ON THE REPRESENTATIONS OF CERTAIN GRADED TENSOR PRODUCTS

ABSTRACT. In this article we obtain a tensor product theorem for a certain class of associative algebras in the braided category of G-graded vector spaces over $\mathbb{C}$ where G is a finite abelian group. The tensor product theorem describes the irreducible modules of the tensor product of G-graded associative algebras in this class as the tensor product of the irreducibles of each of its constituents. Applying this general result to the universal enveloping algebras of basic classical Lie superalgebras or the queer Lie superalgebra or even certain Lie color algebras, we get an extension of a result of Yousofzadeh [16] to this larger class of algebras. Along the way, we also extend to the G-graded setting the results of Lemire in [7] which relates the irreducibility of a representation of a semi-simple Lie algebra $\mathfrak{L}$ to the irreducibility of its weight spaces considered as modules for the centralizer of the cartan subalgebra in the the universal enveloping algebra of $\mathfrak{L}$. We prove it in a more general setting of G-graded associative algebras with a *G-compatible* grading by another abelian group. In the case of the universal enveloping algebras of Lie superalgebras and Lie color algebras belonging to the class mentioned earlier, these results give us the analogues of Lemire's results.

1. INTRODUCTION

The tensor product theorem for associative algebras says that if A and B are unital associative algebras over $\mathbb{C}$ and M and N are finite dimensional modules for A and B, respectively, then $M \otimes N$ is an irreducible $A \otimes B$-module and every finite dimensional irreducible of $A \otimes B$ is of this form. If one works with associative algebras in the braided category of G-graded vector spaces, where G is an abelian group the statement is known to be false. An easy example of this is illustrated in [4] in the case of $G = \mathbb{Z}_2$. It is shown there that the tensor product with itself of the unique irreducible module for the queer algebra $Q(n)$ of $2n \times 2n$-matrices, where $n \geq 2$, is not irreducible for $Q(n) \otimes Q(n)$. However, things improve if we work with the universal enveloping algebras of basic classical Lie superalgebras. In this case, if one restricts to irreducible *finite weight modules*, in particular also to finite dimensional irreducibles, then the statement is true. That is, if $\mathfrak{L}_1$ and $\mathfrak{L}_2$ are basic classical Lie superalgebras then $M(1) \otimes M(2)$ is an irreducible finite weight modules for $\mathfrak{L}_1 \oplus \mathfrak{L}_2$, where $M(i)$ is an irreducible finite weight module for $\mathfrak{L}_i$ for $i = 1, 2$ and all the irreducible $\mathfrak{L}_1 \oplus \mathfrak{L}_2$-modules are of this form. This was proved recently in [16]. It was shown there that if $\mathfrak{L}_i$ has a subalgebra $\mathfrak{h}_i$, with respect to which $(\mathfrak{L}_i, \mathfrak{h}_i)$ forms an *admissible pair* then the statement is true. By the work of Cheng [3], we know that this statement is not expected to be true for arbitrary Lie superalgebras. In fact, we observe in §5 that if we take $\mathfrak{L}_i = \mathfrak{q}(3)$ for $i = 1, 2$ then the tensor product theorem will fail to be true for $\mathfrak{L}_1 \oplus \mathfrak{L}_2$. However, an extension of the above results is still possible. For this, we weaken the conditions defining *admissible pairs* as introduced in [16] and show that the statement is true for the finite weight modules of admissible pairs with the assumption that

MSC subject classification 2020: 15A72, 16W50, 17B10, 17B35, 17B70, 17B75.
Keywords: associative graded algebras, tensor product representations, Finite weight modules, Lie Superalgebras, Lie color algebra.

one of the admissible pairs is *strongly admissible*. The example alluded to earlier shows that this additional assumption cannot be relaxed.

In this paper, we prove the tensor product theorem in a more general setting of G-graded algebras which have an additional G-*compatible grading* by an abelian group Λ. As a corollary of this general result we obtain an alternate proof of a result of Cheng [3] where he shows that if A is an associative algebra and B is a superalgebra then the tensor product theorem is true for irreducible finite dimensional modules. This result then leads to a description of the irreducibles of finite dimensional semisimple Lie superalgebras owing to a classification theorem of such Lie superalgebras due to Kac [6]. In fact, the extension of the tensor product theorem proved here, allows one to easily observe that if $\mathfrak{L}_1, \ldots, \mathfrak{L}_r$ are finitely many classical (basic or strange) simple Lie superalgebras such that at least one of them is basic then the (f.d) irreducible representations of $\mathfrak{L}_1 \oplus \cdots \oplus \mathfrak{L}_r$ can be obtained as the tensor product of the irreducible (f.d) representations of each of the $\mathfrak{L}_i$, $i = 1, \ldots, r$. Thus, in this case, we are able to obtain a more definitive answer for the irreducible representations of finite dimensional semismiple Lie superalgebras than that obtained in [4]. The tensor product theorem for admissible pairs of Lie color algebras and Lie superalgebras are easily obtained as an application of this general result. Lie color algebras are the G-graded analogues of Lie superalgebras. They were first introduced by Ree [11] as "generalized Lie algebras" motivated from particle physics and parastatistics. These objects and their representations were later systematically studied by Scheunert through a series of papers ([13], [14]). Some recent interest related to their representation theory can be found in [1], [2], [15], [17].

The irreducibility of tensor products of irreducibles and whether all the irreducibles of the tensor product of two algebras are in the form of a tensor product of irreducibles is a very important question. For example, the study of the modules of various other classes of algebras, like affine Lie algebras, loop superalgebras *etc.*, are closely connected to understanding the irreducibles of direct sums of simple lie superalgebras ([8], [12]). The study of finite weight cuspidal representations of simple Lie superalgebras crucially depends on the irreducibility of tensor products of representations of the constituent Lie superalgebras ([5], [9]). Hence an extension of the tensor product theorem to a larger class of Lie superalgebras is highly desirable.

In this paper, we also study the relationship of an irreducible finite weight representation of an admissible pair and its non-zero weight spaces. Extending a result of Lemire [7], we obtain a relation between the irreducibles of a G-graded associative algebra with a G-compatible grading by an abelian group Λ and its *weight spaces*. These results when applied to the universal enveloping algebra of an admissible pair of Lie superalgebras or a Lie color algebras leads to an exact analogue of Lemire's results to these settings.

Outline of the paper: In §2, we introduce the preliminary notions and some natural isomorphisms and maps in the braided category of G-vector spaces over $\mathbb{C}$, the field of complex numbers. We also introduce the notion of *admissible pairs* and *strongly admissible pairs* for Lie color algebras (the Lie algebra-like objects in this category). In §3, we introduce various notions and background results to state and prove the tensor product theorem for G-graded associative algebras. Along the way, we also obtain generalizations of Lemire's results to this setting and use them in §4 to obtain analogues of Lemire's results for Lie color algebras, in particular also for Lie superalgebras. In §5 we obtain the tensor product theorem for the Lie color and Lie superalgebra, as an application of the main theorem in §4.

Notation: $\mathbb{C}$ denotes the field of complex numbers; $\mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$ denotes the field with 2 elements.

2. Preliminaries on G-graded structures

2.1. Let G be a finite abelian group with its identity element denoted as $\mathbf{0}$. A map $\epsilon : G \times G \to \mathbb{C}^*$ where $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$ is called a *skew-symmetric bicharacter* on G if the following identities hold, for all $f, g, h \in G$.

i) $\epsilon(f, g+h) = \epsilon(f, g)\epsilon(f, h)$,

ii) $\epsilon(g+h, f) = \epsilon(g, f)\,\epsilon(h, f)$,

iii) $\epsilon(g, h)\,\epsilon(h, g) = 1$.

From the definition of ϵ it follows that $\epsilon(g, \mathbf{0}) = \epsilon(\mathbf{0}, g) = 1$ and $\epsilon(g, g) = \pm 1$, where $\mathbf{0}$ denotes the identity element of G.

Remark 2.1. *We note that when $G = \mathbb{Z}_2$, using properties (i) and (ii) above, a skew-symmetric bicharacter ϵ on $\mathbb{Z}_2$ is completely determined by the value of $\epsilon(\bar{1}, \bar{1}) = \pm 1$. Hence every such bicharacter is either given by $\epsilon(g, h) = (-1)^{gh}$ or it is trivial depending on whether $\epsilon(\bar{1}, \bar{1}) = -1$ or $+1$. More generally, for any cyclic group generated by a, ϵ is completely determined by its value on (a, a) which can take values from ± 1.*

For a skew-symmetric bicharacter ϵ of G, set $G_{\bar{0}} := \{g \in G : \epsilon(g, g) = 1\}$ and $G_{\bar{1}} = G \backslash G_{\bar{0}}$. Note that $G_{\bar{0}}$ is a subgroup of G whose index in G is 1 or 2. Hence identifying the quotient $G/G_{\bar{0}}$ as a subgroup of $\mathbb{Z}_2$, we get a group homomorphism $\theta : G \to \mathbb{Z}_2$ such that $\theta(g) = \bar{0}$ for $g \in G_{\bar{0}}$ and $\theta(g) = \bar{1}$ for $g \in G_{\bar{1}}$. It is easily seen that if $g \in G_{\bar{1}}$ then $-g \in G_{\bar{1}}$.

Define a new bicharacter $\epsilon_0 : G \times G \to \{\pm 1\} \subset \mathbb{C}^*$ given by $\epsilon_0(g, h) = (-1)^{\theta(g)\theta(h)}$. Fix $\delta(g, h) = \epsilon(g, h)^{-1}\epsilon_0(g, h)$. Then δ is also a skew-symmetric bicharacter on G such that $\delta(g, g) = 1$ for all $g \in G$. A *multiplier* associated to δ is the bicharacter $\kappa : G \times G \to \mathbb{C}^*$ given by

$$\kappa(\sum_i r_i g_i, \sum_j s_j g_j) = \prod_{i<j} \delta(g_i, g_j)^{r_i s_j},$$

where $G = \langle g_1 \rangle \times \ldots \times \langle g_r \rangle$ such that $\langle g_i \rangle \cong \mathbb{Z}/n_i\mathbb{Z}$ for some $n_i \in \mathbb{N}$, and r_i, s_j are integers. The above multiplier κ satisfies the condition that $\delta(g, h) = \kappa(g, h)\kappa(h, g)^{-1}$ for $g, h \in G$. So $\kappa(g, h)\kappa(h, g)^{-1}\,\epsilon(g, h) = \epsilon_0(g, h)$. The existence of such a multiplier will be important especially if one wants to move from ϵ to ϵ_0-structures, as will be illustrated later.

For a finite abelian group G, a *G-graded vector space* is a vector space V over $\mathbb{C}$ together with a decomposition into a direct sum of the form $V = \oplus_{g \in G} V_g$, where each V_g is a subspace of V. For a given $g \in G$ the elements of V_g are then called *homogeneous elements of degree* g and we write $|v|$ to denote the degree of v. The category of G-graded vector spaces forms a braided category via the braiding given by $V \otimes W \cong W \otimes V$ given by $v \otimes w \mapsto \epsilon(|v|, |w|) w \otimes v$ for all homogeneous elements $v \in V$, $w \in W$. The *parity functor*, Π_a where $a \in G$, is the parity shift functor on G-graded vector spaces such that $\Pi_a(V_b) = V_{a+b}$.

Any G-graded vector space $V = \oplus_{g \in G} V_g$ can be given an obvious $\mathbb{Z}_2$-grading via $V = V_{G_{\bar{0}}} \oplus V_{G_{\bar{1}}}$, where $V_{G_{\bar{0}}} = \oplus_{g \in G_{\bar{0}}} V_g$ and $V_{G_{\bar{1}}} = \oplus_{g \in G_{\bar{1}}} V_g$. We refer to $V_{G_{\bar{0}}}$ and $V_{G_{\bar{1}}}$ as the *even part* and *odd part* of V and denote them as just $V_{\bar{0}}$ and $V_{\bar{1}}$, respectively.

From now on through the rest of the article we fix a finite abelian group G and a skew-symmetric bicharacter ϵ. All the G-graded vector spaces will be so with respect to this fixed G and ϵ. Also, we fix an ordering of elements of $G_{\bar{0}}$ starting with $\mathbf{0}$. Now noting that $G_{\bar{0}}$ is a subgroup of index 1 or 2, by fixing an element $g' \in G_{\bar{1}}$ we get an induced ordering of the coset $G_{\bar{1}} = g' + G_{\bar{0}}$. We will list the elements of G as $\{\mathbf{0} = g_1, \dots, g_r, g' = g' + g_1, \dots, g' + g_r\}$

2.2. **Lie color algebras.** A *Lie color algebra* $\mathfrak{L} = \oplus_{g\in G}\mathfrak{L}_g$ associated to (G, ϵ) is a G-graded vector space over $\mathbb{C}$ with a bilinear map $[-,-] : \mathfrak{L} \times \mathfrak{L} \to \mathfrak{L}$, which we call an *$\epsilon$-lie bracket*, satisfying the following conditions:

i) $[\mathfrak{L}_g, \mathfrak{L}_h] \subset \mathfrak{L}_{g+h}$ for every $g, h \in G$,

ii) $[x, y] = -\,\epsilon(|x|, |y|)[y, x]$ and

iii) $\epsilon(|z|, |x|)[x, [y, z]] + \epsilon(|x|, |y|)[y, [z, x]] + \epsilon(|y|, |z|)[z, [x, y]] = 0$, for all homogeneous elements $x, y, z \in \mathfrak{L}$.

We shall denote such a Lie color algebra by the pair $(\mathfrak{L}, \epsilon)$ or just by $\mathfrak{L}$, whenever ϵ is understood from the context. Such Lie color algebras are also called "generalized Lie algebras" (as in [13]) or "(G, ϵ)-Lie superalgebras".

2.2.1. *Special examples of Lie color algebras.*

a) In view of the Remark 2.1 above, when $G = \mathbb{Z}_2$, ϵ is either given by $\epsilon(g, h) = (-1)^{gh}$ for $g, h \in \mathbb{Z}_2$ or it is trivial. In the former case, a Lie color algebra is better known as a Lie superalgebra; while in the latter case these are just Lie algebras.

b) Given a G-graded vector space $V = \oplus_{g\in G}V_g$, the $\mathbb{C}$-endomorphisms of V, $\mathrm{End}_{\mathbb{C}}(V)$ is also G-graded, where for each $g \in G$ we have
$$\mathrm{End}_{\mathbb{C}}(V)_g := \{f : V \to V : f(V_h) \subset V_{g+h},\ \ for\ all\ h \in G\}.$$
The *general linear Lie color algebra*, $\mathfrak{gl}_\epsilon(V)$ is defined to be $\mathrm{End}_{\mathbb{C}}(V)$ with the Lie bracket given by $[x, y]_\epsilon = xy - \epsilon(|x|, |y|)yx$, where x, y are homogeneous elements in $\mathrm{End}_{\mathbb{C}}(V)$.

Fixing a homogeneous basis of V to be $\{v_1, \dots, v_n\}$, one can identify $\mathfrak{gl}_\epsilon(V)$ with $n \times n$ matrices, where $n = \dim V$. We shall denote this space of matrices as $\mathfrak{gl}(n, \epsilon)$. It is naturally a Lie color algebra with the elementary matrices E_{ij}, corresponding to the linear transformation taking v_j to v_i, having parity given by $|v_j| - |v_i|$ for $1 \le i, j \le n$.

c) Let $G_{\bar{0}}$, $G_{\bar{1}}$ be as defined earlier. Assume that $|G/G_{\bar{0}}| = 2$ and that there is a $g' \in G_{\bar{1}}$ such that $2g' = \mathbf{0}$. Then we may list the elements of G as $\{g_1, \dots, g_r, g' + g_1, \dots, g' + g_r\}$ where $g_1, \dots, g_r \in G_{\bar{0}}$. Assume further that $dim(V_a) = dim(V_{a+g'})$, for all $a \in G$. So, $n = \dim V$ is even. Let $f \in \mathrm{End}(V)_{g'}$ be a fixed endomorphism such that $f \circ f = -\kappa(g', g')id$. Set $q(V, \epsilon)_a := \{X \in End(V)_a : f \circ X = \epsilon(g', a)X \circ f\}$. Define $q(V, \epsilon) := \bigoplus_{a\in G} q(V, \epsilon)_a$. Then $q(V, \epsilon)$ with the induced bracket from $\mathfrak{gl}_\epsilon(V)$ forms a Lie color algebra, called the *queer Lie color algebra*.

List the homogeneous basis vectors $\{v_1, \dots, v_n\}$ of V such that $i \le j$ implies $|v_i| \le |v_j|$ under the linear ordering of elements of G described above. Let $n = 2m$ for some $m \in \mathbb{N}$. Then, $\{v_1, \dots, v_m\} \in V_{G_{\bar{0}}}$ and $|v_{m+i}| = g' + |v_i|$ for all $i = 1 \dots m$. Take f to be the endomorphism given by $f(v_i) = v_{m+i}$ and $f(v_{m+i}) = -\kappa(g', g')v_i$, for $i = 1, \dots, m$. Then $f \in \mathrm{End}(V)_{g'}$ and under the identification of $\mathfrak{gl}_\epsilon(V)$ with the space of matrices for f as defined above, the Lie color subalgebra of $\mathfrak{q}(V, \epsilon)$ obtained will be denoted as $\mathfrak{q}(m, \epsilon)$. Its elements are of the form $\begin{pmatrix} A & -\kappa(g',g')B \\ B & A \end{pmatrix}$ where A, B are arbitrary $m \times m$-matrices with entries in $\mathbb{C}$.

2.3. **Admissible pairs and their finite weight representations.**

2.3.1. *Admissible pairs.* Let $\mathfrak{L}$ be a Lie color algebra and $\mathfrak{h}$ be a G-graded Lie subalgebra of $\mathfrak{L}$ with $\mathfrak{h}_{\mathbf{0}}$ being its G-graded component corresponding to $\mathbf{0} \in G$. We say $(\mathfrak{L}, \mathfrak{h})$ is an *admissible pair* if the following conditions hold:

i) $\mathfrak{L}$ has a "root space decomposition" with respect to $\mathfrak{h}_{\mathbf{0}}$, *i.e.,* $\mathfrak{L} = \mathfrak{h} \oplus_{\alpha \in \mathfrak{h}_{\mathbf{0}}^*} \mathfrak{L}^\alpha$ where $\mathfrak{L}^\alpha := \{x \in \mathfrak{L} \mid [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{h}_{\mathbf{0}}\}$ and $\mathfrak{L}^0 := \{x \in \mathfrak{L} \mid [h, x] = 0 \text{ for all } h \in \mathfrak{h}_{\mathbf{0}}\} = \mathfrak{h}$; (note that since $h \in \mathfrak{h}_{\mathbf{0}}$ has parity $\mathbf{0}$, each root space $\mathfrak{L}^\alpha$ is itself G-graded; in particular, $\mathfrak{L}^\alpha = \oplus_{a \in G} (\mathfrak{L}^\alpha)_a$),
ii) $R := \{\alpha \in \mathfrak{h}_{\mathbf{0}}^* \mid \mathfrak{L}^\alpha \neq 0\}$ is a root system for $\mathfrak{L}$; (in particular, we have $R = -R$),
iii) for $a \in G$, set $R_a := \{\alpha \in R \mid \mathfrak{L}^\alpha \cap \mathfrak{L}_a \neq 0\}$ then $R_{-a} = -R_a$ and $\cup_{a \in G} R_a = R$; in particular, for $G = \mathbb{Z}_2$, $R_{\bar{0}} = -R_{\bar{0}}$, $R_{\bar{1}} = -R_{\bar{1}}$ and $R = R_{\bar{0}} \cup R_{\bar{1}}$.

If we additionally add the following conditions:

i') $\mathfrak{h} = \mathfrak{h}_{\mathbf{0}}$,
ii') $R_a \cap R_b = \varnothing$ for distinct $a, b \in G$; (note that this condition is met if $\dim \mathfrak{L}^\alpha = 1$ for all $\alpha \in R$).

Then we call such an admissible pair as a *strongly admissible pair*. Note that the notion of admissible pair is less restrictive than the one given in [16] which, in fact, co-incides with the notion of strongly admissible pair. In fact under the above definition, $\mathfrak{q}(n)$ with its standard cartan subalgebra (see Example 2.3.2(b) below) is an admissible pair[1] but not strongly admissible since in this case, $R_{\bar{0}} = R_{\bar{1}}$.

2.3.2. *Examples of admissible pairs.*

a) When $G = \mathbb{Z}_2$, all the basic, classical Lie superalgebra as classified by Kac [6] along with its cartan subalgebra, form strongly admissible pairs.
b) When $G = \mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$, the queer lie superalgebra $\mathfrak{q}(m)$ along with its cartan subalgebra $\mathfrak{h}$ spanned by diagonal matrices $H_i := E_{ii} + E_{m+i,m+i}$ where $i = 1, \ldots, m$ and the off-diagonal matrices $H_{\bar{i}} := E_{im+i} + E_{m+ii}$ forms an admissible pair. In this case, $\mathfrak{h}_{\bar{0}}$ is spanned by H_i where $i = 1, \ldots, m$ and $\mathfrak{q}(m)$ has a root space decomposition with respect to $\mathfrak{h}_{\bar{0}}$ given by $\mathfrak{h} \oplus_{1 \le i,j \le m} \mathfrak{q}(m)_{\varepsilon_i - \varepsilon_j}$ where $\varepsilon_i \in \mathfrak{h}_{\bar{0}}^*$ given by $\varepsilon_i(H_j) = \delta_{i,j} H_j$. Note that the dimension of the root spaces in this case is 2 and also, $\mathfrak{h}_{\bar{0}} \neq \mathfrak{h}$, hence this is not a strongly admissible pair.
c) Let $\mathfrak{L} = \mathfrak{gl}(n, \epsilon)$ be as defined above. Let $\mathfrak{h} = \mathbb{C}\text{-span}\langle H_k = E_{kk} : 1 \le k \le n\rangle$ be the space of all diagonal matrices in L, in particular $\mathfrak{h} = \mathfrak{h}_{\mathbf{0}}$. Further, with respect to $\mathfrak{h}$ the root space decomposition of $\mathfrak{L}$ is: $\mathfrak{gl}(n, \epsilon) = \mathfrak{h} \bigoplus_{1 \le i \ne j \le n} \mathfrak{gl}(n, \epsilon)_{\varepsilon_i - \varepsilon_j}$ where ε_i is the linear functional given by $\varepsilon_i(E_{jj}) = \delta_{ij}$. Hence $(\mathfrak{gl}(n, \epsilon), \mathfrak{h})$ forms a strongly admissible pair.
d) Let $\mathfrak{L} = \mathfrak{q}(m, \epsilon)$ be the color algebra defined above. Let $\mathfrak{h} = \mathfrak{h}_{\bar{0}} \bigoplus \mathfrak{h}_{\bar{1}}$ where $h_{\bar{0}} = \mathbb{C}\text{-span } \langle H_i = E_{ii} + E_{m+i,m+i} : 1 \le i \le m\rangle$, $\mathfrak{h}_{\bar{1}} = \mathbb{C}\text{ -span}\langle H_{\bar{i}} = E_{m+i,i} + \kappa(g', g') E_{i,m+i} : 1 \le i \le n\rangle$. The root space decomposition of $\mathfrak{q}(m, \epsilon)$ is as follows: $\mathfrak{q}(m, \epsilon) = \mathfrak{h} \bigoplus_{1 \le i \ne j \le m} q_\epsilon(V)_{\varepsilon_i - \varepsilon_j}$ where $\varepsilon_i(E_{jj} + E_{m+j,m+j}) = \delta_{ij}$ for $1 \le i, j \le m$. Since $R_a \cap R_{a+g'} \neq \emptyset$, just as in the case of $\mathfrak{q}(m)$, $(\mathfrak{q}(m, \epsilon), \mathfrak{h})$ is an admissible pair which is not a strongly admissible pair.

[1] It may be pointed out here that in the above description, we account for only one α for each root space, since we do not insist that the root spaces be 1-dimensional. This is done to enable us to uniformly include the queer Lie superalgebra as well into the discussion. This is unlike the definition of the roots for $\mathfrak{q}(n)$ as in [4, §1.2.6].

2.3.3. *G-graded representations of $\mathfrak{L}$.* Let $\mathfrak{L}$ be a Lie color algebra with G and ϵ as fixed above. A G-graded vector space is said to be a *G-graded representation* of $\mathfrak{L}$ if there exists a map $\rho : \mathfrak{L} \to \mathfrak{gl}(V, \epsilon)$ which is a homomorphism of Lie color algebras, *i.e.,* ρ is a linear map which is homogeneous of degree $\mathbf{0}$ and $\rho([x, y]) = [\rho(x), \rho(y)]_\epsilon$ where the bracket on the left is the ϵ-Lie bracket on $\mathfrak{L}$ while that on the right is the one described in Example 2.2.1(b). We denote such a representation by the pair (ρ, V). A result of Scheunert describes all such representations of a Lie color algebra as coming from G-graded representations of an associated Lie superalgebra. (Refer [13] for more details.) We thus can generate plenty of examples of such representations.

Remark 2.2. *Let V be a G-graded representation of $\mathfrak{L}$. Then $\Pi_a(V)$ is also a G-graded representation of $\mathfrak{L}$. In fact, for every irreducible G-graded representation V of $\mathfrak{L}$ we get that $\Pi_a(V)$ is also irreducible (though, obviously, not always isomorphic to V as $\mathfrak{L}$-modules.)*

2.3.4. *Finite weight modules for admissible pairs.* Let us now consider an admissible pair $(\mathfrak{L}, \mathfrak{h})$. For such a pair, a G-graded representation (ρ, V) is said to be a *finite weight module* if the following conditions hold:

i) $V = \oplus_{\lambda \in \mathfrak{h}_\mathbf{0}^*} V^\lambda$ where $V^\lambda := \{v \in V \mid \rho(h)(v) = \lambda(h)v$ for all $h \in \mathfrak{h}_\mathbf{0}\}$ is called the λ-*weight space* of V, and
ii) each λ-weight space V^λ is finite dimensional.

For a finite weight module V, we denote the set of $\lambda \in \mathfrak{h}_\mathbf{0}^*$ such that $V^\lambda \neq 0$ as the support of V, denoted as *Support* V

For the examples in 2.3.2 (a) and (b), the super-analogues of the Verma module construction [4, Chapter 2] yields many such examples. In particular, all the finite dimensional irreducibles in these cases are examples of such representations. In [10], examples of finite dimensional irreducibles are obtained for the Lie color algebra described in Example 2.3.2(c). All of these irreducibles are highest weight modules, hence they decompose into finite dimensional weight spaces.

2.4. **G-graded associative algebras and some useful examples.** *All the algebras that we consider will be defined over $\mathbb{C}$; all such algebras are assumed to contain unity, denoted as* 1. An associative algebra A along with a G-grading such that $A_a A_b \subset A_{a+b}$ for all $a, b \in G$ and $x.y = \epsilon(|x|, |y|)yx$ for all homogeneous elements $x, y \in A$ is said be a *G-graded associative algebra*. We *assume that for all G-graded algebras,* $1 \in A_\mathbf{0}$. Special examples of such algebras which will be of interest to us are the tensor algebra and the universal enveloping algebra of a Lie color algebra $\mathfrak{L}$.

2.4.1. *Tensor algebra of $\mathfrak{L}$.* The n-tensor space $T^n(\mathfrak{L})$ is the G-graded space $\mathfrak{L} \otimes \cdots \otimes \mathfrak{L}$ where the G-grading on a vector $v_1 \otimes \cdots \otimes v_n$ where $|v_i| = g_{j_i}$ is $\sum_i g_{j_i}$. The *tensor algebra* $T(\mathfrak{L})$ is the space $\oplus_{n \in \mathbb{N} \cup \{0\}} T^n(\mathfrak{L})$ along with the usual tensor multiplication.

2.4.2. *Universal enveloping algebras.* We now recall the definition of the universal enveloping algebra of the Lie color algebra $\mathfrak{L}$ with ϵ-Lie bracket [-, -]. For this consider the tensor algebra $T(\mathfrak{L})$ which is a G-graded associative algebra with respect to the tensor multiplication. Consider the two-sided ideal $J(\mathfrak{L})$ of $T(\mathfrak{L})$ generated by the tensors of the form $X \otimes Y - \epsilon(|X|, |Y|)Y \otimes X - [X, Y]$ where X, Y are homogeneous elements of $\mathfrak{L}$. The quotient algebra $T(\mathfrak{L})/J(\mathfrak{L})$ is the enveloping algebra of $\mathfrak{L}$, denoted as $U(\mathfrak{L})$. Note that $J(\mathfrak{L})$ is a generated by G-homogeneous elements so it is a G-graded ideal. Hence

$U(\mathfrak{L})$ is a G-graded associative algebra with the multiplication descending from the tensor multiplication on $T(\mathfrak{L})$.

Let $(\mathfrak{L}, \mathfrak{h})$ be an admissible pair as above. Fix a total ordering on the $\mathbb{R}$-span of the roots R such that the set of positive roots with respect to this ordering is $R+$. Let Δ be a set of simple roots of R with respect to this ordering then for each $a \in G$ there exists elements $e_{\alpha,a} \in (\mathfrak{L}^{\alpha})_a, f_{\alpha,a} \in (\mathfrak{L}^{-\alpha})_{-a}$ where $\alpha \in R_a \cap R^+$ such that $\{e_{\alpha,a}, f_{\alpha,a}, \mid a, b \in G, \alpha \in R_a \cap R^+\}$ along with a basis of $\mathfrak{h}$ forms a basis of $\mathfrak{L}$. Fix the lexicographic ordering on $G \times R$ respecting the above ordering on R and the linear ordering on G fixed earlier. Then retaining this ordering on the indexing set $G \times R$, we can totally order the basis of $\mathfrak{L}$ given above as $\{f_{a,\alpha}\}_{\alpha \in R^+ \cap R_a, a \in G} \cup \{h_1, \ldots, h_l\} \cup \{e_{a,\alpha}\}_{\alpha \in R^+ \cap R_a, a \in G}$ where $h_1 \ldots, h_l$ forms a totally ordered basis of $\mathfrak{h}$. Applying a G-graded version of the PBW-theorem [13, Cor. 2], $U(\mathfrak{L})$ has a homogeneous basis given by monomials of the form $x_1^{n_1} \cdots x_k^{n_k}$ where x_i's are elements of the above basis of L arranged such that $x_i \leq x_{i+1}$ whenever $1 \leq i \leq n-1$ and further, $x_i \neq x_{i+1}$ if $\epsilon(|x_i|, |x_i|) = -1$.

2.4.3. *Adjoint action of $\mathfrak{L}$ on $U(\mathfrak{L})$:* The enveloping algebra $U(\mathfrak{L})$ of a Lie color algebra $\mathfrak{L}$ is a G-graded associative algebra so the bracket $[\text{-}, \text{-}]_\epsilon$ is defined by $[x, a]_\epsilon := xa - \epsilon(|x|, |a|)ax$ for all $a \in U(\mathfrak{L})$, defines an action of $\mathfrak{L}$ on $U(\mathfrak{L})$. This we call the adjoint action on $U(\mathfrak{L})$. In particular, for $x \in \mathfrak{L}$ on $U(L)$ is given by $x \mapsto [x, a]_\epsilon$ for all $a \in U(\mathfrak{L})$. If $(\mathfrak{L}, \mathfrak{h})$ is an admissible pair, then with respect to the adjoint action of $\mathfrak{L}$ on it, $U(\mathfrak{L})$ has a weight space decomposition $\oplus_{\lambda \in \mathfrak{h}_{\mathbf{0}}^*} U(\mathfrak{L})^\lambda$ where $U(\mathfrak{L})^\lambda := \{a \in U(\mathfrak{L}) \mid [h, a] = \lambda(h)a \text{ for all } h \in \mathfrak{h}_{\mathbf{0}}\}$. Note that if $(\mathfrak{L}, \mathfrak{h})$ is a strongly admissible pair then from the monomial basis of $U(\mathfrak{L})$ described above, it can be easily seen that $U(\mathfrak{L})^0 \subset U(\mathfrak{L})_{\mathbf{0}}$.

3. Tensor product theorem for G-graded associative algebras

We start with a general result about irreducible representations of associative algebras. A proof of this statement may be found in [4].

Proposition 3.1. *Let A, B be associative algebras with 1 and $C := A \otimes B$ with its natural algebra structure. If M and N are (finite dimensional) irreducible modules of A and B respectively then $M \otimes N$ is an irreducible for C. Conversely, every finite dimensional irreducible module for C is uniquely expressed as a tensor product of the above form.* □

The above statement is known to be false for infinite dimensional modules. Further if A and B are G-graded associative algebras then the above statement is not true for the graded tensor product $A \otimes B$ even for finite dimensional irreducibles (See [4, §3.1.3] for an easy example when $G = \mathbb{Z}_2$.).

Let Λ be an abelian group whose identity element is denoted as 0. An associative algebra A graded by Λ such that $A_\lambda A_\mu \subset A_{\lambda+\mu}$ for all $\lambda, \mu \in \Lambda$, will be called an *ordinary Λ-graded algebra* or just a *graded algebra*. A module for an ordinary Λ-graded algebra A is always assumed to be a Λ-graded A-module. For a Λ-graded A-module, M, and a $\lambda \in \Lambda$ we denote the λ-graded part of M as M_λ. For any graded algebra A, we also assume that $1 \in A_0$. All the ideals and modules considered are taken to be left A-modules.

Keeping notations as above, we have the following:

Lemma 3.2. *Let $\mathcal{M}_0$ be a maximal ideal of A_0. Then there exists a unique Λ-graded maximal ideal $\mathcal{M}$ of A containing $\mathcal{M}_0$.*

Proof. Since $1 \in A$, the existence of a maximal ideal containing $\mathcal{M}_0$ is obtained by a standard Zorn's lemma argument. To prove the uniqueness, first note that $A_0/\mathcal{M}_0$ is an irreducible A_0-module and the inclusion map of A_0 into A is an A_0-homomorphism. Now, let W be another graded ideal in A containing $\mathcal{M}_0$ such that $W \not\subset \mathcal{M}$ then, $\mathcal{M} + W = A$, by the maximality of $\mathcal{M}$. Further, $W \cap \mathcal{M} \cap A_0 = \mathcal{M}_0$, by the maximality of $\mathcal{M}_0$ in A_0. Hence the A_0-homomorphism $A_0 \hookrightarrow A \to \frac{A}{W \cap \mathcal{M}}$ has kernel precisely $\mathcal{M}_0$. The quotient on the right is also Λ-graded since W and $\mathcal{M}$ are both Λ-graded. So, the image of the above map is $A_0 + W \cap \mathcal{M}/W \cap \mathcal{M}$ which is exactly the 0-component of the quotient, namely $(A/W \cap \mathcal{M})_0$. Since $A = W + \mathcal{M}$, we have $A/W \cap \mathcal{M} = W/W \cap \mathcal{M} \oplus \mathcal{M}/W \cap \mathcal{M}$, and

$$(A/W \cap \mathcal{M})_0 = (W/W \cap \mathcal{M} \oplus \mathcal{M}/W \cap \mathcal{M})_0 = (W/W \cap \mathcal{M})_0 \oplus (\mathcal{M}/W \cap \mathcal{M})_0.$$

The second equality comes from noting that both the submodules $W/W \cap \mathcal{M}$ and $\mathcal{M}/W \cap \mathcal{M}$ are Λ-graded. Since $A_0/\mathcal{M}_0$ is irreducible and the above gives a decomposition of $(A/W \cap \mathcal{M})_0$ into A_0-sumbodules, we conclude that one of the summands is 0. Let π_W and π_M denote the projections from $(A/W \cap \mathcal{M})$ onto the corresponding summands. Then $1 + W \cap \mathcal{M} = \pi_W(1) + \pi_{\mathcal{M}}(1)$. Suppose $\pi_M(1) \neq 0$ then the A_0-module generated by it is $A_0\pi_{\mathcal{M}}(1) = \pi_{\mathcal{M}}(A_0.1) = \pi_{\mathcal{M}}((A_0 + W \cap \mathcal{M})/W \cap \mathcal{M}) = (\mathcal{M}/W \cap \mathcal{M})_0$. From our earlier conclusion we then get $(W/W \cap \mathcal{M})_0 = 0$ so that $\pi_W(1) = 0$. In particular, $1 + W \cap \mathcal{M} = \pi_{\mathcal{M}}(1)$, leading to $\mathcal{M}/W \cap \mathcal{M} = A/W \cap \mathcal{M}$ which is a contradiction to $\mathcal{M} \subsetneq A$. Hence $\pi_W(1) = 1 + W \cap \mathcal{M}$, which in turn leads to $W = A$, proving that every proper ideal of A containing $\mathcal{M}_0$ is contained in $\mathcal{M}$. Hence establishing the uniqueness of $\mathcal{M}$. ☐

Remark 3.3. Note that if the quotient $A/\mathcal{M}$ by a left ideal $\mathcal{M}$ is an irreducible graded A-module then $\mathcal{M}$ is a maximal graded ideal since it occurs as the annihilator of a graded A-module. In this case the 0-component of $A/\mathcal{M}$ is $(A_0 + \mathcal{M})/\mathcal{M} \cong A_0/\mathcal{M}_0$, where $\mathcal{M}_0 = \mathcal{M} \cap A_0$. Further, in view of the Proposition 3.4 below, $A_0/\mathcal{M}_0$ is irreducible A_0-module so that $\mathcal{M}_0$ is a maximal ideal of A_0.

Let A be an ordinary Λ-graded algebra and M be a graded module for A. We say M is a *finite weight module* for A if each graded component M_λ, $\lambda \in \Lambda$ is a finite dimensional vector space over $\mathbb{C}$.

The next proposition relates the irreducible, finite weight modules of A with the finite dimensional, irreducible modules of A_0, thus extending the main result of [7] to finite weight modules of ordinary Λ-graded algebras.

Proposition 3.4. *Let M be an irreducible finite weight graded A-module such that $M_\lambda \neq 0$ for some $\lambda \in \Lambda$. Then M_λ is a finite dimensional irreducible module for A_0. Conversely, if K is a finite dimensional irreducible module for A_0 then there is a unique (upto isomorphism) irreducible finite weight module M of A with $M_\lambda \cong K$ as A_0 modules.*

Proof. First note that if M is a (irreducible) graded A-module then the module $\sqcap_\mu M$ is also a (irreducible) graded A-module and vice versa. Here $\sqcap_\mu$ is the shift operator which makes the $(\lambda - \mu)$-component of M into the λ-component of $\sqcap_\mu M$. So we may assume henceforth that $\lambda = 0$.

Let $0 \neq x \in M_0$. Then by the irreducibility of M we know that the A-submodule generated by x, denoted as $\langle x \rangle_A$, is M. Since M is Λ-graded, we have $A_\mu M_0 \subset M_\mu$. Hence, the only $\mu \in \Lambda$, that takes $A_\mu x$ to M_0 is $\mu = 0$. In particular, since $\langle x \rangle_A = M$, we have $\langle x \rangle_{A_0} = M_0$ for any $0 \neq x \in M_0$. This establishes the first assertion. To prove the second assertion, we apply Lemma 3.2 to the annihilator $\mathcal{M}_0$ of $0 \neq x \in K$. By the irreducibility of K, we get that $\mathcal{M}_0$ is a maximal ideal of A_0. Applying the lemma, we get a unique maximal ideal $\mathcal{M}$ of A containing $\mathcal{M}_0$ such that $(A/\mathcal{M})_0 \cong A_0/\mathcal{M}_0$. Thus, $A/\mathcal{M}$ is the required irreducible. To prove the uniqueness, consider M' to be another such irreducible satisfying the condition that $(M')_0 \cong K$. Then the annihilator $\mathcal{M}' \subset A$ of any $0 \neq y \in M'_0$ is a maximal ideal of A on one hand, and on the other hand the annihilator of y in A_0 is $\mathcal{M}' \cap A_0 = \mathcal{M}_0$. By the uniqueness part of Lemma 3.2, we get $\mathcal{M} = \mathcal{M}'$. Hence $M \cong M'$, as required. □

The next result may be regarded as a generalization of the Proposition 3.1 for finite dimensional modules of the tensor product of two graded algebras to "finite weight modules" for graded algebras. Let A, B be two graded algebras over $\mathbb{C}$ with Λ and Ω grading, respectively. Then $A \otimes B$ is an associative graded algebra with a natural $\Lambda \times \Omega$-grading. We describe the irreducible finite weight modules for $A \otimes B$ with respect to this grading.

Theorem 3.5. *Let $A,\ B$ be two associative graded algebras over $\mathbb{C}$ with Λ and Ω grading, respectively.*

(1) Let M and N be irreducible finite weight modules for A and B, respectively. Then $M \otimes N$ is an irreducible finite weight module for $A \otimes B$; and
(2) Every irreducible finite weight module for $A \otimes B$ is uniquely expressed in the above form.

Proof. We prove this more generally below in Theorem 3.11 for G-graded associative algebras. Both the assertions above then follow by taking the trivial G-grading on A and B. □

3.1. **G-graded algebras.** We now consider A to be a G-graded associative algebra. Suppose A further has a grading given by an abelian group Λ, whose identity element is 0. For clarity, we will indicate the Λ-grading as superscripts and the G-grading as subscripts.

We say that the Λ-grading on A is a *G-compatible grading* if A^0 is a G-graded associative algebra and for each $\lambda \in \Lambda$, A^λ is a G-graded A^0-module. In particular, A is also an ordinary $G \times \Lambda$-graded algebra, where $A_{a,\lambda} := (A^\lambda)_a$. With this notation, $A_{\mathbf{0}} = \oplus_{\lambda\in\Lambda} A_{\mathbf{0},0}$ and $A^0 = \oplus_{a\in G} A_{a,0}$.

All modules considered for a $G \times \Lambda$-graded algebra are assumed to be $G \times \Lambda$-graded. In particular, the induced Λ-grading is G-compatible. So if M is such a module for A, then for each $\lambda \in \Lambda$, the component M^λ of M is a G-graded A^0-module. For $(a, \lambda) \in G \times \Lambda$, we denote the (a, λ)-graded part as $M_{a,\lambda}$, and $M^\lambda = \oplus_{a\in G} M_{a,\lambda}$.

With notations and conventions as above, we can apply Lemma 3.2 and Proposition 3.4 to the $G \times \Lambda$-graded algebra A and its graded module M, we get the following Proposition which is different from [16, Proposition 2.4]. The following formulation permits us to handle more cases of the tensor product theorem than those discussed there.

Lemma 3.6. *Let $\mathcal{M}_{\mathbf{0},0}$ be a maximal ideal of $A_{\mathbf{0},0}$. Then there exists a unique $G\times\Lambda$-graded maximal ideal $\mathcal{M}$ of A containing $\mathcal{M}_{\mathbf{0},0}$.* □

Proposition 3.7. *Let M be an irreducible finite weight graded A-module such that $M_{a,\lambda} \neq 0$ for some $(a, \lambda) \in G \times \Lambda$. Then $M_{a,\lambda}$ is a finite dimensional irreducible module for $A_{\mathbf{0},0}$. Conversely, if K is a finite dimensional irreducible module for $A_{\mathbf{0},0}$ then there is a unique (upto isomorphism) irreducible finite weight module M of A with $M_{a,\lambda} \cong K$ as $A_{\mathbf{0},0}$ modules.* □

3.1.1. *Another G-analogue of the above results.* We retain the notations and conventions as in §3.1. Let A be an G-graded associative algebra with a G-compatible grading given by an abelian group Λ and M be a $G \times \Lambda$-graded A-module. For brevity, we also use the notation M_a^λ to denote the graded component $M_{a,\lambda}$, where $a \in G$ and $\lambda \in \Lambda$. Note that if M is a finite weight $G \times \Lambda$-graded A-module then M^λ is finite dimensional (since $|G|$ is finite).

The following lemma and proposition are the G-graded version of Lemma 3.2 and Proposition 3.4. They can also be proved following the same line of arguments given there, so we will skip repeating the proofs here.

Lemma 3.8. *With notations as above, let $\mathcal{M}^0$ be a G-graded maximal ideal of A^0, where $0 \in \Lambda$. Then there exists a unique $G \times \Lambda$-graded maximal ideal $\mathcal{M}$ of A containing $\mathcal{M}^0$. Further, $\mathcal{M} \cap A^0 = \mathcal{M}^0$.* □

Proposition 3.9. *Let M be an irreducible finite weight graded A-module such that $M^\lambda \neq 0$ for some $\lambda \in \Lambda$. Then M^λ is a finite dimensional irreducible module for A^0. Conversely, if K is an irreducible finite dimensional module for A^0 then there is a unique (upto isomorphism) irreducible finite weight module M of A with $M^\lambda \cong K$ as A^0 modules.* □

3.2. **Main theorem.** Owing to the braiding in the category of G-graded vector spaces, the tensor product of two G-graded algebras is itself a G-graded algebra with multiplication given by $a \otimes b.a' \otimes b' = \epsilon(|b|, |a'|)aa' \otimes bb'$. This will be called the *graded tensor product*.

Let A and B be two G-graded associative algebras with G-compatible grading given by abelian groups Λ and Ω, respectively. Then the graded tensor product $A \otimes B$ has a G-compatible $\Lambda \times \Omega$-grading. So it is naturally a $G \times \Lambda \times \Omega$-graded algebra; its ideals considered are also $G \times \Lambda \times \Omega$-graded.

Through the rest of this section we will try to obtain a generalization of Proposition 3.1 to the G-graded setting. In the case of $G = \mathbb{Z}_2$, the following result may be considered an extension of [16, Lemma 2.1]. This formulation allows us to handle more cases than that discussed in [16]. Keeping notations as described above, we have:

Lemma 3.10. *Let A, B be two G-graded associative algebras with G-compatible grading by Λ and Ω, respectively. Consider the following:*

(1) Let $I \subseteq A$ is a maximal $G \times \Lambda$-graded left ideal such that $I^0 = I \cap A^0$ is a maximal left ideal of A^0.
(2) Let $J \subseteq B$ is a maximal $G \times \Omega$-graded left ideal such that $J^0 = J \cap B^0$ is a maximal left ideal of B^0.
(3) Set $\mathcal{I} := (A \otimes J) + (I \otimes B)$, $\mathcal{G} := \Lambda \times \Omega$, and $\mathcal{A} := A \otimes B$ with the induced $\mathcal{G}$-grading which is compatible with the natural G-grading on the tensor product.

If, for every $G \times \mathcal{G}$-graded left ideal T such that $I \subseteq T \subsetneq \mathcal{A}$ we get that $T_{\mathbf{0}}^{(0,0)} := T \cap (A^0 \otimes B^0)_{\mathbf{0}} = \mathcal{I}_{\mathbf{0}}^{(0,0)} = (A^0 \otimes J^0)_{\mathbf{0}} + (I^0 \otimes B^0)_{\mathbf{0}}$, then $\mathcal{I}$ is a maximal $G \times \mathcal{G}$-graded left ideal of $\mathcal{A}$.

Proof. Suppose $\mathcal{I}$ is not maximal. Then there is a $G \times \Lambda \times \Omega$-graded ideal T of $A \otimes B$ such that $T \supsetneq \mathcal{I}$. Let $x \in T \setminus \mathcal{I}$. Fix a $G \times \Lambda$ (resp. $G \times \Omega$) homogeneous basis of A (resp., of B) containing 1, given by the set $\{x_i | i \in \Pi\}$ (resp., $\{y_j | j \in \Gamma\}$) such that $\Pi' \subset \Pi$ and $\Pi^0 \subset \Pi$ (resp., $\Gamma' \subset \Gamma$, $\Gamma^0 \subset \Gamma$) index bases of I and A^0, respectively (resp., J, B^0). The condition that $x \notin \mathcal{I}$ implies that expressing x in terms of the basis elements $x_i \otimes y_j$, there is at least one element where $i \in \Pi \setminus \Pi'$, $j \in \Gamma \setminus \Gamma'$ in such an expression. After re-indexing and re-scaling, if required, we denote this element as $x_1 \otimes y_1$. Note $x_1 \notin I$,

$y_1 \notin J$. Further, since $\mathcal{I} \subset T$, we may assume that $x \in T \setminus \mathcal{I}$ has an expression of the form $x_1 \otimes y_1 + \sum_{r=1}^{s} \alpha_r x_{i_r} \otimes y_{j_r}$ where $\alpha_r \in \mathbb{C}$, $i_r \in \Pi \setminus \Pi'$ and $j_r \in \Gamma \setminus \Gamma'$.

Step 1: *There exists an element of the form* $1 \otimes y_1 + \sum_{r=1}^{s} \alpha_r x_{i_r} \otimes y_{j_r} \in T$ *where* $\alpha_r \in \mathbb{C}$, $i_r \in \Pi \setminus \Pi'$ *and* $j_r \in \Gamma \setminus \Gamma'$ *and* $y_{j_r} \neq y_1$. For this, consider the subset $S_A := \{a \in A \mid a \otimes y_1 + \sum_{r=1}^{s} \alpha_r x_{i_r} \otimes y_{j_r} \in T \text{ where } i_r \in \Pi \setminus \Pi', j_r \in \Gamma \setminus \Gamma', y_{j_r} \neq y_1\}$. It is easy to see that this subset is a non-zero left ideal in A. Also, it contains I since $I \otimes B \subset T$. Now, from the expression for $x \in T$ given above and noting that $x_1 \notin I$, we get $x_1 \in S_A \setminus I$. To see that S_A is a $G \times \Lambda$-graded ideal note that T is a $G \times \Lambda \times \Omega$-graded ideal. So for any $(\lambda, \gamma) \in \Lambda \times \Omega$, the (λ, γ)-component of an expression of the form $a \otimes y_1 + \sum_{r=1}^{s} \alpha_r x_{i_r} \otimes y_{j_r} \in T$, also belongs to T. In particular, taking $\gamma \in \Omega$ such that y_1 is a homogeneous (h_1, γ)-graded element for some $h_1 \in G$, we may assume that all the elements in the above expression have $\Lambda \times \Omega$-grading given by a fixed (λ, γ) for some $\lambda \in \Lambda$. Hence, if $a \in S_A$, all its Λ-graded components also belong to S_A. Thus, S_A is a Λ-graded ideal of A. To show that S_A is G-graded, consider $a = a_{g_1} + \cdots + a_{g_k} \in S_A$ where $a_{g_i} \in A_{g_i}$ for distinct $g_1, \ldots, g_k \in G$. Since we have already seen that S_A is Λ-graded, we may assume that $a \in S_A^{\lambda}$ for some $\lambda \in \Lambda$. Hence there is an expression of the form $a \otimes y_1 + \sum_{r=1}^{s} \alpha_r x_{i_r} \otimes y_{j_r} \in T^{(\lambda, \gamma)}$. Let the $G \times \Lambda$-grading of x_{i_r} in the expression be (h_{i_r}, λ) and the $G \times \Omega$-grading of y_{j_r} be (h'_{j_r}, γ) for some $h_{i_r}, h'_{j_r} \in G$. Since T is a $G \times \Lambda \times \Omega$-graded ideal the $(g_i + h_1, \lambda, \gamma)$-component of the above expression belongs to T, for each i. As the g_i's are distinct and $a = a_{g_1} + \cdots + a_{g_k}$, this component has the form $a_{g_i} \otimes y_1 + \sum_{r: h_{i_r} + h'_{j_r} = g_i + h_1} \alpha_r x_{i_r} \otimes y_{j_r}$. So $a_{g_i} \in S_A$ for each g_i. Thus, S_A is a $G \times \Lambda$-graded ideal of A. By the maximality of I in A, we get that $S_A = A$. In particular, S_A contains 1, *i.e.,* there exists an element of the form $1 \otimes y_1 + \sum_{r=1}^{s} \alpha_r x_{i_r} \otimes y_{j_r} \in T$. Since $\mathcal{I} \subset T$, we may assume that in the above expression $i_r \in \Pi \setminus \Pi'$ and $j_r \in \Gamma \setminus \Gamma'$.

Step 2: *There exists an element of the form* $1 \otimes 1 + \sum_{r=1}^{s} \alpha_r x_{i_r} \otimes y_{j_r} \in T$ *where* $\alpha_r \in \mathbb{C}$, $i_r \in \Pi \setminus \Pi'$ *and* $j_r \in \Gamma \setminus \Gamma'$. To see this, consider the set $S_B := \{b \in B \mid 1 \otimes b + \sum_{r=1}^{s} \alpha_r x_{i_r} \otimes y_{j_r} \in T \text{ where } i_r \in \Pi \setminus \Pi', j_r \in \Gamma \setminus \Gamma'\}$. Repeating the argument as in Step 1 with the role of $y_1 \in B$ replaced by $1 \in A$, we note that the above subset of B is a $G \times \Omega$-graded left ideal containing J. By Step 1, $y_1 \in S_B \setminus J$. Hence by maximality of J in B, S_B is the whole of B. This establishes the assertion of Step 2.

Final Step: *There exists an element of the form* $1 \otimes 1 + \sum_{r=1}^{s} \alpha_r x_{i_r} \otimes y_{j_r} \in T_{\mathbf{0}}^{(0,0)}$ *where* $\alpha_r \in \mathbb{C}$, $i_r \in \Pi^0 \setminus (\Pi' \cap \Pi^0)$ *and* $j_r \in \Gamma^0 \setminus (\Gamma' \cap \Gamma^0)$. *The latter element is clearly not in* $\mathcal{I}_{\mathbf{0}}^{(0,0)}$. *This gives the required contradiction to the hypothesis that* $T_{\mathbf{0}}^{(0,0)} = \mathcal{I}_{\mathbf{0}}^{(0,0)}$. For showing the existence of such an element in $T_{\mathbf{0}}^{(0,0)}$, we note that T is given to be $G \times \mathcal{G}$-graded. Hence the $(\mathbf{0}, (0,0))$-component of the element obtained in Step 2, containing $1 \otimes 1$, belongs to $T_{\mathbf{0}}^{(0,0)}$. The rest of the claim follows from this. $\square$

We are now in a position to apply the above results to obtain the main theorem.

Theorem 3.11. *Let* A, B *be two* G*-graded associative algebras over* $\mathbb{C}$ *with* G*-compatible grading given by* Λ *and* Ω*, respectively. Assume that* $A^0 = A^0_{\mathbf{0}}$*. Then we have the following:*

(1) let M *and* N *be irreducible finite weight modules for* A *and* B*, respectively. Then* $M \otimes N$ *is an irreducible finite weight module for* $A \otimes B$*.*
(2) every irreducible finite weight module for $A \otimes B$ *is uniquely expressed in the above form.*

Proof. To prove Part (1) of the Theorem, we use Lemma 3.10 to show that the annihilator of $M \otimes N$ in $A \otimes B$ is maximal. Applying the shift operator as done in the beginning of the proof of Proposition 3.4, we may assume without loss of generality that $M^0 \neq 0$, $N^0 \neq 0$. Let I denote the annihilator in A of $0 \neq v \in M^0$ and J denote the annihilator in B of $0 \neq w \in N^0$. Since M and N are G-graded modules as well, we may assume that v and w are homogeneous of degree $c, d \in G$ respectively. In view of Remark 3.3, we also note that the annihilators are maximal graded (left) ideals. Fix a homogeneous basis of A (resp., of B) containing 1, given by the set $\{x_i | i \in \Pi\}$ (resp., $\{y_j | j \in \Gamma\}$). Since M and N are irreducible A , B-modules respectively, we have $Av = M$ and $Bw = N$. So we can choose $\Pi' \subset \Pi$, (resp., $\Gamma' \subset \Gamma$) such that $\{x_i v \mid i \in \Pi'\}$ and $\{y_j w \mid i \in \Gamma'\}$ index a basis of M (resp., N).

Claim 1: *Annihilator in $A \otimes B$ of $v \otimes w$, denoted as $\mathcal{I}$, is equal to the left ideal $I \otimes B + A \otimes J$.* The containment of $I \otimes B + A \otimes J$ in $\mathcal{I}$ is obvious. To show the other way containment let us consider an $a \in \mathcal{I}$. The element a may be expressed as $\sum_{p \in \Gamma} c_p \otimes y_p$ for some $c_p \in A$. Since $a.(v \otimes w) = 0$, we get $\sum_{p \in \Gamma} c_p v \otimes y_p w = 0$. In terms of the basis of M, we express $c_p v$ as $\sum_{q \in \Pi'} \alpha_q^p x_q v$, leading to the relation

$$\sum_{p \in \Gamma} (\sum_{q \in \Pi'} \alpha_q^p x_q v \otimes y_p w) = 0$$

in $M \otimes N$. It is immediate from the expression for $c_p v$ obtained above that, for each $p \in \Gamma$, $d_p := c_p - \sum_{q \in \Pi'} \alpha_q^p x_q$ annihilates v, hence is in I. Interchanging the summations in the above relation, we get $\sum_{q \in \Pi'} (x_q v \otimes \sum_{p \in \Gamma} \alpha_q^p y_p w) = 0$. As the elements $\{x_q v\}_{q \in \Pi'}$ are linearly independent in M, it follows that $\sum_{p \in \Gamma} \alpha_q^p y_p w = 0$, implying that $\sum_{p \in \Gamma} \alpha_q^p y_p \in J$. We can thus express a as $\sum_{p \in \Gamma} (d_p \otimes y_p) + \sum_{p \in \Gamma} \sum_{q \in \Pi'} \alpha_q^p x_q \otimes y_p$. The first term belongs to $I \otimes B$. By interchanging summations, the second term can be re-written as $\sum_{q \in \Pi'} (x_q \otimes \sum_{p \in \Gamma} \alpha_q^p y_p)$ which belongs to $A \otimes J$. Thus $a \in I \otimes B + A \otimes J$, as required.

Claim 2: *$\mathcal{I}$ is a maximal $G \times \Lambda \times \Omega$-graded ideal in $A \otimes B$.* To prove that $\mathcal{I}$ is a maximal $G \times \Lambda \times \Omega$-graded ideal, we first note that by the assumption that $A^0 = A^0_{\mathbf{0}}$, we have $\mathcal{I}^{(0,0)}_{\mathbf{0}} = (A^0 \otimes J^0)_{\mathbf{0}} + (I^0 \otimes B^0)_{\mathbf{0}} = A^0_{\mathbf{0}} \otimes J^0_{\mathbf{0}} + I^0_{\mathbf{0}} \otimes B^0_{\mathbf{0}} = \mathcal{I}^{(0,0)}_{(\mathbf{0},\mathbf{0})} := \mathcal{I} \cap A^0_{\mathbf{0}} \otimes B^0_{\mathbf{0}}$. The latter is the annihilator of $v \otimes w$ in $A^0_{\mathbf{0}} \otimes B^0_{\mathbf{0}}$. By Proposition 3.7 applied to M and to N, followed by the Proposition 3.1 applied to $A^0_{\mathbf{0}}$ and $B^0_{\mathbf{0}}$ we get, $M^0_c \otimes N^0_d$ is an irreducible $A^0_{\mathbf{0}} \otimes B^0_{\mathbf{0}}$-module. So its annihilator $\mathcal{I}^{(0,0)}_{\mathbf{0},\mathbf{0}}$ is a maximal ideal in $A^0_{\mathbf{0}} \otimes B^0_{\mathbf{0}} = (A^0 \otimes B^0)_{\mathbf{0}}$. By the latter equality, it is also a maximal (G-graded) ideal in $(A^0 \otimes B^0)_{\mathbf{0}}$. For any $G \times \Lambda \times \Omega$-graded ideal of $A \otimes B$ containing $\mathcal{I}$, we have $T^{(0,0)}_{\mathbf{0}} \supset \mathcal{I}^{(0,0)}_{\mathbf{0}} = \mathcal{I}^{(0,0)}_{\mathbf{0},\mathbf{0}}$. So by the maximality of $\mathcal{I}^{(0,0)}_{\mathbf{0},\mathbf{0}}$ in $(A^0 \otimes B^0)_{\mathbf{0}}$, we get equality everywhere. Hence the required condition of Lemma 3.10 is satisfied. Claims 1 and 2 put together proves (1).

In order to establish (2), let us consider an irreducible, finite weight module L of $A \otimes B$ (with respect to the $\Lambda \times \Gamma$-grading). Applying the parity functor Π_a if required, followed by the Proposition 3.7 applied to the G-graded algebra $A \otimes B$ and its irreducible G-module L, we get $L^{(\lambda(1),\lambda(2))}_{\mathbf{0}} \neq 0$ is an irreducible $(A^0 \otimes B^0)_{\mathbf{0}}$-module for some $(\lambda(1), \lambda(2)) \in \Lambda \times \Omega$. Under the assumption $A^0 = A^0_{\mathbf{0}}$, we know that $(A^0 \otimes B^0)_{\mathbf{0}} = A^0_{\mathbf{0}} \otimes B^0_{\mathbf{0}}$. So applying the second assertion of Proposition 3.1 to $A^0_{\mathbf{0}} \otimes B^0_{\mathbf{0}}$ and its irreducible module $L^{(\lambda(1),\lambda(2))}_{\mathbf{0}}$, we get that $L^{(\lambda(1),\lambda(2))}_{\mathbf{0}} \cong K(1) \otimes K(2)$ for certain unique irreducible finite dimensional modules $K(1)$ and $K(2)$ of $A^0_{\mathbf{0}}$ and $B^0_{\mathbf{0}}$, respectively. We now apply the second assertion of Proposition 3.7 individually to $K(1)$ and to $K(2)$, to get irreducible finite weight modules $M(1)$, $M(2)$ of A and B, respectively, such that $M(i)^{\lambda(i)}_{\mathbf{0}} \cong K(i)$, for $i = 1, 2$. Applying the assertion of Part (1) to $M(1)$, $M(2)$, we get that $M(1) \otimes M(2)$ is an irreducible $A \otimes B$-module. By Proposition 3.7, this implies that $(M(1)^{\lambda(1)} \otimes M(2)^{\lambda(2)})_{\mathbf{0}}$ is an irreducible $(A^0 \otimes B^0)_{\mathbf{0}}$-module. However, $M(1)^{\lambda(1)}_{\mathbf{0}} \otimes M(2)^{\lambda(2)}_{\mathbf{0}} \subset (M(1)^{\lambda(1)} \otimes M(2)^{\lambda(2)})_{\mathbf{0}}$ as $A^0_{\mathbf{0}} \otimes B^0_{\mathbf{0}}$-module hence also as $(A^0 \otimes B^0)_{\mathbf{0}}$. Thus equality holds by the irreducibility of $(M(1)^{\lambda(1)} \otimes M(2)^{\lambda(2)})_{\mathbf{0}}$ as a $(A^0 \otimes B^0)_{\mathbf{0}}$-module. The isomorphisms above put together with the above equality, gives $L^{(\lambda(1),\lambda(2))}_{\mathbf{0}} \cong (M(1)^{\lambda(1)} \otimes M(2)^{\lambda(2)})_{\mathbf{0}}$ as $(A^0 \otimes B^0)_{\mathbf{0}} = A^0_{\mathbf{0}} \otimes B^0_{\mathbf{0}}$-modules . Now applying the uniqueness part of Proposition 3.7 readily gives $L \cong M(1) \otimes M(2)$. The uniqueness of $M(1)$ and $M(2)$ as obtained above, comes from the uniqueness part of Proposition 3.7 applied separately to $K(1)$, $K(2)$. $\square$

As a consequence of this, taking A to be an associative algebra with the trivial $\mathbb{Z}_2$-grading and $\Lambda = \{0\} = \Omega$, we obtain an alternate proof of the following result from Cheng [3]:

Corollary 3.12. *Let A be an associative algebra and B be an associative superalgebra. Let M and N be irreducible finite dimensional modules for A and B, respectively. Then $M \otimes N$ is an irreducible finite dimensional module for $A \otimes B$. Further, every irreducible finite dimensional module for $A \otimes B$ is uniquely expressed in the above form.*

4. Weight spaces of G-graded representations and irreducibility

Let $(\mathfrak{L}, \mathfrak{h})$ be an admissible pair for a Lie color algebra $\mathfrak{L}$. All the modules for $\mathfrak{L}$ considered here are assumed to be G-graded.

Let V be an irreducible G-graded representation of $\mathfrak{L}$. Then using the decomposition of $U(\mathfrak{L})$ into $\mathfrak{h}_{\mathbf{0}}$-weight spaces as given in §2.4.3, it is easy to see that V will be a weight module (not necessarily a finite weight module), if it has at least one non-zero weight vector, *i.e.,* if there is a $v \neq 0$ in V such that $h.v = \lambda(h)v$ for some $\lambda \in \mathfrak{h}_{\mathbf{0}}^*$ and for all $h \in \mathfrak{h}_{\mathbf{0}}$. (Indeed, $V = U(\mathfrak{L})v$ hence $U(\mathfrak{L})^\mu v$ has weight $\mu + \lambda$, so $V = \oplus_{\mu \in \mathfrak{h}_{\mathbf{0}}^*} V^\mu$.)

The aim of this section is to show that if V is an irreducible finite weight module of $\mathfrak{L}$ then V^λ for each $\lambda \in$ *Support* (V) is irreducible for $\mathcal{C}$, the centralizer of $\mathfrak{h}_{\mathbf{0}}$ in $U(\mathfrak{L})$ which also coincides with the 0-weight space $U(\mathfrak{L})^0$. The $(\mathbf{0}, 0)$-component of $U(\mathfrak{L})$ is then the $\mathbf{0}$-component of $\mathcal{C}$, denoted as $\mathcal{C}_{\mathbf{0}}$. By applying Propositions 3.7 and 3.9 to V, we note that for every irreducible finite weight module the non-zero weight spaces V^λ are G-graded irreducible $\mathcal{C}$-modules and that each non-zero G-graded component V^λ_a of V^λ is an irreducible $\mathcal{C}_{\mathbf{0}}$-module.

Let $\lambda \in \mathfrak{h}_{\mathbf{0}}^*$. Note that $\mathfrak{h}_{\mathbf{0}} \subset \mathcal{C}_{\mathbf{0}}$. The $\mathcal{C}$ (resp., $\mathcal{C}_{\mathbf{0}}$)-modules obtained above are called λ-weighted modules for $\mathcal{C}$ (resp., $\mathcal{C}_{\mathbf{0}}$). In general, a $\mathcal{C}$ (resp., $\mathcal{C}_{\mathbf{0}}$)-module K is said to be *λ-weighted module for $\mathcal{C}$ (resp., $\mathcal{C}_{\mathbf{0}}$)* if

i) K is irreducible $\mathcal{C}$ (resp., $\mathcal{C}_{\mathbf{0}}$)-module, and
ii) $h \in \mathfrak{h}_{\mathbf{0}}$ acts on K by the scalar $\lambda(h)$.

Applying results from the previous section to $U(\mathfrak{L})$ we show that every λ-weighted module for $\mathcal{C}$ (resp., $\mathcal{C}_{\mathbf{0}}$) is obtained by the above construction. It may be remarked here that when $G = \mathbb{Z}_2$ and $\epsilon = \epsilon_0$, the results of this section give the analogues of Lemire's results for admissible pairs of Lie superalgebras.

Theorem 4.1. *Let $(\mathfrak{L}, \mathfrak{h})$ be an admissible pair. Let $\mathcal{C}$ denote the centralizer $\mathfrak{h}_{\mathbf{0}}$ in $U(\mathfrak{L})$ and $\mathcal{C}_{\mathbf{0}}$ denote the $\mathbf{0}$-component of $\mathcal{C}$.*

i) *Let M be an irreducible finite weight module for $\mathfrak{L}$ and $\lambda \in$ Support (M). Then M^λ is a finite dimensional irreducible module for $\mathcal{C}$ and M_a^λ is a finite dimensional irreducible module for $\mathcal{C}_{\mathbf{0}}$ for each $a \in G$ such that $M_a^\lambda \neq 0$.*
ii) *Let $(\rho_1, V_1), (\rho_2, V_2)$ be two irreducible finite weight modules and $\lambda \in \mathfrak{h}_{\mathbf{0}}^*$ such that $(V_1)_{a_1}^\lambda \neq 0$ and $(V_2)_{a_2}^\lambda \neq 0$ for some $a_1, a_2 \in G$. If $(V_1)^\lambda \cong (V_2)^\lambda$ as $\mathcal{C}$-modules then $V_1 \cong V_2$ as $U(\mathfrak{L})$-module. Similarly, if $a_1 = a_2 = a$ and $(V_1)_a^\lambda \cong (V_2)_a^\lambda$ as $\mathcal{C}_{\mathbf{0}}$-modules then $V_1 \cong V_2$ as $U(\mathfrak{L})$-module.*
iii) *Every finite dimensional λ-weighted (irreducible) module of $\mathcal{C}$ (resp., $\mathcal{C}_{\mathbf{0}}$) is isomorphic to V^λ (resp., $V_{\mathbf{0}}^\lambda$) for some irreducible representation (ρ, V).*

Proof. The parts (i) follows by applying the first parts of Proposition 3.9 and Proposition 3.7 respectively to the $U(\mathfrak{L})$-module M. Part(iii) is obtained by the converse part of Proposition 3.9 and Proposition 3.7 applied to the $U(\mathfrak{L})$-module M. Part (ii) is an immediate consequence of the uniqueness part of Lemma 3.8 and Lemma 3.6 applied to the annihilators of V_1 and V_2. □

5. Tensor product theorem

The following theorem can be obtained as an immediate application of Theorem 3.11 to $U(\mathfrak{L}_1)$, $U(\mathfrak{L}_2)$, where $\mathfrak{L}_1$, $\mathfrak{L}_2$ are Lie color algebras or Lie superalgebras.

Theorem 5.1. *With notations as above, let $(\mathfrak{L}_1, \mathfrak{h}_1)$ and $(\mathfrak{L}_2, \mathfrak{h}_2)$ be two admissible pairs with corresponding root systems $R(1)$, $R(2)$ respectively. Assume that at least one of the above pairs is strongly admissible. Set $\mathfrak{L} := \mathfrak{L}_1 \bigoplus \mathfrak{L}_2$.*

(1) *For $i = 1, 2$, let $M(i)$ be an irreducible finite weight module for $\mathfrak{L}_i$ then $M := M(1) \otimes M(2)$ is an irreducible finite weight module for $\mathfrak{L}$.*
(2) *Each irreducible finite weight module of $\mathfrak{L}$ is uniquely expressed in the form $M(1) \otimes M(2)$ for some irreducible finite modules $M(1)$, $M(2)$ of $\mathfrak{L}_1, \mathfrak{L}_2$ respectively.*

Proof. Assume $(\mathfrak{L}_1, \mathfrak{h}_1)$ to be a strongly admissible pair. In this case $U(\mathfrak{L}_1)^0 = U(\mathfrak{L}_1)^0 \cap U(\mathfrak{L}_1)_{\mathbf{0}}$ (see §2.4.3) as required in Theorem 3.11. Hence the two assertions immediately follow by applying Theorem 3.11 to $U(\mathfrak{L}_1)$, $U(\mathfrak{L}_2)$ and the modules $M(1)$ and $M(2)$. □

Note. In the above theorem, the assumption that one of the admissible pairs is strongly admissible cannot be dropped as illustrated by the following example. We retain notations as in 2.3.2(b). Take $\mathfrak{L}_j = \mathfrak{q}(3)$ for $j = 1, 2$. Let $\mathfrak{h}$ denote the Cartan subalgebra of $\mathfrak{q}(3)$ and $\mathfrak{b} = \oplus_{1 \leq i < j \leq 3} \mathfrak{q}(3)_{\varepsilon_i - \varepsilon_j}$ be the standard Borel subalgebra. Set $\lambda_j := \lambda$ for $j = 1, 2$, where $\lambda \in \mathfrak{h}_{\bar{0}}^*$ is given by $\lambda(H_i) = i$ for $i = 1, 2, 3$. We follow the construction as in [4, §1.5.4] to obtain the irreducible highest weight module corresponding to λ by first constructing an irreducible $\mathfrak{h}$-module W_λ, extending it trivially to a module for $\mathfrak{b}$, then inducing it to $\mathfrak{L}_j$, followed by taking its quotient by its unique maximal submodule. We denote this irreducible module of $\mathfrak{L}_j$ as $M(j)$, for $j = 1, 2$. By [4, Theorem 2.18], $M(j)$, $j = 1, 2$ are finite dimensional modules. As illustrated in *loc. cit.*, the Clifford algebra $\mathcal{C}_\lambda$ associated to the bilinear form $\langle \cdot, \cdot \rangle_\lambda$ given by $\langle v, w \rangle_\lambda := \lambda([v, w])$ for $v, w \in \mathfrak{h}_{\bar{1}}$ when restricted to the quotient space $\mathfrak{h}_{\bar{1}}/\text{Rad}\, \langle \cdot, \cdot \rangle_\lambda$ is isomorphic to $U(\mathfrak{h})/I_\lambda$, where I_λ is the ideal generated by Rad $\langle \cdot, \cdot \rangle_\lambda$ and $a - \lambda(a)$ for $a \in \mathfrak{h}_{\bar{0}}$. The irreducible $\mathfrak{h}$-module W_λ is identified with the unique irreducible of the Clifford algebra $\mathcal{C}_\lambda$ considered as an $\mathfrak{h}$-module under the above isomorphism. Then W_λ has dimension $2^{\lceil m/2 \rceil}$ where $m = \dim \mathfrak{h}_{\bar{1}}/\text{Rad}\langle \cdot, \cdot \rangle_\lambda$ and $\lceil m/2 \rceil$ is the smallest integer greater than or equal to $m/2$. Since $\text{Rad}\langle \cdot, \cdot \rangle_\lambda = 0$ for the above λ, we get $m = 3$ and so dimension of W_λ is 4. The $\mathfrak{h}$-module W_λ is the λ_i-weight space of $M(i)$ for $i = 1, 2$. Since both $M(1)$ and $M(2)$ are highest weight modules, the tensor product $M(1) \otimes M(2)$ is a finite dimensional weight module for the admissible pair $(\mathfrak{L}_1 \oplus \mathfrak{L}_2, \mathfrak{h} \oplus \mathfrak{h})$. The dimension of its (λ_1, λ_2)-weight space is 16. Also, $M(1) \otimes M(2)$ is generated as a $\mathfrak{L}_1 \oplus \mathfrak{L}_2$-module by its (λ_1, λ_2)-weight space. So, suppose $M(1) \otimes M(2)$ is an irreducible $\mathfrak{L}_1 \oplus \mathfrak{L}_2$-module then its (λ_1, λ_2)-weight space has to be an irreducible $\mathfrak{h} \oplus \mathfrak{h}$-module (indeed, the $U(\mathfrak{L}_1) \otimes U(\mathfrak{L}_2)$-module generated by any proper $\mathfrak{h} \oplus \mathfrak{h}$-submodule of the (λ_1, λ_2)-weight space $W_\lambda \otimes W_\lambda$, gives rise to a proper submodule of $M(1) \otimes M(2)$, contradicting the irreducibility of the latter). On the other hand, extending the construction of [4, §1.5.4] to $\mathfrak{q}(3) \oplus \mathfrak{q}(3)$, we note that the irreducible $\mathfrak{h} \oplus \mathfrak{h}$-module corresponding to (λ_1, λ_2), denoted as $W_{(\lambda_1,\lambda_2)}$, is obtained as the unique irreducible module for the Clifford algebra $\mathcal{C}_{(\lambda_1,\lambda_2)} \cong \mathcal{C}_{m'}$ where $m' = \dim\big((\mathfrak{h}_{\bar{1}} \oplus \mathfrak{h}_{\bar{1}})/\text{Rad}\ \langle\cdot,\cdot\rangle_{(\lambda_1,\lambda_2)}\big)$. Since $\text{Rad}\langle\cdot,\cdot\rangle_{(\lambda_1,\lambda_2)} = \text{Rad}\ \langle\cdot,\cdot\rangle_\lambda \oplus \text{Rad}\ \langle\cdot,\cdot\rangle_\lambda$, the latter being 0, we get $m' = 2m$. Hence the dimension of $W_{(\lambda_1,\lambda_2)}$ which is $2^{\lceil m'/2 \rceil} = 8$. Since the irreducible $\mathfrak{h} \oplus \mathfrak{h}$-module corresponding to $(\lambda_1, \lambda_2) \in (\mathfrak{h}_{\bar{0}} \oplus \mathfrak{h}_{\bar{0}})^*$ is unique, by dimension consideration the (λ_1, λ_2)-weight space of $M(1) \otimes M(2)$ is not an irreducible $\mathfrak{h} \oplus \mathfrak{h}$-module. In particular, $M(1) \otimes M(2)$ is not irreducible for $\mathfrak{L}_1 \oplus \mathfrak{L}_2$.

References

[1] A. Berele: Invariant theory and trace identities associated with Lie color algebras, J. Algebra 310 (2007).

[2] X.W. Chen, S.D. Silvestrov, F. Van Oystaeyen: Representations and Cocycle Twists of Color Lie Algebras. Algebr. Represent. Theor. 9, (2006).

[3] S.-J. Cheng, Differentiably simple Lie superalgebras and representations of semisimple Lie superalgebras, J. Algebra, 173(1) (1995).

[4] S.-J. Cheng, W. Wang, Dualities and representations of Lie superalgebras, AMS Graduate Texts in Mathematics, 441 .

[5] T. Ferguson, M.Gorelik, D. Grantcharov, Bounded highest weight modules over $\mathfrak{osp}(1, 2n)$, in Lie Algebras, Lie Superalgebras, Vertex Algebras and Related Topics, Proc. of Symp. Pure Math., 2019.

[6] V.G. Kac, Lie superalgebras, Adv. Math. no. 26, (1977).

[7] F. W. Lemire, Weight spaces and irreducible representations of simple Lie algebras, Proc. Am. Math. Soc., 22 (1999).

[8] H. Li, On certain categories of modules for affine Lie algebras, Math. Z., 248, (2004)

[9] O. Mathieu, Classification of irreducible weight modules, Ann. Inst. Fourier, 50 (2000).

[10] D. Moon, The centralizer algebras of Lie color algebras, Comm. Algebra, 27(7) (1999).

[11] R. Ree: Generalized Lie elements, Canad. J. Math. 12, (1960).

[12] A. Savage, Equivariant map superalgebras, Math. Z., 277, (2014).

[13] M. Scheunert: Generalized Lie algebras, J. Math. Phys., 20(4) (1979).

[14] M. Scheunert: Casimir elements of ϵ-Lie algebras, J. Math. Phys., 24(11) (1983).

[15] Y. Su, K. Zhao, L. Zhu: Simple Lie color algebras of Weyl type Isr. J. Math. 137, (2003).

[16] M. Yousofzadeh, Tensor product representations, J. Algebra 606 (2022).

[17] K. Zhao: Simple Lie color algebras from graded associative algebras, J. Algebra 269 (2), (2003).