v) = \lambda v$, for some $\lambda \in F$, whenever F is an algebraically closed field.

Let $M \in M_{n\times n}(F)$ be a matrix and let F' be a field containing F. Let $T : (F')^n \to (F')^n$ be a linear transformation defined by setting $X \mapsto MX$. If T has an eigenvector v, with eigenvalue $\lambda \in F$, then there exists an eigenvector v' for the linear transformation $T' : F^n \to F^n$ which sends $X \mapsto MX$. The reason for this lies in the simple fact that $\det(T - \mu\,\mathrm{id})$ and $\det(T' - \mu\,\mathrm{id})$ is equal to $\det(M - \mu\,\mathrm{id})$, for any $\mu \in F'$.

For any $\lambda \in F$, λ-*eigenspace* for T is the kernel of the linear transformation $T - \lambda\,\mathrm{id}$. A generalised λ-eigenspace for the linear transformation $T : V \to V$ is defined by setting

$$\{v \in V : (T - \lambda\,\mathrm{id})^{n(v)} v = 0, n(v) \in \mathbb{N}\}.$$

A generalised λ-eigenspace for T is denoted by V_λ, when T is clear from the context. Let us look at the following example

$$M = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Generalised 1-eigenspace of M is equal to $\langle e_1, e_2, e_3, e_6\rangle$ and generalised 2-eigenspace of the matrix is $\langle e_4, e_5\rangle$.

12.3. **Upper triangular form.** Let us continue to assume that F is an algebraically closed field and V is finite dimensional vector space over F. We have

$$T(T-\lambda)w = (T-\lambda)^2 w + \lambda(T-\lambda)w.$$

Thus, the image of $T-\lambda$ is stable under the action of the operator T. Assume that λ is an eigenvalue for T, and let W be the image of $T-\lambda$. Thus, we obtain non-trivial subspace W such that $TW \subset W$. Thus, using induction, we obtain a sequence of subspaces

$$\cdots \subset W_2 \subset W_1 \subset W_0 = V \tag{12.1}$$

such that $T(W_i) \subseteq W_i$. Using induction on the dimension of vector space V, we may refine the sequence (12.1) such that $\dim_F(W_i) - \dim_F(W_{i+1}) = 1$. Now choose v_i such that $w_i \in W_{i+1}\backslash W_i$. The tuple $(w_1, w_2, \ldots, w_n)$ is a basis for the F-vector space V and in this basis the matrix of the linear transformation T is upper triangular form. The following discussion can be translated into matrices as the following lemma:

Lemma 12.1. *Let F be an algebraically closed field and let $M \in \mathrm{M}_{n\times n}(F)$. There exists a matrix $g \in \mathrm{GL}_n(F)$ such that gMg^{-1} is an upper triangular matrix.*

Two matricies A and B in $M_{n\times n}(F)$ are said to be *similar* if there exists a $P \in \mathrm{GL}_n(F)$ such that $PAP^{-1} = B$. The question, we want to understand is a criterion to determine when two matrices are similar. The relation that A is similar to B is an equivalence relation. The equivalence classes are called the adjoint classes, or similarity classes. Clearly if A and B are similar then they have the same characteristic polynomial. However, the converse is not true. Consider the following matrices

$$M_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, M_2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, M_3 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, M_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

None of these matrices are similar, however, they all have the characteristic polynomial. It is instructive to work out the invariant subspaces of each of these matrices.

13. Jordan canonical form

13.1. Let us recall some standard facts on polynomials in one variables. Let f and g be two elements in $F[X]$. Recall that there exists $q \in F[X]$ and $r \in F[X]$ such that $f = qg + r$, where $deg(r) < \deg(g)$. Given any two polynomials f and g, there exists a unique monic polynomial h such that $h|f$ and $h|g$ and for any $h' \in F[X]$ such that $h'|f$ and $h'|g$, we have $h|h'$. The polynomial h' is called the greatest common divisor of f and g, and is denoted by (f, g). Recall the Euclidean algorithm which consisting og the following sequence of steps:

$$\begin{aligned} f &= q_0 g + r_0, \deg(r_0) < \deg(g), \\ g &= q_1 r_0 + r_1, \deg(r_1) < \deg(r_0), \\ r_0 &= q_2 r_1 + r_2, \deg(r_2) < \deg(r_1), \\ r_1 &= q_3 r_2 + r_3, \deg(r_3) < \deg(r_2), \\ &\vdots \\ r_{n-1} &= q_{n+1} r_n. \end{aligned}$$

Note that r_n is the greatest common divisor of f and g. Note that $r_n = fp + gq$, for some $p, q \in F[X]$. Similarly, if $f_1, f_2, \ldots, f_n$ are n polynomials such that g is their g.c.d, then there exists $q_1, q_2, \ldots, q_n$ such that

$$f_1 q_1 + f_2 q_2 + \cdots + f_n q_n = 1. \tag{13.1}$$

Another way to see this is the concept of an ideal of $F[X]$. A vector subspace I of $F[X]$ is called an ideal if for any $f \in F[X]$ and $g \in I$, we have $fg \in I$. Any ideal of I is equal to $\{qf : f \in F[X]\}$, for some $q \in F[X]$. To see this, consider the integer $N = \min\{deg(f) : f \in I\}$. Let $q \in F[X]$ be a monic polynomial such that $\deg(q) = N$. For any $f \in I$ we may write $f = qq' + r$, where $\deg(r) < \deg(q)$. This implies that $r = 0$, otherwise $r \in I$ with $r \neq 0$ and $\deg(r) < \deg(q)$. Now, consider the space $I := \{fp + gq : p, q \in F[X]\}$. Note that I is an ideal of $F[X]$, and there exists a $q \in F[X]$ such that $\{qh : h \in F[X]\}$ is equal to I. Note that $f, g \in I$, and hence $q|f$ and $q|g$, for any other $q'|f$ and $q'|g$, we get that $q'|q$, since $q = fs + gt$, for some $s, t \in F[X]$. Thus, we get that q is the g.c.d of f, g, and $q = fs + gt$, for some $s, t \in F[X]$.

Given any linear transformation $T : V \to V$ on a finite dimensional vector space V over the field F, there exists a polynomial $f \in F[X]$ such that $f(T) = 0$. Now consider the ideal I defined by setting

$$I = \{f \in F[X] : f(T) = 0\}$$

There exists a unique monic polynomial $m_T(t)$ such that $I = \{m_T q : q \in F[X]\}$. The polynomial $m_F(t) \in F[X]$ is called the minimal polynomial of T.

Before, we begin the discussion on eigenspaces, let us prove a general observation:

Lemma 13.1. *Let V be a vector space over a field F, and let $T : V \to V$ be a linear transformation. Let W_1 and W_2 be two subspaces of V and $P_1, P_2 \in F[X]$ such that $P_1(T)(w_1) = 0$, for all $w_1 \in W_1$ and $P_2(t)(w_2) = 0$ for all $w_2 \in W_2$. If g.c.d of P_1 and P_2 is 1, then $W_1 \cap W_2 = \{0\}$*

Proof. Let $w \in W_1 \cap W_2$. Since g.c.d of P_1 and P_2 is 1, there exists $Q_1, Q_2 \in F[X]$ such that $P_1 Q_1 + P_2 Q_2 = 1$. Thus, we get that $w = P_1(T)Q_1(T)(w) + P_2 Q_2(T)(w)$, but the righthand side is zero. □

13.2. Let V be an n-dimensional vector space over an algebraically closed field F, and let $T : V \to V$ be a linear transformation. Let $\{\lambda_1, \lambda_2, \ldots, \lambda_k\}$ be the roots of the minimal polynomial $m_T(t)$ of T.

Lemma 13.2. *We have $V = V_{\lambda_1} \oplus V_{\lambda_2} \oplus \cdots \oplus V_{\lambda_k}$.*

Proof. Let $m_T(t)$ be the minimal polynomial of T. Let

$$m_T(t) = (t - \mu_1)_1^n (t - \mu_2)^{n_2} \cdots (t - \mu_l)^{n_l}.$$

Let $R_i(t) = m_T(t)/(t - \mu_i)^{n_i}$. Note that the g.c.d of $R_1, R_2, \ldots, R_l$ is equal to one, and hence there exists $S_1, S_2, \ldots, S_l \in F[X]$ such that

$$R_1 S_1 + R_2 S_2 + \cdots + R_l S_l = 1.$$

Thus, given any vector v, we may write

$$v = R_1(T)S_1(T)(v) + R_2(T)S_2(T)(v) + \cdots + R_l(T)S_l(T)(v)$$

Note that $R_i(T)S_i(T)(v) \in V_{\mu_i}$, since $(T - mu_i)^{n_i} R_i(T)S_i(T) = m_T(T)S_i(T)$. Using Lemma 13.1 we get that $V_{\mu_i} \cap (V_{\mu_1} + \cdots + V_{\mu_{i-1}} + V_{\mu_{i+1}} + \cdots + V_{\mu_l})$ is zero subspace. Thus, we get that

$$V = V_{\mu_1} \oplus V_{\mu_2} \oplus \cdots \oplus V_{\mu_k}.$$

□

Lemma 13.3. *Let $\mu \in F$ and let V_μ be a generalised eigenspace of V. The space V_μ is non-zero if and only if μ is a root of $m_T(t)$.*

Proof. Let $\lambda \in F$ such that $m_T(\lambda) = 0$. Set $R(t) = m_T(t)/(t - \mu)$, and note that $\deg(R) < \deg(m_T)$. If $R(T)(v) = 0$, for all $v \in V$, then it would contradict the minimality of degree of m_T among the polynomials $P \in F[T]$ such that $P(T) = 0$. Hence there exists a $v \in V$ such that $R(T)(v) \neq 0$. Thus the vector $R(T)(v)$ is an eigenvalue of T. So, we get that $V_\lambda \neq 0$. Applying Lemma 13.1 to the subspaces $W_1 = V_\mu$ and $W_2 = V$, with $P_1(t) = (t - \lambda)^m$ and $P_2(t) = m_T(t)$, we get that V_μ is non-zero if and only if μ is a root of $m_T(t)$. □

Note that V_λ is an invariant subspace of V, and the operator $T - \lambda$ is a nilpotent operator on V_λ. So, in the next section we will study similarity of nilpotent operators and invariant subspaces of nilpotent operators in general.

13.3. **Nilpotent operators.** During the lectures, we assumed that F is algebraically closed. However, such assumptions are not needed for this section. We give a modified treatment here. Let V be a finite dimensional vector space over a field F, and let $T : V \to V$ be a nilpotent linear transformation. Here we do not have any assumptions on the field F. The integer m is called the order of nilpotency of T if $T^m = 0$ and T^{m-1} is non-zero operator. Observe that

$$\ker(T) \subset \ker(T^2) \subset \ker(T^3) \subset \cdots \subset \ker(T^{m-1}).$$

and

$$\operatorname{img}(T^{m-1}) \subset \cdots \subset \operatorname{img}(T^3) \subset \operatorname{img}(T^2) \subset \operatorname{img}(T)$$

Since T is nilpotent and $\operatorname{img}(T)$ is a T-invariant subspace, we get that $\operatorname{img}(T^{i+1})$ is strictly contained in $\operatorname{img}(T^i)$ or $T^i = 0$. Assume that $\ker(T^i) = \ker(T^{i+1})$. Now let $v \in \operatorname{img}(T^i)$, so $v = T^i(w)$. If $T(v) = 0$, then $T^{i+}(w) = 0$, but $w \in ker(T^i)$ which implies that $v = T^i(w) = 0$. Thus the map T restricted to image of T^i is injective map. Thus, we get the following if $\operatorname{img}(T^i)$ is non-zero subspace, then $\ker(T^i)$ is a proper subspace of $\ker(T^{i+1})$.

Let $\{v_1, v_2, \ldots, v_n\}$ be a maximal set with respect to the property that $\langle v_1, v_2, \ldots, v_n\rangle \cap \ker(T^{m-1}) = 0$. Note that $\{T(v_1), T(v_2), \ldots, T(v_n)\}$ are contained in $\ker(T^{m-1})$ and $\langle T(v_1), T(v_2), \ldots, T(v_n)\rangle \cap \ker(T^{m-2})$ is the zero vector space. To see this, consider a linear relation $\sum_{i=1}^n a_i T(v_i) \in \ker(T^{m-2})$, then $T^{m-1}(\sum_{i=1}^n a_i v_i) = 0$, and hence $a_i = 0$, for all $i \in [n]$.

Lemma 13.4. *Let V be a finite dimensional vector space over a field F and let $T : V \to V$ be a nilpotent linear transformation of V. There exists $v_1, v_2, \ldots, v_n \in V$ such that the tuple*

$$(v_1, T(v_1), \ldots, v_2, T(v_2), \ldots, v_n, T(v_n), \ldots)$$

forms a basis of the vector space V.

Proof. We will prove a stronger result using induction on the dimension of V and degree of nilpotency of T.

Claim 1. *If $v_1, v_2, \ldots, v_n$ be any vectors such that $\langle v_1, v_2, \ldots, v_n\rangle \cap \ker(T^{m-1}) = \{0\}$, there exists set of vectors $\{w_1, w_2, \ldots, w_m\}$ in V containing $\{v_1, v_2, \ldots, v_n\}$ such that*

$$(w_1, T(w_1), \ldots, w_2, T(w_2), \ldots, v_n, T(w_m), \ldots)$$

forms a basis of V.

Let $v_1, v_2, \ldots, v_n \in V$ such that $\langle v_1, v_2, \ldots, v_n\rangle \cap \ker(T^{m-1}) = \{0\}$. Note that $\ker(T^{m-1})$ is a proper T-invariant subspace of V. Using the above discussion and the induction hypothesis, there exists $w_1, w_2, \ldots w_m$ which contains $\{T(v_1), T(v_2), \ldots, T(v_n)\}$ such that

$$(w_1, T(w_1), \ldots, w_2, T(w_2), \ldots, v_n, T(w_n), \ldots)$$

forms a basis of $\ker(T^{m-1})$. Thus, we get a Now, the tuple

$$(v_1, v_2, \dots, w_1, T(w_1), \dots, w_2, T(w_2), \dots, v_n, T(w_n), \dots)$$

forms a basis of V. Thus, we prove the claim which implies the lemma. □

Let $\lambda \in F$ and $n \in \mathbb{N}$. A jordan block, $J_n(\lambda) \in M_{n\times n}(F)$ is the matrix

$$\begin{pmatrix} \lambda & 0 & 0 & \dots & 0 \\ 1 & \lambda & 0 & 0\dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 1 & \lambda \end{pmatrix}$$

The block matrix of the form (with λ_i's here need not be distinct):

$$\begin{pmatrix} J_{n_1}(\lambda_1) & 0 & 0 & 0 & 0 \\ 0 & J_{n_2}(\lambda_2) & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & J_{n_k}(\lambda_k) \end{pmatrix}$$

is called the Jordan canonical form. Using the above lemma's we get the following two theorems.

Theorem 13.5. *Given any matrix $M \in M_{n\times n}(\mathbb{C})$, there exists a $P \in \mathrm{GL}_n(\mathbb{C})$ such that PAP^{-1} is in the Jordan canonical form. Moreover, two matrices A and B have the same jordan canonical form if and only if there exists a $P \in \mathrm{GL}_n(\mathbb{C})$ such that $PAP^{-1} = B$.*

We conclude the section with the most important result of this course namely:

Theorem 13.6. *Let V be a finite dimensional vector space over an algebraically closed field F. Let $T : V \to V$ be any linear transformation of V. There exists a unique pair (T_s, T_n) with $T_s, T_n \in \mathrm{Hom}_F(V, V)$, T_s is diagonalisable and T_n is nilpotent such that $T = T_s + T_n$ and $T_sT_n = T_nT_s$. Moreover, there exists two polynomials $S, N \in F[X]$ such that $S(T) = T_s$ amd $N(T) = T_n$.*

14. Lecture: Jordan canonical form continued.

14.1. Let us begin the proof of the Theorem 13.5. Let $T : F^n \to F^n$ be the linear transformation which sends $X \mapsto AX$, where $A \in M_{n\times n}(F)$. Using the generalised eigenspace decomposition in Lemma 13.2, we get that

$$V = V_{\lambda_1} \oplus \cdots \oplus V_{\lambda_k},$$

where $\lambda_1, \lambda_2, \ldots, \lambda_k$ are eigenvalues for the matrix A. Here V_λ is the set of vectors $X \in F^n$ such that $(A-\lambda)^n X = 0$, for some n possibly depending on the vector X. Note that the operator $T_i' = T - \lambda_i$ acts a nilpotent operator on the vector space V_{λ_i}, for $i \in [k]$. So, using Lemma 13.4 we get a basis

$$B_i = (w_1, T_i'(w_1), \ldots, w_2, T_i'(w_2), \ldots, w_n, T_i'(w_n), \ldots)$$

for the vector space V_{λ_i}.

We note that

$$\begin{aligned} T(w_i) &= \lambda_i w_i + (T-\lambda_i)w_i = \lambda_i w_i + T_i'(w_i), \\ TT_i'(w_i) &= T(T-\lambda_i)w_i = \lambda_i(T-\lambda_i)(w_i) + (T-\lambda_i)^2 w_i = \lambda_i(T_i')(w_i) + (T_i')^2(w_i), \\ T(T_i')^2(w_i) &= T(T-\lambda_i)^2 w_i = \lambda_i(T-\lambda_i)^2 + (T-\lambda_i)^3 w_i = \lambda_i(T_i')^2 w_i + (T_i')^3(w_i), \end{aligned}$$

the general case being:

$$T(T_i')^l(w_i) = T(T-\lambda_i)^l(w_i) = \lambda_i(T-\lambda_i)^l(w_i) + (T-\lambda_i)^{l+1}(w_i) = \lambda_i(T_i')^l(w_i) + (T_i')^{l+1}(w_i).$$

Let $B = (B_1, B_2, \ldots, B_n)$ be a basis for the vector space F^n. In the basis B the matrix of T is in Jordan canonical form. If P is the change of basis matrix from the standard basis to B, then we get that PAP^{-1} is in the Jordan canonical form.

14.2. Let $m_T(t)$ be the minimal polynomial of T. Let $m_T(t) = \prod_{i=1}^k (t-\lambda_i)^{m_i}$ be the minimal polynomial of T, and let $R_i(t) = m_T(t)/(t-\lambda_i)^{m_i}$. Note that there exists $S_i \in F[t]$ such that

$$R_1S_1 + R_2S_2 + \cdots + R_kS_k = 1.$$

Note that $\ker(R_iS_i)$ is equal to $\oplus_{i\neq j} V_{\lambda_j}$ and $\mathrm{img}(R_iS_i) = V_{\lambda_i}$. Moreover, R_iS_i acts as identity on V_{λ_i}. Thus, we get that

$$T_s = \sum_{i=1}^k \lambda_k R_i(T)S_i(T).$$

We note that $T - T_s$ is nilpotent operator. Thus, we get the existence of T_s and T_n.

Now, assume that T_s' and T_n' are two linear operators which are diagonalisable and nilpotent respectively with $T_s'T_n' = T_s'T_n'$ and $T = T_s' + T_n'$. Let V_μ be the μ-eigenspace of T_s'. Since T is diagonalisable, we get that

$$V = V_{\mu_1} \oplus V_{\mu_2} \oplus \cdots \oplus V_{\mu_k}.$$

Note that $T_n'(V_\mu) \subseteq V_\mu$, and hence $T(V_\mu) \subset V_\mu$. Note that V_μ is the generalised μ-eigenspace of V. So, we get that $T_s' = T_s$.

14.3. **Caley–Hamilton's Theorem.** The following theorem follows as a easy consequence of the existence of Jordan canonical form.

Theorem 14.1. *Let V be a finite dimensional vector space over a field F, and let $T : V \to V$ be a linear transformation. Let $P_T(t)$ be the polynomial $\det(T - t\,\mathrm{id})$. Then, we get that $P_T(T) = 0$.*

14.4. **Invariant subspaces.** Let V be a finite dimensional vector space over an algebraically closed field F, and let $T \in \mathrm{Hom}_F(V,V)$. Let W be an invariant subspace of V. Let $W = \oplus_\mu W_\mu$, where W_μ is the generalised μ-eigenspace. of W. Clearly W_μ is contained in V_μ. Note that W_μ are invariant subspaces of $T-\mu$, and $T-\mu$ is a nilpotent operator when restricted to W_μ. Let us first try to understand invariant subspaces of a nilpotent operator on any vector space.

14.5. **Exercises.**

14.5.1. Let $M, N \in M_{n\times n}(\mathbb{R})$ such that $M = PNP^{-1}$ for some $P \in \mathrm{GL}_n(\mathbb{C})$. Does there exists a $Q \in \mathrm{GL}_n(\mathbb{R})$ such that $M = QNQ^{-1}$.

14.5.2. Let $\mathfrak{sl}_n$ be the set of trace zero matrices of $M_{n\times n}(\mathbb{C})$. Let $M \in \mathfrak{sl}_n$. We define the map $T : \mathfrak{sl}_n \to \mathfrak{sl}_n$ be the linear transformation $X \mapsto MX - XM$. Show that M is diagonalisable if and only if T is diagonalisable. Show that M is nilpotent if and only if T is nilpotent. In general describe the Jordan decomposition of the operator T.

14.5.3. Let V be a finite dimensional complex vector space and let $B : V \times V \to F$ be a non-degenerate ϵ-bilinear form, for $\epsilon \in \{\pm 1\}$. Let $\mathfrak{g}$ be the set of linear transformations $\{T \in \mathrm{Hom}_F(V, V) : T^* = -T\}$. Let $T \in \mathfrak{g}$. Show that T_s and T_n belong to $\mathfrak{g}$.

14.5.4. For any invertible complex matrix P, show that there exists a polynomial $f \in \mathbb{C}[t]$ such that $f(P)^2 = P$. Use this to show that if $X, Y \in \mathfrak{g}$, as in the previous exercise, are similar if and only if there exists an isometry T of B (as defined in the previous exercise) such that $TXT^{-1} = Y$.