INVARIANT THEORY OF THE QUEER COLOR GROUP

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Abstract. In this paper, we introduce the notion of the queer color group, analogous to that of the queer supergroup over the infinite Grassmann algebra. We obtain a Schur-Weyl duality theorem for this group and thereby construct an explicit spanning set of invariants of the associated symmetric algebra of the mixed tensor space of a G-graded vector space where G is a finite abelian group. As a consequence, we obtain a generating set for the polynomial invariants, under the simultaneous action of the queer color group on color analogues of several copies of matrices. We also introduce the notion of concomitants for the queer color group analogous to that of Procesi in [16] and obtain a generating set for the algebra of concomitants. These results generalize those of Berele in [3] to the color setting.

1. Introduction

Lie color algebras were introduced as 'generalized Lie algebras' in 1960 by Ree [17], being also called color Lie superalgebras (see [4]). Since then, this kind of algebra has been an object of constant interest in mathematics, being also remarkable for the important role played in theoretical physics, especially in conformal field theory and supersymmetries. Lie color algebras have close relation with Lie superalgebras. Any Lie superalgebra is a Lie color algebra graded by the simplest nontrivial abelian group $\mathbb{Z}_2$, while any Lie color algebra graded by a finitely generated abelian group can be transformed to a Lie superalgebra under a suitable choice of a 'multiplier'. Unlike Lie algebras and Lie superalgebras, structures and representations of Lie color algebras are far from being well understood and also there is no general classification result on simple Lie color algebras. Some recent interest related to their representation theory and related graded ring theory can be found in [5], [10], [20], [21].

The double centralizer theorem for the general linear lie color algebra is known due to results of Fischman and Montgomery in [7]. Using this, in [2], Berele was able to obtain a Schur-Weyl duality for the associated general linear lie color group. This leads to a spanning set for the space of polynomial invariants on several copies of matrices. These results were generalised in [15] to the invariants of mixed tensor spaces. In this paper we extend these

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results to the queer color group, which we define analogous to the queer supergroup over the infinite Grassmann algebra.

In order to define the queer analogue of the general linear color group, we use the definition of a queer lie color algebra $\mathfrak{q}_\epsilon(V)$ introduced in [13]. This definition is motivated by the work of Scheunert in [19] where he gives a close connection between the representations of lie color algebras and their associated lie superalgebras. The double centralizer theorem in the color case is obtained in [13] from its superalgebra counterpart via this connection. We use this notion of queer lie color algebra and its centralizer theorem to define the queer color group and prove the Schur-Weyl duality theorem for this group. Namely, for a G-graded vector space V, we set $U := V \otimes_{\mathbb{C}} \Lambda_\epsilon$ where Λ_ϵ is an infinite dimensional G-graded algebra playing the same role as the infinite Grassmann algebra in the super setting. The *queer color group* is defined as the group of invertible elements in the degree preserving part of $\mathfrak{q}_\epsilon(V) \otimes \Lambda_\epsilon \subset \mathrm{End}_{\Lambda_\epsilon}(U)$. We denote this group by $Q_\epsilon(U)$. The proof given here of the Schur-Weyl duality for the queer color group follows the line of argument as in the supersetting given in [9].

Having set up the Schur-Weyl duality for the queer color group we are in a position to discuss its invariant theory. We approach this by first looking at the invariants for the mixed tensor space in the same spirit as in [14]. The motivation to looking at mixed tensor spaces comes from classification problems, as illustrated in [6],[8],[11],[12],[18]. However in this case the additional motivation is to also obtain in the G-graded setting results analogous to that of Procesi in [16]. These results are new and extend the results of [15] to the queer case.

Setting U_b^t to be the space $(U^{\otimes b} \otimes U^{*\otimes t})$. By a mixed tensor space we mean a space of the form $\oplus_{i=1}^{s} U_{b_i}^{t_i}$ where t_i, b_i are in $\mathbb{N} \cup \{0\}$ for all $i = 1, \ldots s$. The case of invariants of the color analogues of d copies of matrices may be seen as a special case of the above by taking $t_i = 1 = b_i$ for all $i = 1 \ldots, s$. We show in Theorem 4.1 that the ring of $Q_\epsilon(U)$-invariants of the G-graded symmetric algebra of the dual of the mixed tensor space is generated by "coloured invariants". We then obtain in Corollary 4.7 that in the special case of $t_i = 1 = b_i$ for all $i = 1 \ldots, s$, these invariants form a spanning set for the $Q_\epsilon(U)$-polynomial invariants of $(\mathrm{End}_{\Lambda_\epsilon}(U))^s$. Viewing a superspace as a special case of a G-graded space, the invariants that we obtain here coincide with those obtained in [14, Theorem 6.1].

The notion of Λ_ϵ-valued polynomials over U is motivated from [9]. It was introduced in [15] to describe polynomial invariants for the general linear color group. We consider the Λ_ϵ-module of maps from the graded component of U corresponding to the identity element of G to Λ_ϵ. This module is denoted as $\mathcal{F}(U_\circ, \Lambda_\epsilon)$. Then using a G-graded analogue of the restitution map from the r-fold tensor space of U^* to $\mathcal{F}(U_\circ, \Lambda_\epsilon)$, we call the image under this map to be the space of homogeneous polynomials of degree r. The G-graded algebra of polynomials on U is then taken to be the direct sum of these spaces of homogeneous polynomials. We show that this algebra is isomorphic to the symmetric algebra of U^* under the restitution map, analogous to that done for the general linear color group in [15]. In this paper, we use the above notion of Λ_ϵ-valued polynomials on $\oplus_{i=1}^{s} U_{b_i}^{t_i}$ and obtain a generating set for the polynomial invariants of a G-graded mixed tensor space. We then show that in the special case when the mixed tensor space corresponds to several copies of the endomorphism space of U, the coloured invariants are just queer trace monomials, whenever they are non-zero. This may be regarded as the G-graded analogue of Procesi's result in [16].

We use the above notion of polynomials on a G-graded vector space to define the notion of a polynomial map between G-graded vector spaces. We then define polynomial concomitants for $Q_\epsilon(U)$ analogous to that of Procesi [16]. We prove in Theorem 5.4 that the ring of $Q_\epsilon(U)$-concomitants is generated by the invariant polynomials and the projections. This extends results of Berele in [3] to the color setting.

Notation: Throughout this paper we work over the field of complex numbers $\mathbb{C}$. All modules and algebras are defined over $\mathbb{C}$ and in addition all the modules are of finite dimension. We write $\mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$ and use its standard field structure. For a finite abelian group G, we denote the identity element of G by $\circ$, while we reserve the symbols $0, 1$ to denote the usual complex numbers that they represent.

2. Preliminaries

2.1. **Definition:** Let G be a finite abelian group. A map $\epsilon : G\times G \to \mathbb{C}^*$ where $\mathbb{C}^* := \mathbb{C}\setminus\{0\}$ is called a skew-symmetric bicharacter on G if the following identities hold, for all $f, g, h \in G$.

(1) $\epsilon(f, g + h) = \epsilon(f, g)\epsilon(f, h)$,

(2) $\epsilon(g + h, f) = \epsilon(g, f)\,\epsilon(h, f)$,

(3) $\epsilon(g, h)\,\epsilon(h, g) = 1$.

From the definition of ϵ it follows that $\epsilon(g, \circ) = \epsilon(\circ, g) = 1$ and $\epsilon(g, g) = \pm 1$, where $\circ$ denotes the identity element of G.

For a bicharacter ϵ of G, we set $G_{\bar{0}} := \{g \in G : \epsilon(g, g) = 1\}$ and $G_{\bar{1}} = G \setminus G_{\bar{0}}$. Then there exists a group homomorphism $\theta : G \to \mathbb{Z}_2$ such that $\theta(g) = \bar{0}$ for $g \in G_{\bar{0}}$ while $\theta(g) = \bar{1}$ for $g \in G_{\bar{1}}$. Moreover, $G_{\bar{0}}$ is a subgroup of G of index 1 or 2. It easily follows that if $g \in G_{\bar{1}}$ then $-g \in G_{\bar{1}}$.

Define a new bicharacter $\epsilon_0 : G \times G \to \{\pm 1\} \subset \mathbb{C}^*$ given by $\epsilon_0(g, h) = (-1)^{\theta(g)\theta(h)}$. Fix $\delta(g, h) = \epsilon(g, h)^{-1}\epsilon_0(g, h)$. Then δ is also a bicharacter on G such that $\delta(g, g) = 1$ for all $g \in G$. A *multiplier* associated to δ is the bicharacter $\kappa : G \times G \to \mathbb{C}^*$ given by

$$\kappa(\sum_i r_i g_i, \sum_j s_j g_j) = \prod_{i<j} \delta(g_i, g_j)^{r_i s_j},$$

where $G = \langle g_1\rangle \times \ldots \times \langle g_r\rangle$ such that $\langle g_i\rangle \cong \mathbb{Z}/n_i\mathbb{Z}$ for some $n_i \in \mathbb{N}$, and r_i, s_j are integers. The above multiplier κ satisfies the condition that $\delta(g, h) = \kappa(g, h)\kappa(h, g)^{-1}$ for $g, h \in G$.

Definition: For a finite abelian group G, a G-graded vector space is a vector space V together with a decomposition into a direct sum of the form $V = \oplus_{g\in G} V_g$, where each V_g is a vector subspace of V. For a given $g \in G$ the elements of V_g are called homogeneous elements of degree g and we write $|v|$ to denote the degree of v.

Any finite dimensional G-graded vector space $V = \oplus_{g\in G} V_g$ can be given a $\mathbb{Z}_2$-grading via $V = V_{G_{\bar{0}}} \oplus V_{G_{\bar{1}}}$, where $V_{G_{\bar{0}}} = \oplus_{g\in G_{\bar{0}}} V_g$ and $V_{G_{\bar{1}}} = \oplus_{g\in G_{\bar{1}}} V_g$.

Definition: Fix a pair (G, ϵ), where G is a finite abelian group and ϵ is a skew-symmetric bicharacter on G. A Lie color algebra $L = \oplus_{g\in G} L_g$ associated to (G, ϵ) is a G-graded $\mathbb{C}$-vector space with a graded bilinear map $[-,-] : L \times L \to L$ satisfying

1. $[L_g, L_h] \subset L_{g+h}$ for every $g, h \in G$,
2. $[x, y] = -\,\epsilon(|x|, |y|)[y, x]$ and
3. $\epsilon(|z|, |x|)[x, [y, z]] + \epsilon(|x|, |y|)[y, [z, x]] + \epsilon(|y|, |z|)[z, [x, y]] = 0$ for all homogeneous elements $x, y, z \in L$.

2.2. Given a G-graded vector space $V = \oplus_{g\in G} V_g$, the $\mathbb{C}$-endomorphisms of V, $\mathrm{End}_{\mathbb{C}}(V)$ is also G-graded, where

$$\mathrm{End}_{\mathbb{C}}(V)_g = \{f : V \to V : f(V_h) \subset V_{g+h}, \text{ for all } h \in G\}.$$

The general linear Lie color algebra, $\mathfrak{gl}_\epsilon(V)$ is defined to be $\mathrm{End}_{\mathbb{C}}(V)$ with the Lie bracket given by $[x, y]_\epsilon = xy - \epsilon(|x|, |y|)yx$, where x, y are homogeneous elements in $\mathrm{End}_{\mathbb{C}}(V)$.

The k-fold tensor product $V^{\otimes k}$ of V is also G-graded; $V^{\otimes k} = \oplus_{g\in G}(V^{\otimes k})_g$, where

$$(V^{\otimes k})_g = \bigoplus_{g=g_1+\cdots+g_k} V_{g_1} \otimes \cdots \otimes V_{g_k}.$$

We have an action of the symmetric group S_k on $V^{\otimes k}$ as follows:

The group S_k is generated by the transpositions $s_1, s_2, \cdots s_{k-1}$, where $s_i = (i, i+1)$. Then

$$s_i.(v_1 \otimes v_2 \otimes \cdots \otimes v_k) = \epsilon(g, h)(v_1 \otimes v_2 \otimes \cdots \otimes v_{i+1} \otimes v_i \otimes \cdots \otimes v_k),$$

where $v_i \in V_g$ and $v_{i+1} \in V_h$. More generally, for $\sigma \in S_k$ and $\underline{v} = v_1 \otimes v_2 \otimes \cdots \otimes v_k$ where each v_i is a homogeneous element of V,

$$\sigma.\underline{v} = \gamma(\underline{v}, \sigma^{-1})(v_{\sigma^{-1}(1)} \otimes v_{\sigma^{-1}(2)} \otimes \cdots \otimes v_{\sigma^{-1}(k)}),$$

where $\gamma(\underline{v}, \sigma) = \prod_{(i,j)\in Inv(\sigma)} \epsilon(|v_i|, |v_j|)$, with $Inv(\sigma) = \{(i, j) : i < j \text{ and } \sigma(i) > \sigma(j)\}$.

We then extend the action to $V^{\otimes k}$ by linearity. We then have the following lemma.

Lemma 2.1. *For two permutations $\sigma, \tau \in S_k$, we have $\gamma(\underline{v}, \sigma\tau) = \gamma(\sigma^{-1}\underline{v}, \tau)\gamma(\underline{v}, \sigma)$.*

Proof. Let $w_i = v_{\sigma(i)}$ for all i. Then $w_{\tau(i)} = v_{\sigma\tau(i)}$ for all i.

The action of σ on $\underline{v}$ is given by $\sigma.\underline{v} = \gamma(\underline{v}, \sigma^{-1})(v_{\sigma^{-1}(1)} \otimes v_{\sigma^{-1}(2)} \otimes \cdots \otimes v_{\sigma^{-1}(k)})$

We have $\tau^{-1}\sigma^{-1}.\underline{v} = \gamma(\underline{v}, \sigma\tau)(v_{\sigma\tau(1)} \otimes v_{\sigma\tau(2)} \otimes \cdots \otimes v_{\sigma\tau(k)})$.

On the other hand $\tau^{-1}\sigma^{-1}.\underline{v} = \gamma(\underline{v}, \sigma)\tau^{-1}.(v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(k)}) = \gamma(\underline{v}, \sigma)\tau^{-1}.\underline{w}$

$= \gamma(\underline{v}, \sigma)\gamma(\underline{w}, \tau)(w_{\tau(1)} \otimes w_{\tau(2)} \otimes \cdots \otimes w_{\tau(k)})$

$= \gamma(\underline{v}, \sigma)\gamma(\sigma^{-1}\underline{v}, \tau)(v_{\sigma\tau(1)} \otimes v_{\sigma\tau(2)} \otimes \cdots \otimes v_{\sigma\tau(k)})$

So we have the required identity. □

By fixing a set of homogeneous vectors $v_1, \cdots, v_k$ of V such that $|v_i| = a_i$ for all i, we set I to be the tuple $(a_1, \ldots, a_k)$ in G^k. The symmetric group S_k acts on G^k via $\sigma.(a_1, \ldots, a_k) := (a_{\sigma^{-1}(1)}, \ldots, a_{\sigma^{-1}(k)})$ for $\sigma \in S_k$. By defining $\gamma(I, \sigma) := \gamma(v_1 \otimes \cdots \otimes v_k, \sigma)$, the above relation may be rephrased as, $\gamma(\sigma^{-1}I, \tau)\gamma(I, \sigma) = \gamma(I, \sigma\tau)$.

On the other hand the general linear Lie color algebra, $\mathfrak{gl}_\epsilon(V)$ acts on $V^{\otimes k}$ by twisted derivation:

$$x.(v_1 \otimes v_2 \otimes \cdots \otimes v_k) = \sum_{i=1}^{k} (\prod_{j<i} \epsilon(\alpha, g_j)) v_1 \otimes v_2 \otimes \cdots \otimes x.v_i \otimes \cdots \otimes v_k,$$

where $x \in \mathfrak{gl}_\epsilon(V)_\alpha$ and each $v_j \in V_{g_j}$.

The group G also acts on $V^{\otimes k}$ by

$$g(v_1 \otimes v_2 \otimes \cdots \otimes v_k) = \prod_i \epsilon(g, g_i)(v_1 \otimes v_2 \otimes \cdots \otimes v_k),$$

where $g \in G$ and each $v_i \in V_{g_i}$.

2.3. Fix a pair (G, ϵ), where G is a finite abelian group and ϵ is a skew-symmetric bicharacter on G. We now define a graded algebra Λ_ϵ which generalizes the infinite Grassmann algebra in the super setting. Let $X = \cup_{g\in G} X_g$ be a G-graded set, where each X_g is countably infinite. Let Λ be the free $\mathbb{C}$-algebra generated by X and we define $\Lambda_\epsilon := \Lambda/I$, where I is the ideal generated by the elements $xy - \epsilon(g, h)yx$, for all $g, h \in G$ and for all $x \in X_g$ and $y \in X_h$. Then Λ_ϵ is also G-graded and in fact when $G = \mathbb{Z}_2$ and X contains only degree 1 elements, Λ_ϵ is the Grassmann algebra on the vector space with basis X.

If $X = \{x_1, x_2, \cdots\}$, then the set $\{x_{i_1} x_{i_2} \cdots x_{i_r} : i_1 \leq i_2 \leq \cdots \leq i_r\}$ is a basis of Λ_ϵ. For a fixed linear ordering of the elements of X, we set $\Lambda_\epsilon(N)$ to be the linear span of the basis vectors $\{x_{i_1} x_{i_2} \cdots x_{i_r} : 1 \leq i_1 \leq i_2 \leq \cdots \leq i_r \leq N\}$ where $N \in \mathbb{Z}_{\geq 0}$. We then have the following filtration:

$$\Lambda_\epsilon(N) \subset \Lambda_\epsilon(N+1), \text{ for all } N \geq 0.$$

2.4. Given a G-graded vector space $V = \oplus_{g\in G} V_g$, we set $U := V \otimes_{\mathbb{C}} \Lambda_\epsilon$. Then U is also a G-graded Λ_ϵ-bimodule: a compatible left action of Λ_ϵ can be given on U by setting $\lambda.u = \epsilon(g, h)u\lambda$ where $\lambda \in \Lambda_\epsilon$ and $u \in U$ are of degrees g and h respectively.

Let $\mathrm{End}_{\Lambda_\epsilon}(U) := \{T \in \mathrm{End}_{\mathbb{C}}(U) : T(u\lambda) = T(u)\lambda$ for all $u \in U, \lambda \in \Lambda_\epsilon\}$. Then $\mathrm{End}_{\Lambda_\epsilon}(U)$ is also G-graded in a natural way via

$$\mathrm{End}_{\Lambda_\epsilon}(U)_g = \{T \in \mathrm{End}_{\Lambda_\epsilon}(U) : T(U_h) \subseteq U_{g+h}, \text{ for all } h \in G\}.$$

There is a natural embedding $V \hookrightarrow U$ given by $v \mapsto v \otimes 1$. Any $\mathbb{C}$-linear map $T : V \to V$ extends uniquely to an element of $\mathrm{End}_{\Lambda_\epsilon}(U)$ and we have $\mathrm{End}_{\Lambda_\epsilon}(U) \cong \mathrm{End}_{\mathbb{C}}(V) \otimes_{\mathbb{C}} \Lambda_\epsilon$. More generally, for two G-graded vector spaces V and V', set $U := V \otimes_{\mathbb{C}} \Lambda_\epsilon$ and $U' := V' \otimes_{\mathbb{C}} \Lambda_\epsilon$, we denote by $\mathrm{Hom}_{\Lambda_\epsilon}(U, U')$ to be the set of $\mathbb{C}$-linear maps from U to U' which commute with the right action of Λ_ϵ. Then $\mathrm{Hom}_{\mathbb{C}}(V, V') \otimes_{\mathbb{C}} \Lambda_\epsilon \cong \mathrm{Hom}_{\Lambda_\epsilon}(U, U')$. In particular, if we denote by U^* the G-graded space $\mathrm{Hom}_{\Lambda_\epsilon}(U, \Lambda_\epsilon)$, then $U^* \cong V^* \otimes_{\mathbb{C}} \Lambda_\epsilon$.

With notation as above, it may be easily seen that $U \otimes_{\Lambda_\epsilon} U' \cong (V \otimes_{\mathbb{C}} V') \otimes_{\mathbb{C}} \Lambda_\epsilon$ via the map defined on the homogeneous elements by $v \otimes \lambda \otimes v' \otimes \lambda' \mapsto v \otimes v' \otimes \epsilon(|\lambda|, |v'|)\lambda\lambda'$. Here $|\lambda|$, $|v'|$ denote the G-grading of λ and v' respectively.

2.5. **Symmetric algebra on a graded vector space.** Let G be a finite abelian group and ϵ is a skew-symmetric bicharacter on G. Let $V = \oplus_{g \in G} V_g$ be a G-graded vector space over $\mathbb{C}$. The tensor algebra on V is $T(V) = \oplus_{k \geq 0} V^{\otimes k}$.

The symmetric algebra on V is defined to be $S(V) = T(V)/I(V)$, where $I(V)$ is the ideal of $T(V)$ generated by elements of the form $v \otimes w - \epsilon(g, h) w \otimes v$, where v and w are homogeneous elements of degrees g and h respectively. Note that the ideal $I(V)$ is both $\mathbb{Z}$-graded as well as G-graded.

We note that if we write $V = \oplus_{g \in G} V_g = V_{G_{\bar{0}}} \oplus V_{G_{\bar{1}}}$, where $V_{G_{\bar{0}}} = \oplus_{g \in G_{\bar{0}}} V_g$ and $V_{G_{\bar{1}}} = \oplus_{g \in G_{\bar{1}}} V_g$ then since $\epsilon(|v|, |v|) = -1$ for $v \in V_{G_{\bar{1}}}$ we have $v \otimes v = 0$ in $S(V)$.

We define the d-th symmetric power of V, written $S^d(V)$ to be the image of $V^{\otimes d}$ in $S(V)$. Since $I(V)$ is $\mathbb{Z}$-graded as well as G-graded, we have $S(V) = \oplus_{d \geq 0} S^d(V)$ and each $S^d(V)$ is G-graded. If $V_{G_{\bar{1}}} = V$ then $S(V)$ is denoted as $\bigwedge(V)$ and it is the exterior algebra of V.

We then have the following lemma which will be used in the proof of the main theorem. The proof can be found in [14].

Lemma 2.2. *(1) Given any map $f : V \to W$, between two G-graded vector spaces, there is a unique map of $\mathbb{C}$-algebras $S(f) : S(V) \to S(W)$ carrying V to W.*

(2) For a G-graded vector space V, we have $S(V \otimes_{\mathbb{C}} \Lambda_\epsilon) = S(V) \otimes_{\mathbb{C}} \Lambda_\epsilon$.

(3) For two G-graded vector spaces V and W, we have $S(V \oplus W) = S(V) \otimes S(W)$.

(4) If V is a G-graded vector space and $\{v_1, v_2, \cdots, v_n\}$ is a basis of V consisting of homogeneous elements, then the set of all monomials of the form $v_1^{i_1}..v_n^{i_n}$ such that $\sum_{j=1}^n i_j = d$ and $0 \leq i_j \leq 1$ for v_j in $V_{G_{\bar{1}}}$, form a basis of $S^d(V)$.

Remark 2.3. In the above lemma, for $U = V \otimes_{\mathbb{C}} \Lambda_\epsilon$, $S(U)$ is defined as the quotient of $T(U)$ by the ideal generated by the elements of the form $v \otimes w - \varepsilon(g, h) w \otimes v$ where $v, w \in V$

are regarded as elements of U. Further, in view of (2) above, all the other statements of the lemma also hold after extending scalars to Λ_ϵ.

2.6. For $V = \oplus_{g\in G} V_g$ and $U = V \otimes_{\mathbb{C}} \Lambda_\epsilon$, we set

$$M_\epsilon(U) := \{T : U \to U : T(U_g) \subseteq U_g, \ \textit{for all } g \in G\}.$$

Definition: The general linear color group $\mathrm{GL}_\epsilon(U)$ is defined to be the group of invertible elements in $M_\epsilon(U)$.

2.7. The group $GL_\epsilon(U)$ acts on the k-fold tensor product $U^{\otimes k}$ diagonally:

$$T.(u_1 \otimes \cdots \otimes u_k) = (Tu_1 \otimes \cdots \otimes Tu_k).$$

We denote by η the resulting homomorphism: $\eta : \mathrm{GL}_\epsilon(U) \to End_{\Lambda_\epsilon}(U^{\otimes k})$.

The action of the symmetric group on $V^{\otimes k}$ defined in §2.2 can be extended to an Λ_ϵ-linear action on $U^{\otimes k}$.

As $U = V \otimes \Lambda_\epsilon$ is a Λ_ϵ-bimodule, $U^{\otimes k}$ is also a Λ_ϵ-bimodule. The action of S_k and Λ_ϵ on $U^{\otimes k}$ commute with each other. So the S_k action on $U^{\otimes k}$ extends to an action of the group algebra $\Lambda_\epsilon[S_k]$. We denote by ρ the resulting homomorphism: $\rho : \Lambda_\epsilon[S_k] \to \mathrm{End}_{\Lambda_\epsilon}(U^{\otimes k})$.

In [2], Berele proved a version of Schur-Weyl duality for the above action of the general linear color group.

Theorem 2.4. *Let A be the subalgebra of $\mathrm{End}_{\Lambda_\epsilon}(U^{\otimes k})$ generated by $\eta(\mathrm{GL}_\epsilon(U))$. Then the centralizer of A in $\mathrm{End}_{\Lambda_\epsilon}(U^{\otimes k})$ is $\rho(\Lambda_\epsilon[S_k])$.*

2.8. **Isomorphisms.** In this subsection we list out all the isomorphisms we require in this paper. Let V and W be G-graded vector spaces over $\mathbb{C}$ and $U = V \otimes \Lambda_\epsilon, U' = W \otimes \Lambda_\epsilon$.

1. We have $U \otimes U^* \cong \mathrm{End}(U)$ defined by $(v, \phi) \mapsto T_{(v,\phi)}$ where $T_{(v,\phi)}(w) = v\phi(w)$ for all $v, w \in U$ and $\phi \in U^*$. Under this identification, we define a product operation on $U \otimes U^*$ as $(v \otimes \alpha).(w \otimes \beta) = v\alpha(w) \otimes \beta$ making the identification an isomorphism of G-graded algebras.

The above isomorphism leads to the following isomorphism for each $k \in \mathbb{Z}_{\geq 0}$:

$$\mathrm{End}(U)^{\otimes k} \cong (U \otimes U^*)^{\otimes k}.$$

2. We have an Λ_ϵ-linear isomorphism $U \otimes U' \to U' \otimes U$ defined by $v \otimes w \mapsto \epsilon(|v|, |w|) w \otimes v$, where v and w are homogeneous elements of V and W respectively.

3. We have an evaluation map $ev : (U^*)^{\otimes k} \times U^{\otimes k} \to \Lambda_\epsilon$ defined by

$$(f_1 \otimes f_2 \otimes \cdots \otimes f_k, v_1 \otimes v_2 \otimes \cdots \otimes v_k) \mapsto p_\epsilon(\underline{f}, \underline{v}) f_1(v_1) \cdots f_k(v_k),$$

where $\underline{f} = f_1 \otimes f_2 \otimes \cdots \otimes f_k$, $\underline{v} = v_1 \otimes v_2 \otimes \cdots \otimes v_k$ for homogeneous elements f_i and v_i, and

$$p_\epsilon(\underline{f}, \underline{v}) = \prod_{j=1}^{k-1} (\prod_{i=j+1}^{k} \epsilon(|f_i|, |v_j|)).$$

4. The above evaluation map ev is a non-degenerate bilinear form and hence leads to an isomorphism between $(U^*)^{\otimes k}$ and $(U^{\otimes k})^*$ which is $\mathrm{GL}_\epsilon(U)$ equivariant.

5. Let τ be the permutation which takes $(1, 2, \cdots , k, k+1, \cdots , 2k)$ to $(\tau(1), \tau(2), \cdots , \tau(k), \tau(k+1), \cdots , \tau(2k)) = (1, 3, 5, \cdots , 2k - 1, 2, 4, 6, \cdots , 2k)$. The symmetric group S_{2k} acts on $(U \oplus U^*)^{\otimes k}$. Under this action, the permutation τ takes $U^{\otimes k} \otimes (U^*)^{\otimes k}$ to $(U \otimes U^*)^{\otimes k} = (U \otimes U^*) \otimes (U \otimes U^*) \otimes \cdots \otimes (U \otimes U^*)$. Since the action of S_{2k} and $\mathrm{GL}_\epsilon(U)$ commute with each other, we get a $\mathrm{GL}_\epsilon(U)$-isomorphism between $U^{\otimes k} \otimes (U^*)^{\otimes k}$ and $(U \otimes U^*)^{\otimes k}$.

Let $\underline{v} = v_1 \otimes v_2 \otimes \cdots \otimes v_k$ and $\underline{f} = f_1 \otimes f_2 \otimes \cdots \otimes f_k$. Then the isomorphism is defined by $\underline{v} \otimes \underline{f} \mapsto p_\epsilon(\underline{v}, \underline{f})(v_1 \otimes f_1) \otimes (v_2 \otimes f_2) \otimes \cdots \otimes (v_k \otimes f_k)$.

6. let $V_1, \ldots , V_k$ be G-graded vector spaces over $\mathbb{C}$ and let $W_i := V_i \otimes_{\mathbb{C}} \Lambda_\epsilon$. Let $W = W_1 \oplus \cdots \oplus W_k$ and τ be the permutation as defined above. Then the group S_{2k} acts naturally on the $2k$-fold tensor product $(W \oplus W^*)^{\otimes 2k}$. Under this action the element $\tau \in S_{2k}$ induces an isomorphism between the subspaces $W_1^* \otimes \cdots \otimes W_k^* \otimes W_1 \otimes \cdots \otimes W_k$ and $W_1^* \otimes W_1 \otimes \cdots \otimes W_k^* \otimes W_k$ of $(W \oplus W^*)^{\otimes 2k}$. This isomorphism followed by the map $W_1^* \otimes W_1 \otimes \cdots \otimes W_k^* \otimes W_k \to \Lambda_\epsilon$ given by $\alpha_1 \otimes w_1 \otimes \cdots \otimes \alpha_k \otimes w_k \mapsto \prod_i \alpha_i(w_i)$ gives a non-degenerate pairing $ev : W_1^* \otimes \cdots \otimes W_k^* \otimes W_1 \otimes \cdots \otimes W_k \to \Lambda_\epsilon$. Note that the non-degeneracy comes from noticing that this map is obtained by extending scalars to Λ_ϵ of a non-degenerate pairing over $\mathbb{C}$. The isomorphism induced by this non-degenerate pairing will be denoted by $\iota : W_1^* \otimes \cdots \otimes W_k^* \to (W_1 \otimes \cdots \otimes W_k)^*$ which is also $\mathrm{GL}_\epsilon(U)$-equivariant.

It may be noted here that under the above isomorphism, the dual basis element $T_{w_1 \otimes \cdots \otimes w_n} \in (W_1 \otimes \cdots \otimes W_k)^*$ corresponds to $p_\epsilon(\underline{w}, \underline{w}) T_{w_1} \otimes \cdots \otimes T_{w_k} \in W_1^* \otimes \cdots \otimes W_k^*$ where $\underline{w} = w_1 \otimes \ldots \otimes w_k$ is a basis vector of $W_1 \otimes \cdots \otimes W_k$.

7. Let $\nu \in S_{2k}$ be the permutation $(1\,, 2)(3\,, 4) \cdots (2k - 1\,, 2k)$ and $\mathrm{ev}^{\otimes k} : (U^* \otimes U)^{\otimes k} \to \Lambda_\epsilon$ denote the evaluation map $(f_1 \otimes v_1) \otimes (f_2 \otimes v_2) \otimes \cdots \otimes (f_k \otimes v_k) \mapsto f_1(v_1) \cdots f_k(v_k)$. We have a non-degenerate bilinear form $\mathrm{End}(U^{\otimes k}) \otimes (U^{\otimes k} \otimes (U^*)^{\otimes k}) \to \Lambda_\epsilon$ given by $\langle A, \underline{v} \otimes \underline{f} \rangle := \mathrm{ev}^{\otimes k}[(\nu.\tau).(A(\underline{v}) \otimes \underline{f})]$, where $\underline{v} = v_1 \otimes v_2 \otimes \cdots \otimes v_k$ and $\underline{f} = f_1 \otimes f_2 \otimes \cdots \otimes f_k$. This gives a $\mathrm{GL}_\epsilon(U)$ isomorphism

$$\Theta : \mathrm{End}(U^{\otimes k}) \cong (U^{\otimes k} \otimes (U^*)^{\otimes k})^*$$

8. The isomorphism Θ followed by the isomorphisms given in (1) and (5) lead to a $\mathrm{GL}_\epsilon(U)$-isomorphism $\mathrm{End}(U^{\otimes k}) \cong (\mathrm{End}(U)^{\otimes k})^*$.

2.9. **Color trace.** With the above identification of $\mathrm{End}_{\Lambda_\epsilon}(U)$ with $U \otimes_{\Lambda_\epsilon} U^*$, we define the color trace map $ctr : \mathrm{End}_{\Lambda_\epsilon}(U) \to \Lambda_\epsilon$ given by

$$ctr(u \otimes f) = \epsilon(|u|, |f|)f(u)$$

for two homogeneous elements $u \in U$ and $f \in U^*$ and extend to all of $U \otimes_{\Lambda_\epsilon} U^*$ by linearity.

It can be easily seen that $ctr(AB) = \epsilon(|A|, |B|)ctr(BA)$ for two homogoneous elements $A, B \in \mathrm{End}_{\Lambda_\epsilon}(U)$ and as a consequence we get that $ctr(ABA^{-1}) = \epsilon(|A|, |B|)ctr(B)$ for all $B \in \mathrm{End}_{\Lambda_\epsilon}(U)$ and for all invertible $A \in \mathrm{End}_{\Lambda_\epsilon}(U)$.

By choosing a basis $\{v_1, \cdots, v_r, v_{r+1} \cdots, v_n\}$ of V, where the first r of them are in $V_{G_{\bar{0}}}$ and the last $n-r$ are in $V_{G_{\bar{1}}}$ and identifying $\mathrm{End}_{\Lambda_\epsilon}(U)$ with the space of matrices we get that

$$ctr(A) = \sum_{i=1}^{r} a_{ii} - \sum_{i=r+1}^{n} a_{ii}.$$

When we restrict the color trace function to $M_\epsilon(U)$ it becomes an ordinary trace function from $M_\epsilon(U)$ to $(\Lambda_\epsilon)_{G_0}$ in the sense that $ctr(AB) = ctr(BA)$. Since $M_\epsilon(U) \subset \mathrm{End}_{\Lambda_\epsilon}(U)_0$ we get that $ctr(ABA^{-1}) = ctr(B)$ for all $A \in \mathrm{GL}_\epsilon(U)$ and $B \in \mathrm{End}_{\Lambda_\epsilon}(U)$.

3. Queer Lie Color algebra and Queer Lie color group

3.1. Let G be a finite abelian group and ϵ be a skew-symmetric bicharacter on G. Assume that $[G, G_{\bar{0}}] = 2$ and let $g \in G_{\bar{1}}$ such that $2g = \circ$. Fixing an ordering for the elements of $G_{\bar{0}}$, we list the elements of G as $\{\circ, g_1, g_2, \ldots, g, g+g_1, g+g_2, \ldots\}$ with the elements in $G_{\bar{0}}$ appearing first. Let V be a G-graded vector space such that $dim(V_a) = dim(V_{a+g})$ for all $a \in G$. Note that the dimension of V is $2m$ where $\sum_{a \in G_{\bar{0}}} V_a$ has dimension m. We fix a basis $\{e_i\}_{i=1}^{2m}$ of V, arranged such that the first m vectors are from $V_{G_{\bar{0}}}$ and the rest are from $V_{G_{\bar{1}}}$; further, ordered linearly so that $i < j$ implies $|e_i|$ appears before $|e_j|$ under the fixed ordering of elements of G. For the rest of the article we fix this notation.

Suppose P is a linear map such that

(1) P maps V_a to V_{a+g} for all $a \in G$.

(2) $P^2 = -\kappa(g, g)id$ where κ is the multiplier as defined in §2.1.

Note that $P|_{V_a} : V_a \to V_{a+g}$ is an isomorphism for all $a \in G$ and $|P| = g$.

We could define $P(e_i) = e_{m+i}$ and $P(v_{m+i}) = -\kappa(g, g)v_i$, for $i = 1, \ldots, m$. In a uniform way, this could be expressed as

$$P(e_i) = (-\kappa(g, g))^{\theta(|e_i|)} e_{(m+i) \ mod \ 2m},$$

where $\theta : G \to \mathbb{Z}_2$ such that $\theta(g) = \bar{0}$ for $g \in G_{\bar{0}}$ and $\theta(g) = \bar{1}$ for $g \in G_{\bar{1}}$.

We define the queer Lie Color algebra to be $\mathfrak{q}_\epsilon(V) := \oplus_{a \in G} \mathfrak{q}_\epsilon(V)_a$, where

$$\mathfrak{q}_\epsilon(V)_a = \{x \in \mathrm{End}(V)_a : P(xv) = \epsilon(g, a)xP(v), \text{ for all } v \in V\}.$$

By choosing a basis of V and denoting the matrix of $x \in End(V)_a$ with respect to this basis by A, the above relation translates to

$$PA = \epsilon(g,a)AP.$$

The queer Lie color algebra $\mathfrak{q}_\epsilon(V)$ is a subalgebra of $\mathfrak{gl}_\epsilon(V)$ and $ctr(A) = 0$ for all $A \in \mathfrak{q}_\epsilon(V)$ (see [13, Proposition 3.13]).

3.2. Let S_k be the symmetric group on k letters. We know that it is generated by the transpositions $s_1, \cdots, s_{k-1}$, where $s_i = (i, i+1)$ for $1 \le i \le k-1$.

Let H_k be the associative algebra generated by $s_1, \cdots, s_{k-1}$ and $c_1, \cdots, c_{k-1}, c_k$ with the following defining relations:

$s_i^2 = 1, s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, s_i s_j = s_j s_i \quad (|i-j| > 1),$

$c_i^2 = -1, c_i c_j = -c_j c_i, \ (i \neq j)$

$s_i c_i s_i = c_{i+1}, s_i c_j = c_j s_i, \ j \neq i, i+1.$

The generators $s_1, \cdots, s_{k-1}$ are of degree 0 and the subalgebra generated by them is isomorphic to the group algebra $\mathbb{C}[S_k]$ of S_k, and the generators $c_1, \cdots, c_{k-1}, c_k$ are of degree g and the $\mathbb{C}$-subalgebra generated by them is isomorphic to the Clifford algebra $\mathbf{C}l_k$.

Then the algebra H_k is isomorphic to $\mathbb{C}[S_k] \ltimes \mathbf{C}l_k$, which is $\mathbb{C}[S_k] \otimes \mathbf{C}l_k$ as a vector space with the multiplication given by

$$(\sigma \otimes c_{i_1} \cdots c_{i_l})(\tau \otimes c_{j_1} \cdots c_{j_m}) = \sigma\tau \otimes c_{\tau^{-1}(i_1)} \cdots c_{\tau^{-1}(i_l)} c_{j_1} \cdots c_{j_m},$$

where $1 \le i_s, j_t \le k$.

Note that H_k has as basis the elements which can be written in the form $\sigma \otimes c_1^{\epsilon_1} \cdots c_k^{\epsilon_k}$ where $\sigma \in S_k$ and $\epsilon_i \in \mathbb{Z}_2$ for all i.

3.3. There is a left action of H_k on $V^{\otimes k}$ as follows:

The generators s_j act on $V^{\otimes k}$ by

$$s_j.(v_1 \otimes \cdots \otimes v_k) = \epsilon(|v_j|, |v_{j-1}|) v_1 \otimes \cdots \otimes v_{j-1} \otimes v_{j+1} \otimes v_j \otimes v_{j+2} \otimes \cdots \otimes v_k.$$

The generators c_j act on $V^{\otimes k}$ by

$$c_j.(v_1 \otimes \cdots \otimes v_k) = \prod_{i=1}^{j-1} \epsilon(g, |v_i|) v_1 \otimes \cdots \otimes v_{j-1} \otimes P(v_j) \otimes v_{j+1} \otimes v_{j+2} \otimes \cdots \otimes v_k.$$

From the above action, we may note that for $\varepsilon_i \in \mathbb{Z}_2$, for $i = 1, \ldots, k$,

$$c_1^{\varepsilon_1} \cdots c_k^{\varepsilon_k}.(v_1 \otimes \cdots \otimes v_k) = \prod_{i>j} \epsilon(g_{\varepsilon_i}, |v_j|) P^{\varepsilon_1} v_1 \otimes \cdots \otimes P^{\varepsilon_k} v_k,$$

where

$$g_{\varepsilon_i} = \begin{cases} \circ & \text{if } \varepsilon_i = 0; \\ g & \text{if } \varepsilon_i = 1. \end{cases}$$

The action of a basis element $(\sigma \otimes c_1^{\varepsilon_1} \dots c_k^{\varepsilon_k}) \in H_k$ on $V^{\otimes k}$ is given by

$(\sigma \otimes c_1^{\varepsilon_1} \dots c_k^{\varepsilon_k}).(v_1 \otimes \cdots \otimes v_k) = \prod_{i>j} \epsilon(g_{\varepsilon_i}, |v_j|)\gamma((|v_1|+g_{\varepsilon_1}, \cdots, |v_k|+g_{\varepsilon_k}), \sigma^{-1})(P^{\varepsilon_{\sigma^{-1}(1)}} v_{\sigma^{-1}(1)} \otimes \dots \otimes P^{\varepsilon_{\sigma^{-1}(k)}} v_{\sigma^{-1}(k)})$.

So we get a G-graded algebra homomorphism

$$\Phi_k : H_k \to \mathrm{End}_{\mathbb{C}}(V^{\otimes k}).$$

On the other hand the queer Lie color algebra, $\mathfrak{q}_\epsilon(V)$ acts on $V^{\otimes k}$ by twisted derivation:

$$x.(v_1 \otimes v_2 \otimes \cdots \otimes v_k) = \sum_{i=1}^{k} (\prod_{j<i} \epsilon(\alpha, g_j)) v_1 \otimes v_2 \otimes \cdots \otimes x.v_i \otimes \cdots \otimes v_k,$$

where $x \in \mathfrak{q}_\epsilon(V)_\alpha$ and each $v_j \in V_{g_j}$.

We denote by Ψ the resulting homomorphism: $\psi : \mathfrak{q}_\epsilon(V) \to \mathrm{End}_{\mathbb{C}}(V^{\otimes k})$.

The group G also act on $V^{\otimes k}$ by

$$g(v_1 \otimes v_2 \otimes \cdots \otimes v_k) = \prod_i \epsilon(g, g_i)(v_1 \otimes v_2 \otimes \cdots \otimes v_k),$$

where $g \in G$ and each $v_i \in V_{g_i}$.

We then have the double centralizer theorem for the queer Lie color algebra.

Theorem 3.1 (Moon [13])**.** *Let $A = \Phi(H_k)$ and let B be the subalgebra of $\mathrm{End}_{\mathbb{C}}(V^{\otimes k})$ generated by $\psi(\mathfrak{q}_\epsilon(V))$. Then A and B are centralizers of each other.*

3.4. For $V = \oplus_{g \in G} V_g$ and $U = V \otimes_{\mathbb{C}} \Lambda_\epsilon$, we set

$$M_\epsilon(U) := \{T : U \to U : T(U_g) \subseteq U_g, \ for \ all \ g \in G\}.$$

Recall that the general linear color group $\mathrm{GL}_\epsilon(U)$ is defined as the group of invertible elements in $M_\epsilon(U)$.

The map $P \in \mathrm{End}_{\mathbb{C}}(V)$ defined earlier extends to a map in $\mathrm{End}_{\Lambda_\epsilon}(U)$. We still denote it by P. For $A \in \mathrm{End}_{\Lambda_\epsilon}(U)_0$ we define the *queer trace* as $qtr(A) := ctr(PA)$.

Definition: The *queer Lie color algebra over* Λ_ϵ is defined to be $\mathfrak{q}_\epsilon(U) := \{A \in \mathrm{End}_{\Lambda_\epsilon}(U)_0 : PA = AP\}$ with the ordinary Lie bracket and the *queer Lie color group* $Q_\epsilon(U)$ is defined to be the elements of $\mathrm{GL}_\epsilon(U)$ which commute with P.

Remark 3.2. *All the isomorphisms mentioned in §2.8 are $\mathrm{GL}_\epsilon(U)$-equivariant, hence they are also $Q_\epsilon(U)$-equivariant.*

The group $Q_\epsilon(U)$ acts on the k-fold tensor product $U^{\otimes k}$ diagonally:

$$T.(u_1 \otimes \cdots \otimes u_k) = (Tu_1 \otimes \cdots \otimes Tu_k).$$

We denote by Ψ the resulting homomorphism: $\Psi : Q_\epsilon(U) \to \mathrm{End}_{\Lambda_\epsilon}(U^{\otimes k})$.

The action of H_k on $V^{\otimes k}$ defined in §3.3 can be extended to a Λ_ϵ-linear action on $U^{\otimes k}$ as follows:

For $\sigma \in S_k$ and $\underline{u} = u_1 \otimes u_2 \otimes \cdots \otimes u_k \in U^{\otimes k}$ where each u_i is a homogeneous element of U,

$$\sigma.\underline{u} = \gamma(\underline{u}, \sigma^{-1})(u_{\sigma^{-1}(1)} \otimes u_{\sigma^{-1}(2)} \otimes \cdots \otimes u_{\sigma^{-1}(k)}),$$

where $\gamma(\underline{u}, \sigma) = \prod_{(i,j)\in Inv(\sigma)} \epsilon(|u_i|, |u_j|)$, with $Inv(\sigma) = \{(i, j) : i < j \text{ and } \sigma(i) > \sigma(j)\}$, and for the generators $c_j \in H_k$,

$$c_j.(u_1 \otimes \cdots \otimes u_k) = \prod_{i=1}^{j-1} \epsilon(P, u_i) u_1 \otimes \cdots \otimes u_{j-1} \otimes P(u_j) \otimes u_{j+1} \otimes u_{j+2} \otimes \cdots \otimes u_k.$$

We then have the following:

Theorem 3.3. *Let A be the subalgebra of $\mathrm{End}_{\Lambda_\epsilon}(U^{\otimes k})$ generated by $\Psi(Q_\epsilon(U))$. Then the centralizer of A in $\mathrm{End}_{\Lambda_\epsilon}(U^{\otimes k})$ is $\Phi(\Lambda_\epsilon[H_k])$.*

Proof. First note that if $x \in \mathfrak{q}_\epsilon(V)$ is homogeneous and $e \in \Lambda_\epsilon$ is homogeneous of degree $-|x|$, with $e^2 = 0$, then $I + xe$ is invertible with inverse $I - xe$. Again $I + xe$ commutes with P since x and e have the same degree and each commutes with it. Hence, $xe \in \mathfrak{q}_\epsilon(U)$ and $I + xe \in Q_\epsilon(U)$.

By the above note it is clear that if $\phi \in \mathrm{End}_\epsilon(U^{\otimes k})$ commutes with $Q_\epsilon(U)$ then it also commutes with $\mathfrak{q}_\epsilon(U)$. On the other hand, $Q_\epsilon(U) \subset \mathfrak{q}_\epsilon(U)$ so the converse is also true. So it suffices to show that

$$\mathrm{End}_{\mathfrak{q}_\epsilon(U)}(U^{\otimes k}) = \Phi(\Lambda_\epsilon[H_k]).$$

However, $\Phi(\Lambda_\epsilon[H_k])$ can be identified with $\Phi(H_k)\otimes_{\mathbb{C}}\Lambda_\epsilon$, so we will show that $\mathrm{End}_{\mathfrak{q}_\epsilon(U)}(U^{\otimes k}) = \mathrm{End}_{\mathfrak{q}_\epsilon(V)}(V^{\otimes k}) \otimes \Lambda_\epsilon$. The result then follows from Theorem 3.1.

Given $\mathrm{End}_{\Lambda_\epsilon}(U^{\otimes k}) \cong \mathrm{End}_{\mathbb{C}}(V^{\otimes k}) \otimes \Lambda_\epsilon$, an element $\phi \in \mathrm{End}_{\Lambda_\epsilon}(U^{\otimes k})$ can be expressed as $\sum_i \phi_i \otimes \lambda_i$ where $\phi_i \in \mathrm{End}_{\mathbb{C}}(V^{\otimes k})$ and λ_i's are in Λ_ϵ can be assumed to be linearly independent over $\mathbb{C}$ and $\phi_i \otimes \lambda_i$ is homogeneous for each i. Now ϕ commutes with $\mathfrak{q}_\epsilon(U) \subset M_\epsilon(U)$ implies that $\phi \circ (X \otimes \lambda) - (X \otimes \lambda) \circ \phi = 0$ on $U^{\otimes k}$ where $X \otimes \lambda$ is an arbitrary homogeneous element of $\mathfrak{q}_\epsilon(U)$. Thus we have,

$$\sum_i (\phi_i \otimes \lambda_i) \circ X \otimes \lambda - \sum_i X \otimes \lambda \circ (\phi_i \otimes \lambda_i) = 0.$$

Using the relation $(Y \otimes \mu) \circ (X \otimes \lambda) = Y \circ X \otimes \epsilon(|\mu|, |X|)\mu\lambda$ in $\mathrm{End}_{\Lambda_\epsilon}(U^{\otimes k})$, we get from the above equality that

$$\sum_i (\phi_i \circ X \otimes \epsilon(|\lambda_i|, |X|)\lambda_i\lambda - \sum_i X \circ \phi_i \otimes \epsilon(|\lambda|, |\phi_i|)\epsilon(|\lambda|, |\lambda_i|)\lambda_i\lambda) = 0 \text{ for infinitely many } \lambda.$$

Using the condition that $|X| = -|\lambda|$ and that $\epsilon(-|X|, |\phi_i|) = \epsilon(|\phi_i|, |X|)$ obtained from the relations in Definition 2.1(2),(3), we reduce the above equation to

$$\sum_i (\phi_i \circ X - \epsilon(|\phi_i|, |X|)X \circ \phi_i) \otimes \epsilon(|\lambda|, |\lambda_i|)\lambda_i\lambda = 0.$$

Since λ_i are linearly independent over $\mathbb{C}$ and Λ_ϵ is of infinite degree, λ can be chosen of sufficiently large degree N such that $\lambda\lambda_i$ are linearly independent for all i. Hence the left hand-side element of $\mathrm{End}_{\mathbb{C}}(V^{\otimes k}) \otimes \Lambda_\epsilon$ is 0 if and only if $\phi_i \circ X - \epsilon(|\phi_i|, |X|)X \circ \phi_i = 0$ for each i. In particular, $\phi_i \in \mathrm{End}_{\mathfrak{q}_\epsilon(V)}(V^{\otimes k})$ for all i, as required. $\square$

4. Mixed tensor Invariants of $Q_\epsilon(U)$

Let G, V, ϵ, V be as defined above. By a mixed tensor space we mean a direct sum of the form $\oplus_{i=1}^s (V^{\otimes b_i} \otimes V^{*\otimes t_i})$ for an $s \in \mathbb{N}$ and $t_i, b_i \in \mathbb{N} \cup \{0\}$. For simplicity of notation we denote $V^{\otimes b_i} \otimes V^{*\otimes t_i}$ by $V_{b_i}^{t_i}$. Since each summand in the mixed tensor space has a G-grading, there is a natural G-grading inherited by their direct sum $\oplus_{i=1}^s (V_{b_i}^{t_i})$. Taking $U = V \otimes_{\mathbb{C}} \Lambda_\epsilon$ and $U^* = Hom_{\Lambda_\epsilon}(U, \Lambda_\epsilon) \cong V^* \otimes_{\mathbb{C}} \Lambda_\epsilon$, the mixed tensor space over Λ_ϵ is the G-graded space $\oplus_{i=1}^s (U^{\otimes b_i} \otimes_{\Lambda_\epsilon} U^{*\otimes t_i})$ for an $s \in \mathbb{N}$ and $t_i, b_i \in \mathbb{N} \cup \{0\}$. It may be seen that $\left(\oplus_{i=1}^s (V_{b_i}^{t_i})\right) \otimes_{\mathbb{C}} \Lambda_\epsilon \cong \oplus_{i=1}^s (U^{\otimes b_i} \otimes_{\Lambda_\epsilon} U^{*\otimes t_i})$. We shall denote this mixed tensor space over Λ_ϵ by W.

Fixing an ordering for the elements of $G_{\bar{0}}$, we then list the elements of G as $\{\circ, g_1, g_2, \ldots, g, g+g_1, g+g_2, \ldots\}$ with the elements in $G_{\bar{0}}$ appearing first. We fix a basis of V, $\{e_i\}_{i=1}^{2m}$, arranged such that the first m vectors are from $V_{G_{\bar{0}}}$ and the rest are from $V_{G_{\bar{1}}}$; further, ordered linearly so that $i < j$ implies $|e_i|$ appears before $|e_j|$ under the fixed ordering of elements of G. Let $\{e_i^*\}_{i=1}^{2m}$ be the dual basis corresponding to $\{e_i\}_{i=1}^{2m}$. We denote the image in U and U^* of the above basis elements under the embedding $V \hookrightarrow U$ (and $V^* \hookrightarrow U^*$ respectively) also by the same notation. The mixed tensor space W then has a basis given in terms of the above bases of U and U^*. Denote the element dual to the basis vector $e_{l_1} \otimes \cdots \otimes e_{l_{b_i}} \otimes e_{u_1}^* \otimes \cdots \otimes e_{u_{t_i}}^* \in U_{b_i}^{t_i}$ by $T(i)_{l_1 \ldots l_{b_i}}^{u_1 \ldots u_{t_i}}$. We denote the corresponding element in W^* also by the same notation. Notice that $T(i)_{l_1 \ldots l_{b_i}}^{u_1 \ldots u_{t_i}}$ defines a linear map on W via the projection $p_i : W \to U_{b_i}^{t_i}$, *i.e.,* $T(i)_{l_1 \ldots l_{b_i}}^{u_1 \ldots u_{t_i}}(v_1, \ldots, v_s) = T(i)_{l_1 \ldots l_{b_i}}^{u_1 \ldots u_{t_i}}(v_i)$ for $(v_1, \ldots, v_s) \in W$. The G-grading of the element $T(i)_{l_1 \ldots l_{b_i}}^{u_1 \ldots u_{t_i}} \in W^*$ is given by $\sum_{i=1}^{b_i} h_{l_i} - \sum_{i=1}^{t_i} h_{u_i}$ where h_a are such that $e_a \in V_{h_a}$. The set of all $T(i)_{l_1 \ldots l_{b_i}}^{u_1 \ldots u_{t_i}}$, where $i = 1, \ldots, s$ and u_j, l_j are from $\{1, \cdots, 2m\}$, forms a Λ_ϵ-basis of W^*.

4.1. **Symmetric algebra of the mixed tensor space.** Let $S(W^*)$ be the symmetric algebra of W^*. We denote by $\varpi : T(W^*) \to S(W^*)$ the natural map from the tensor algebra of W^* to $S(W^*)$. We know that $S(W^*)$ has a $\mathbb{Z}$-grading given by $\oplus_{r\in\mathbb{N}\cup\{0\}}S^r(W^*)$ where $S^r(W^*)$ is the image under ϖ of $T^r(W^*)$. The restriction of ϖ to $T^r(W^*)$ is denoted as ϖ_r. It is easily seen that $S(W^*)$ has a Λ_ϵ-basis consisting of monomials in $T(i)^{u_1\dots u_{t_i}}_{l_1\dots l_{b_i}}$, $i = 1,\dots,s$ and u_j, l_j are from $\{1,\cdots,m+n\}$ where the variables $T(i)^{u_1\dots u_{t_i}}_{l_1\dots l_{b_i}}$ are arranged in order such that basis vectors of $(\oplus_{i=1}^s V_{b_i}^{t_i *})_{g_l}$ come before those of $(\oplus_{i=1}^s V_{b_i}^{t_i *})_{g_k}$ whenever $l < k$ and among them arranged from $i = 1\dots,s$; the degree of each variable $T(i)^{u_1\dots u_{t_i}}_{l_1\dots l_{b_i}} \in (\oplus_{i=1}^s V_{b_i}^{t_i *})_{G_{\bar{1}}}$ in such a monomial being either 0 or 1. The monomials among the above whose total degree is r, form a basis for $S^r(W^*)$ over Λ_ϵ.

4.2. $Q_\epsilon(U)$**-invariants of the symmetric algebra** $S(W^*)$**.** In this subsection we give a spanning set for the space of $Q_\epsilon(U)$-invariants of $S(W^*)$. The following argument is similar to the argument given in [15, §3.2]. Let $\sum_i m_i b_i = N$ and $\sum_i m_i t_i = N'$. We have the following sequence of $Q_\epsilon(U)$-invariant isomorphisms:

$$\left(U^{\otimes(\sum_i m_i b_i)} \otimes (U^*)^{\otimes(\sum_i m_i t_i)}\right)^* \to (\otimes_{i=1}^s (U_{b_i}^{t_i})^{\otimes m_i})^* \to \otimes_{i=1}^s((U_{b_i}^{t_i})^*)^{\otimes m_i} \tag{4.1}$$

The first isomorphism is induced on the duals by the permutation action of $\mu \in S_{N+N'}$ on the subspace $U^{\otimes N} \otimes U^{*\otimes N'}$ of $(U \oplus U^*)^{\otimes(N+N')}$ where μ takes $(1,\dots,N+N')$ to $(1,\dots b_1, N+1,\dots,N+t_1, b_1+1,\dots 2b_1,\cdots)$. The isomorphism above is $Q_\epsilon(U)$-equivariant since the symmetric group action on $(U \oplus U^*)^{\otimes(N+N')}$ commutes with the $Q_\epsilon(U)$-action induced on it. Let $k = \sum_i m_i$ and $W = W_1 \oplus \cdots \oplus W_k$, where $W_i = U_{b_i}^{t_i}$. Then the group S_{2k} acts naturally on the $2k$-fold tensor product $(W \oplus W^*)^{\otimes 2k}$. The second isomorphism in the above sequence is the inverse of ι, as described in (6) of §2.8, between $W_1^* \otimes \cdots \otimes W_k^*$ and $(W_1 \otimes \cdots \otimes W_k)^*$. More explicitly, the isomorphism is given on the dual basis by

$$T_{w_1\otimes\cdots\otimes w_{\sum_i m_i}} \mapsto p_\epsilon(\underline{w},\underline{w})T(1)_{w_1} \otimes \cdots \otimes T(s)_{w_{\sum_i m_i}}$$

where $w_i \in W_i$ is a basis vector and $T(i)_{w_i}$ is the dual vector in W_i^*; here for $w_i = e_{l_1} \otimes \cdots \otimes e_{l_{b_i}} \otimes e^*_{u_1} \otimes \cdots \otimes e^*_{u_{t_i}} \in U_{b_i}^{t_i}$ the notation $T(i)_{w_i}$ represents the linear map $T(i)^{u_1\dots u_{t_i}}_{l_1\dots l_{b_i}}$ by our earlier notation. The value of $p_\epsilon(\underline{w},\underline{w})$ for $\underline{w} = (w_1,\dots,w_k)$ where $w_i \in V_{|w_i|}$ is given by $\prod_{i=1}^{k-1}(\prod_{i<j\le k}\epsilon(|w_i|,|w_j|))$. This isomorphism is $Q_\epsilon(U)$-equivariant since the symmetric group action on $(W \oplus W^*)^{\otimes 2k}$ commutes with the $Q_\epsilon(U)$-action induced on it.

By a standard argument, the space $U^{\otimes N} \otimes U^{*\otimes N'}$ has $Q_\epsilon(U)$-invariants if and only if $N = N'$, and so does its dual, $(U^{\otimes N} \otimes U^{*\otimes N'})^*$. When $N = N'$, the latter space can be identified with $\mathrm{End}_{\Lambda_\epsilon}(U^{\otimes N})$ via the non-degenerate pairing $(\mathrm{End}_{\Lambda_\epsilon}(U^{\otimes N})) \otimes (U^{\otimes N} \otimes U^{*\otimes N}) \to \Lambda_\epsilon$ described in §2.8 leading to a $Q_\epsilon(U)$-linear isomorphism,

$$\Theta : \mathrm{End}_{\Lambda_\epsilon}(U^{\otimes N}) \to (U^{\otimes N} \otimes U^{*\otimes N})^*.$$

The $Q_\epsilon(U)$-invariants of $\mathrm{End}_{\Lambda_\epsilon}(U^{\otimes N})$ is just the centralizer $\mathrm{End}_{Q_\epsilon(U)}(U^{\otimes N})$ which by Theorem 3.3, is spanned by the images of the elements of H_k. Now using the same argument as in the proof of [15, Theorem 3.2], we see that $S(W^*)^{Q_\epsilon(U)}$ is spanned by the images of the elements of H_k, as k varies in $\mathbb{N}$.

For $\sigma \in S_N$ and $c_1^{\varepsilon_1}\dots c_N^{\varepsilon_N} \in \mathbf{C}l_N$ which we also denote in short-hand by $c^{\boldsymbol{\varepsilon}}$, the linear map on $U^{\otimes N}\otimes U^{*\otimes N}$, associated to the element $\sigma\otimes c^{\boldsymbol{\varepsilon}} \in H_N$ under the map Ψ followed by Θ is given by

$$\sum_{L=(l_1,\dots,l_N)} \hat{\gamma}(L,\boldsymbol{\varepsilon})\gamma((\sigma(L^{\boldsymbol{\varepsilon}}),\sigma(L^{\boldsymbol{\varepsilon}})^*),\tau^{-1})\gamma(\tau.(\sigma(L^{\boldsymbol{\varepsilon}}),\sigma(L^{\boldsymbol{\varepsilon}})^*),\nu^{-1})T_{L,\sigma(L^{\boldsymbol{\varepsilon}})^*}$$

where $\boldsymbol{\varepsilon} = (\varepsilon_1,\dots,\varepsilon_N)$, $(l_1,\dots,l_N)^* := (l_1^*,\dots,l_N^*)$, indicating that these are the indexes corresponding to the basis vectors $e^*_{l_1},\dots,e^*_{l_N}$ in the dual space U^*, $(l_1,\dots,l_N)^{\boldsymbol{\varepsilon}} := (\varepsilon_1 m + l_1,\dots,\varepsilon_N m + l_N)$ where $m = \dim V_{\bar{0}}$ and the indices are taken modulo $(2m)$; note that $|l_i + \varepsilon_i m| = g_{\varepsilon_i} + |l_i|$, $|(l_i+\varepsilon_i m)^*| = -(g_{\varepsilon_i}+|l_i|)$ and $\hat{\gamma}((l_1,\dots,l_N),\boldsymbol{\varepsilon}) := (-\kappa(g,g))^{\sum_{i=1}^N \varepsilon_i |v_{l_i}|}\prod_{j<i}\epsilon(g_{\varepsilon_i,|v_{l_j}|})$.

Going through the isomorphisms in (4.1) and projecting via the map $\varpi_{m_1}\otimes\cdots\otimes\varpi_{m_s}$ the above linear map corresponds to the following element of $S(W^*)$:

$$\sum_{L=(l_1,\dots,l_N)} \hat{q}_{\sigma,\boldsymbol{\varepsilon}}(L,m_1,\dots,m_s)\prod_{i=1}^{s}\prod_{j=1}^{m_i} T(i)^{l_{(\sum_{p<i} m_p t_p + (j-1)t_i+1)}\cdots l_{(\sum_{p<i} m_p t_p + j t_i)}}_{l'_{(\sum_{p<i} m_p b_p + (j-1)b_i+1)}\cdots l'_{(\sum_{p<i} m_p b_p + j b_i)}} \tag{4.2}$$

where $l_i \in \{1,\dots,m,m+1,\cdots,2m\}$ for all $i = 1,\dots,N$, $l'_j = l_{\sigma(j)} + \varepsilon_j m(\mod 2m)$ for $j = 1\dots,N$. The value of $\hat{q}_{\sigma,\boldsymbol{\varepsilon}}(L,m_1,\dots,m_s)$ is given by the formula

$$p(\underline{w},\underline{w})\gamma(\mu^{-1}.(L,\sigma(L^{\boldsymbol{\varepsilon}})^*),\mu^{-1})\hat{\gamma}(L,\boldsymbol{\varepsilon})\gamma((\sigma(L^{\boldsymbol{\varepsilon}}),\sigma(L^{\boldsymbol{\varepsilon}})^*),\tau^{-1})\gamma(\tau.(\sigma(L^{\boldsymbol{\varepsilon}}),\sigma(L^{\boldsymbol{\varepsilon}})^*),\nu^{-1}).$$

This element in $S(W^*)$ is denoted as $\varphi_{\sigma\otimes c^{\boldsymbol{\varepsilon}}}$. Given an s-tuple $(m_1,\dots,m_s)$ of non-negative integers such that $\sum_{k=1}^s m_k t_k = \sum_{k=1}^s m_k b_k = N$, as σ varies in S_N and $\boldsymbol{\varepsilon}$ varies in $\mathbb{Z}_2^N$, the elements $\varphi_{\sigma\otimes c^{\boldsymbol{\varepsilon}}}$ will be referred to as 'coloured invariants' [1].

We have thus obtained the following:

Theorem 4.1. *Given a mixed tensor space $W = \oplus_{i=1}^s U_{b_i}^{t_i}$, the coloured invariants as in (4.2) spans $S(W^*)^{Q_\epsilon(U)}$.* □

Remark 4.2. We say that a variable

$$T(i)^{r_{(\sum_{p<i} m_p t_p + (j-1)t_i+1)}\cdots r_{(\sum_{p<i} m_p t_p + j t_i)}}_{r_{\sigma(\sum_{p<i} m_p b_p + (j-1)b_i+1)}\cdots r_{\sigma(\sum_{p<i} m_p b_p + j b_i)}} \in S(W^*)$$

is *even* or *odd* depending on whether the degree of the variable is in $G_{\bar{0}}$ or $G_{\bar{1}}$. With this terminology, we note that in the above formula the monomials with repeated odd degree variables will be identically 0 in $S(W^*)$. In particular, those indices $r_1,\dots,r_N$ in the sum in

[1]In (4.2), the monomials should be considered modulo the anti-commutativity relations in $S(W^*)$. See Remark 4.2 below.

(4.2) are to be dropped for which there is a $1 \leq i \leq s$ and $1 \leq j < j' \leq m_i$ such that the above variable corresponding to these values of i, j, j' turns out to be of odd degree and,

$$(r_{\sigma(\sum_{p<i} m_p b_p + (j-1)b_i + 1)}, \ldots, r_{\sigma(\sum_{p<i} m_p b_p + j b_i)}) = (r_{\sigma(\sum_{p<i} m_p b_p + (j'-1)b_i + 1)}, \ldots, r_{\sigma(\sum_{p<i} m_p b_p + j' b_i)}),$$
$$(r_{\sum_{p<i} m_p t_p + (j-1)t_i + 1}, \ldots, r_{\sum_{p<i} m_p t_p + j t_i}) = (r_{\sum_{p<i} m_p t_p + (j'-1)t_i + 1}, \ldots, r_{\sum_{p<i} m_p t_p + j' t_i}).$$

4.2.1. *Coloured picture invariants on* $\mathrm{End}_{\Lambda_\epsilon}(U)^s$ *in terms of color traces.* For an s-tuple $(m_1, \ldots, m_s)$ such that $\sum_{i=1}^s m_i = r$, let $T_\sigma \in (\otimes_{i=1}^s (U_{b_i}^{t_i})^{\otimes m_i})^*$ be the linear map corresponding to $\sigma \in S_N$ obtained via the map Ψ followed by the isomorphisms Θ and (4.1). The coloured picture invariants defined in §4.2 are the images in $S^{m_1,\ldots,m_s}(W^*)$ of these linear maps T_σ, $\sigma \in S_N$.

When $t_i = b_i = 1$ for all $i = 1, \ldots s$, under the identification $\mathrm{End}_{\Lambda_\epsilon}(U) \cong U \otimes_{\Lambda_\epsilon} U^*$ the mixed tensor space W can be identified with $\mathrm{End}_{\Lambda_\epsilon}(U)^s$; further the composition operation in $\mathrm{End}_{\Lambda_\epsilon}(U)$ corresponds to the product on $U \otimes U^*$ given by $(v \otimes \alpha).(w \otimes \beta) = v\alpha(w) \otimes \beta$. Then the color trace monomial ctr_σ associated to σ is defined as $ctr_\sigma(v_1 \otimes \phi_1 \otimes \cdots \otimes v_N \otimes \phi_N) = ctr(v_{i_1} \otimes \phi_{i_1}.v_{i_2} \otimes \phi_{i_2}.\cdots.v_{i_r} \otimes \phi_{i_r}) ctr(v_{j_1} \otimes \phi_{j_1}.v_{j_2} \otimes \phi_{j_2}.\cdots.v_{j_t} \otimes \phi_{j_t}) \cdots$ where $\sigma^{-1} = (i_1\ i_2\ \ldots i_r)(j_1\ j_2 \cdots j_t)\cdots$. The proof of the following is analogous to that of Lemma 3.8 of [14]:

Lemma 4.3. *For a* $\sigma \in S_N$ *such that* $\sigma^{-1} = (i_1\ i_2\ \ldots i_r)(j_1\ j_2 \cdots j_t) \cdots \in S_N$, *the* $\mathrm{GL}_\epsilon(U)$-*invariant map* T_σ *(as described above) corresponds to the trace monomials* ctr_σ *up to a scalar. Further, both the maps agree when restricted to the degree* $\circ$ *part,* $((U_1^1)_\circ)^{\otimes N}$.

□

From the above Lemma, the coloured invariants can be described in terms of color traces as follows:

Lemma 4.4. *For a* $\sigma \in S_N$ *and* $c_1^{\varepsilon_1} \ldots c_N^{\varepsilon_N} \in \mathbf{Cl}_N$, *we have*

$$ev(\nu\tau.(\sigma\widehat{c_1^{\varepsilon_1} \ldots c_N^{\varepsilon_N}}).\tau^{-1})(u_1 \otimes \phi_1 \otimes \cdots \otimes u_N \otimes \phi_N) = m.ctr_\sigma(P^{\varepsilon_1}u_1 \otimes \phi_1, \ldots, P^{\varepsilon_N}u_N \otimes \phi_N),$$

where $m \in \mathbb{C} \setminus \{0\}$ *and* $(u_1 \otimes \phi_1 \otimes \cdots \otimes u_N \otimes \phi_N) \in ((U_1^1)_0)^{\otimes N}$.

Proof. We have

$$(\sigma.c_1^{\varepsilon_1} \ldots c_N^{\varepsilon_N}).(u_1 \otimes \cdots \otimes u_N) =$$
$$\gamma((|u_1| + g_{\varepsilon_1}, \cdots, |u_N| + g_{\varepsilon_N}), \sigma^{-1})[\prod_{i>j} \epsilon(g_{\varepsilon_i}, |u_j|)](P^{\varepsilon_{\sigma^{-1}(1)}} u_{\sigma^{-1}(1)} \otimes \ldots \otimes P^{\varepsilon_{\sigma^{-1}(k)}} u_{\sigma^{-1}(N)}).$$

Consider the action of $c_1^{\varepsilon_1} \ldots c_N^{\varepsilon_N}$ followed by σ from the left on $\tau^{-1}.(u_1 \otimes \phi_1 \otimes \cdots \otimes u_N \otimes \phi_N)$ where $|u_i| = \alpha_i = -|\phi_i|$. This gives the vector

$$\gamma((\alpha_1 + g_{\varepsilon_1}, \cdots, \alpha_N + g_{\varepsilon_N}), \sigma^{-1})[\prod_{i>j} \epsilon(g_{\varepsilon_i}, \alpha_j)](P^{\varepsilon_{\sigma^{-1}(1)}} u_{\sigma^{-1}(1)} \otimes \ldots \otimes P^{\varepsilon_{\sigma^{-1}(N)}} u_{\sigma^{-1}(k)} \otimes \phi_1 \otimes \ldots \otimes \phi_N).$$

Applying $\nu\tau$ to the above vector followed by the evaluation we get $\prod_{i=1}^{N}\phi_i(P^{\varepsilon_{\sigma^{-1}(i)}}v_{\sigma^{-1}(i)})$, up to a scalar. Here the scalar is given by

$$\beta=\gamma((\alpha_1+g_{\varepsilon_1},\cdots,\alpha_N+g_{\varepsilon_N}),\sigma^{-1})[\prod_{i>j}\epsilon(g_{\varepsilon_i},\alpha_j)]\gamma((\sigma.(\alpha_1+g_{\varepsilon_1},\ldots,\alpha_N+g_{\varepsilon_N}),-\alpha_1,\ldots,-\alpha_N),(\nu\tau)^{-1}).$$

Hence we note that

$$ev(\nu\tau.(\sigma\otimes\widehat{c_1^{\varepsilon_1}\ldots c_N^{\varepsilon_N}}).\tau^{-1})(u_1\otimes\phi_1\otimes\cdots\otimes u_N\otimes\phi_N))=ev((\nu\tau.\widehat{\sigma}.\tau^{-1}).(P^{\varepsilon_1}u_1\otimes\phi_1\otimes\cdots\otimes P^{\varepsilon_N}u_N\otimes\phi_N)).$$

From Lemma 4.3 we get that this is equal to $\beta.ctr_\sigma(P^{\varepsilon_1}u_1\otimes\phi_1,\ldots,P^{\varepsilon_N}u_N\otimes\phi_N)$. □

Identifying $(U_1^1)_\circ$ with $M_\epsilon(U)$, we also have the following analogue of [14, Proposition 6.3]. The proof is the same line of argument used in Theorem 4.1 to observe that the linear maps $ev(\nu\tau.(\sigma\widehat{c_1^{\varepsilon_1}\ldots c_N^{\varepsilon_N}}).\tau^{-1})$ of the above lemma span the $Q_\epsilon(U)$-invariant linear functionals on $\mathrm{End}_{\Lambda_\epsilon}(U)^{\otimes N}$. Let $T_{\sigma,\boldsymbol{\varepsilon}}$ be the linear map $ev(\nu\tau.(\sigma\widehat{c_1^{\varepsilon_1}\ldots c_N^{\varepsilon_N}}).\tau^{-1})$ of the above lemma.

Theorem 4.5. *The space of $Q_\epsilon(U)$-invariant linear functionals on $M_\epsilon(U)^{\otimes N}$ is spanned by colour trace monomials given by*

$$ctr_{\sigma,\boldsymbol{\varepsilon}}(A_1\otimes\cdots\otimes A_N)=ctr(P^{\varepsilon_{i_1}}A_{i_1}\cdots P^{\varepsilon_{i_a}}A_{i_a})ctr(P^{\varepsilon_{j_1}}A_{j_1}\cdots P^{\varepsilon_{j_b}}A_{j_b})\cdots$$

where $\sigma=(i_1,i_2,\ldots,i_a)(j_1,\ldots,j_b)\cdots$ is a permutation of S_N written as product of disjoint cycles and $\boldsymbol{\varepsilon}=(\varepsilon_1\ldots,\varepsilon_n)\in\mathbb{Z}_2^N$

Proof. We already observed above that $T_{\sigma,\boldsymbol{\varepsilon}}$ span $[\mathrm{End}_\epsilon(U)^{\otimes N*}]^{Q_\epsilon(U)}$ and so they also span $[M_\epsilon(U)^{\otimes N*}]^{Q_\epsilon(U)}$. Further by the above Lemma, these $T_{\sigma,\boldsymbol{\varepsilon}}$ are given by color trace monomials, upto a scalar. □

4.3. **Polynomial invariants of $W_\circ$.** We recall the notion of polynomials from [15]. Let W be the mixed tensor space as defined in the beginning of this section. Then the graded component of W corresponding to the identity element $\circ\in G$ is denoted as $W_\circ$. Let us consider the Λ_ϵ-module of all functions from $W_\circ\to\Lambda_\epsilon$, denoted by $\mathcal{F}(W_\circ,\Lambda_\epsilon)$. Let $F^r:T^r(W^*)\to\mathcal{F}(W_\circ,\Lambda_\epsilon)$ be given by $F^r(\boldsymbol{\alpha})(v)=ev(\boldsymbol{\alpha}\otimes v\otimes\cdots\otimes v)$ where $ev:T^r(W^*)\otimes T^r(W)\to\Lambda_\epsilon$ is as defined in §2.8. Let $F^\bullet:=\oplus_{r=0}^{\infty}F^r:T(W^*)\to\mathcal{F}(W_\circ,\Lambda_\epsilon)$. This map factors through $S(W^*)$. We continue to denote the induced map from $S(W^*)\to\mathcal{F}(W_\circ,\Lambda_\epsilon)$ also by $F^\bullet$, and its restriction to $S^r(W^*)$ as F^r. The map $F^\bullet$ is an isomorphism onto its image. The image of $F^\bullet$ is denoted as $\mathcal{P}(W_\circ)$. The elements of $\mathcal{P}(W_\circ)$ are called *polynomials on* $W_\circ$. The map $F^r:S^r(W^*)\to\mathcal{F}(W_\circ,\Lambda_\epsilon)$ is called the restitution map and its image is $\mathcal{P}^r(W_\circ)$. A $Q_\epsilon(U)$-action can be given on $\mathcal{P}^r(W_\circ)$ via the isomorphism $F^\bullet$. The invariants in $\mathcal{P}(W_\circ)$ for the induced action of $Q_\epsilon(U)$ are called the invariant polynomials on $W_\circ$. The invariant polynomials of degree r are the images under F^r of $S^r(W^*)^{Q_\epsilon(U)}$.

For a tuple $(m_1,\ldots,m_s)$ such that $\sum_{i=1}^{s}m_i=r$, let $T_{\sigma,\boldsymbol{\varepsilon}}\in(\otimes_{i=1}^{s}(U_{b_i}^{t_i})^{\otimes m_i})^*$ be the linear map described above, corresponding to $\sigma\in S_N$ and $\boldsymbol{\varepsilon}\in\mathbb{Z}/2\mathbb{Z}^N$. The coloured invariants defined in (4.2) are the images in $S^{m_1,\dots,m_s}(W^*)$ of these linear maps T_σ, $\sigma \in S_N$. Then the proof of [15, Proposition 3.5] also establishes the following:

Proposition 4.6. *The coloured invariant $\varphi_{\sigma\otimes c^{\boldsymbol{\varepsilon}}} \in S^{(m_1,\dots,m_s)}(W^*)$ maps under restitution F^r to the element of $\mathcal{F}(W_\circ, \Lambda_\epsilon)$ given by $\boldsymbol{u} \mapsto T_{\sigma,\boldsymbol{\varepsilon}}(u_1^{\otimes m_1}\otimes\cdots\otimes u_s^{\otimes m_s})$ where $\boldsymbol{u} = (u_1 \dots, u_s) \in W_\circ$.* □

Corollary 4.7. *The space of $Q_\epsilon(U)$-invariant polynomials on $\mathrm{End}_{\Lambda_\epsilon}(U)^s$ is spanned by the color trace monomials given by*

$$ctr(P^{\varepsilon_{i_1}} A_{i_1} \cdots P^{\varepsilon_{i_a}} A_{i_a}) ctr(P^{\varepsilon_{j_1}} A_{j_1} \cdots P^{\varepsilon_{j_b}} A_{j_b}) \cdots$$

where $A_1, \dots, A_s \in \mathrm{End}_{\Lambda_\epsilon}(U)$, $\sigma = (i_1, i_2, \dots, i_a)(j_1, \dots, j_b)\cdots$ is a permutation of S_N written as product of disjoint cycles, $\boldsymbol{\varepsilon} = (\varepsilon_1 \dots, \varepsilon_N) \in \mathbb{Z}_2^N$ and $f : \{1, \dots, N\} \to \{1, \dots, s\}$, for each $N \in \mathbb{N}$.

Proof. Follows from Proposition 4.6 and the Lemma 4.4 in view of the Theorem 4.1. □

Lemma 4.8. *For $A \in \mathfrak{q}_{\Lambda_\epsilon}(U)$, $ctr(A) = 0$.*

Proof. We have $P^2 = -\kappa(g, g)I$. Let $A \in \mathfrak{q}_{\Lambda_\epsilon}(U)$. Since A commutes with P, we have $PAP = P^2A = -\kappa(g, g)A$. On the other hand since PA and P are each of degree g, we have

$$ctr(PAP) = \epsilon(g, g)ctr(P^2A) = \kappa(g, g)ctr(A),$$

as $\epsilon(g, g) = -1$. So $ctr(A) = 0$. □

Remark: Suppose $A_1, A_2, \cdots, A_k$ commute with P. We then have $P^{\varepsilon_1}A_1 \cdots P^{\varepsilon_k}A_k = P^{\varepsilon_1+\cdots+\varepsilon_k}A_1 \cdots A_k$. So,

$$ctr(P^{\varepsilon_1}A_1 \cdots P^{\varepsilon_k}A_k) = ctr(P^{\varepsilon_1+\cdots+\varepsilon_k}A_1 \cdots A_k) = (-1)^t\kappa(g, g)^t.ctr(A_1 \cdots A_k) = 0$$

if $\varepsilon_1 + \cdots + \varepsilon_k = 2t$ for some t. If $\varepsilon_1 + \cdots + \varepsilon_k = 2t + 1$ for some t then we have $ctr(P^{\varepsilon_1}A_1 \cdots P^{\varepsilon_k}A_k) = (-1)^t\kappa(g, g)^t.ctr(PA_1 \cdots A_k) = (-1)^t\kappa(g, g)^t.qtr(A_1 \cdots A_k)$.

Let $\mathfrak{I}$ be the subspace of elements of $(U_1^1)_\circ$ which commute with P. We then have the following corollary.

Corollary 4.9. *The space of $Q_\epsilon(U)$-invariant polynomials on $\mathrm{End}_{\Lambda_\epsilon}(U)^s$ restricted to $\mathfrak{I}^s$ is spanned by the queer trace monomials given by*

$$qtr(A_{f(i_1)} \cdots A_{f(i_r)})qtr(A_{f(i_{r+1})} \cdots A_{f(i_t)}) \cdots$$

where $A_1, \dots, A_s \in \mathfrak{I}$, $(i_1 \dots i_r)(i_{r+1} \dots i_t) \cdots \in S_N$ and a map $f : \{1, \dots, N\} \to \{1, \dots, s\}$, for each $N \in \mathbb{N}$.

Proof. Follows from Corollary 4.7 in view of the above remark. □

5. Concomitants on $\mathfrak{q}_\epsilon(U)^k$

For G-graded vector spaces M and M' over $\mathbb{C}$, we set $M_\epsilon := M \otimes_{\mathbb{C}} \Lambda_\epsilon$ and $M'_\epsilon = M' \otimes_{\mathbb{C}} \Lambda_\epsilon$. The $\circ$-component of M_ϵ and M'_ϵ are denoted as $(M_\epsilon)_\circ$ and $(M'_\epsilon)_\circ$ respectively. A morphism $\phi : (M_\epsilon)_\circ \to (M'_\epsilon)_\circ$ is said to be a *polynomial map* if the map induced by ϕ from $\mathcal{F}((M'_\epsilon)_\circ) \to \mathcal{F}((M_\epsilon)_\circ)$ given by $\alpha \mapsto \alpha \circ \phi$ takes $\mathcal{P}((M'_\epsilon)_\circ) \to \mathcal{P}((M_\epsilon)_\circ)$. We denote the latter map as ϕ^*. In view of the isomorphism of the symmetric algebras $S(M_\epsilon{}^*)$ and $S(M'^*_\epsilon)$ with the G-graded algebras $\mathcal{P}((M_\epsilon)_\circ)$ and $\mathcal{P}((M'_\epsilon)_\circ)$ respectively, it may be noted that ϕ is a polynomial map if and only if $T_i \circ \phi$ is an element of $\mathcal{P}((M_\epsilon)_\circ)$ for all the dual basis vectors, $T_i \in (M'_\epsilon)^*$.

Remark 5.1. (1) It is easily seen that, for G-graded vector spaces M, N, M', N', the projections $(M_\epsilon)_\circ \times (M'_\epsilon)_\circ \to (M_\epsilon)_\circ$, $(M_\epsilon)_\circ \times (M'_\epsilon)_\circ \to (M'_\epsilon)_\circ$ are polynomial maps; and if $f : (M_\epsilon)_\circ \to (N_\epsilon)_\circ$, $g : (M'_\epsilon)_\circ \to (N'_\epsilon)_\circ$ are polynomial maps then so is $(f, g) : (M_\epsilon)_\circ \times (M'_\epsilon)_\circ \to (N_\epsilon)_\circ \times (N'_\epsilon)_\circ$.

(2) If N, N' are G-graded vector subspaces of M, M' respectively and $\phi : (M_\epsilon)_\circ \to (M'_\epsilon)_\circ$ is a polynomial map then the restriction of ϕ to $(N_\epsilon)_\circ$ is also a polynomial map; indeed, restriction to $(N_\epsilon)_\circ$ induces a surjection from $S(M^*_\epsilon)$ onto $S(N^*_\epsilon)$, hence the restriction map $res : \mathcal{P}((M_\epsilon)_\circ) \to \mathcal{P}((N_\epsilon)_\circ)$ given by $f \mapsto f|_{(N_\epsilon)_\circ}$ is a surjection, so $f \circ \phi|_{(N_\epsilon)_\circ} = (res \circ \phi^*)(f) \in \mathcal{P}((N_\epsilon)_\circ)$ for all $f \in \mathcal{P}((M'_\epsilon)_\circ)$. Further, if $\phi((N_\epsilon)_\circ) \subset (N'_\epsilon)_\circ$, then $\phi : (N_\epsilon)_\circ \to (N'_\epsilon)_\circ$ is also a polynomial map, since $res : \mathcal{P}((M'_\epsilon)_\circ) \to \mathcal{P}((N'_\epsilon)_\circ)$ is a surjection and $f|_{(N'_\epsilon)_\circ} \circ \phi|_{(N_\epsilon)_\circ} = f \circ \phi|_{(N_\epsilon)_\circ}$.

Lemma 5.2. *Let $M_\epsilon(U) = (\mathrm{End}_{\Lambda_\epsilon}(U))_\circ$ be as defined in §3.4. Then the maps describing the addition and composition operations in $M_\epsilon(U)$ are polynomial maps.*

Proof. That the addition operation is a polynomial map follows from the earlier observation that the projections X_1, X_2 from $M_\epsilon(U) \times M_\epsilon(U) \to M_\epsilon(U)$ are polynomial maps and that $\alpha \circ (X_1 + X_2) = \alpha \circ X_1 + \alpha \circ X_2$ for $\alpha \in \mathcal{P}(M_\epsilon(U))$.

For the claim about the composition map, first let $T \in \mathrm{End}_{\Lambda_\epsilon}(U)$. Denote the basis elements of U (obtained from that of V) by v_a, $a = 1, \dots, 2m$. Then $T(v_b) = \sum_a v_a t_{ab}$. Let T_{ab} be the elementary transformation given by $v_c \mapsto \delta_{cb} v_a$ as $c = 1 \dots, 2m$. Then we can express T in terms of the basis T_{ab} as $\sum_{a,b} \epsilon(|v_b|, |t_{ab}|) T_{ab} t_{ab}$. Further, if $\gamma_{ab} \in (\mathrm{End}_{\Lambda_\epsilon}(U))^*$ be the basis element dual to T_{ab}, then $\gamma_{cd}(T) = \epsilon(|t_{cd}|, |v_d|) t_{cd}$. Note that if $T \in M_\epsilon(U)$ then $T(U_g) \subset U_g$ for each $g \in G$, so $|t_{ab}| = |v_b| - |v_a|$ in the above expression and $|T_{ab}| = -|t_{ab}| = |v_a| - |v_b|$. Now using the relation that $\lambda T = \epsilon(|\lambda|, |T|) T \lambda$ for $\lambda \in \Lambda_\epsilon$, one can check that $\gamma_{cb}(S \circ T) = \sum_a \epsilon(|T_{ca}|, |t_{ab}|) \gamma_{ca}(S) \gamma_{ab}(T)$. Denoting the dual basis vector associated to $(T_{ab}, 0) \in \mathrm{End}_{\Lambda_\epsilon}(U) \times \mathrm{End}_{\Lambda_\epsilon}(U)$ and $(0, T_{cd}) \in \mathrm{End}_{\Lambda_\epsilon}(U) \times \mathrm{End}_{\Lambda_\epsilon}(U)$ by α_{ab} and β_{cd} respectively, we can rewrite the above expression as $\gamma_{cb}(S \circ T) = (\sum_a \epsilon(|T_{ca}|, |t_{ab}|) \alpha_{ca} \beta_{ab})(S, T)$, establishing that the composition on $M_\epsilon(U)$ is a polynomial map. $\square$

Identifying $\mathrm{End}_{\Lambda_\epsilon}(U)$ with $U \otimes U^*$ (see §2.8(1)), we note that $\mathfrak{q}_\epsilon(U)$ is identified with the subspace $\mathfrak{I}$ given in Corollary 4.9.

Corollary 5.3. *In $\mathfrak{q}_\epsilon(U)$, the maps describing the addition and composition are polynomial maps.*

Proof. Follows immediately from the Remark 5.1 and the above Lemma. □

A $Q_\epsilon(U)$-*concomitant* $\phi : \mathfrak{q}_\epsilon(U)^k \to \mathfrak{q}_\epsilon(U)$ is a polynomial map such that

$$\phi(Y.(A_1, \dots, A_k)) = Y.\phi(A_1, \dots, A_k)$$

for $(A_1, \dots, A_k) \in \mathfrak{q}_\epsilon(U)^k$ and $Y \in Q_\epsilon(U)$.

By the previous lemma and the observations preceding it, it may be noted that the set of all $Q_\epsilon(U)$-concomitants from $\mathfrak{q}_\epsilon(U)^k \to \mathfrak{q}_\epsilon(U)$ forms a G-graded ring with respect to the pointwise multiplication and pointwise addition.

Consider the projection operator X_i onto the i-th coordinate, *i.e.*, $X_i : \mathfrak{q}_\epsilon(U)^k \to \mathfrak{q}_\epsilon(U)$ is given by $X_i(A_1, \dots, A_i, \dots, A_k) = A_i$ where $i = 1, \dots, k$. We then have the following:

Theorem 5.4. *The algebra of $Q_\epsilon(U)$-concomitants is generated by the projections $X_1, \dots, X_k$ and the products of even number of queer trace monomials.*

Proof. Let f be a polynomial concomitant. Consider $\bar{f}(A_1, ...A_k, A_{k+1}) := qtr(f(A_1, , , A_k)A_{k+1})$. By the non-degeneracy of the color trace bilinear form it follows that, if f, g are polynomial concomitants such that $\bar{f} = \bar{g}$ on $\mathfrak{q}_\epsilon(U)^{k+1}$ then $f = g$. It is also clear from the observations made above that $\bar{f}$ is a polynomial invariant if f is a polynomial concomitant. Using Corollary 4.9, we can write $\bar{f} = \sum_{i_1,..i_j \neq k+1} \lambda_{i_1,...i_j} qtr(A_{i_1} ...A_{i_j} A_{k+1})$ where $\lambda_{i_1,...i_j}$ is product of queer trace monomials in which A_{k+1} does not appear. Now from the definition of projection operators and the above identity for $\bar{f}$ we have that $f = \sum_{i_1,...i_j} \lambda_{i_1,...i_j} X_{i_1} ...X_{i_j}$.

Note that each queer trace monomial appearing in $\lambda_{i_1,...,i_j}$ in the above expression has degree g, so $\lambda_{i_1,...,i_j}$ has degree g times the number of monomials in $\lambda_{i_1,...,i_j}$; further, $\bar{f}$ and $qtr(A_{i_1} ...A_{i_j} A_{k+1})$ also have degree g. This can happen only if the number of monomials in $\lambda_{i_1,...,i_j}$ is even. Hence the theorem. □

An expression of the form $\lambda_{i_1,...i_j} X_{i_1} ...X_{i_j}$ where $\lambda_{i_1,...i_j}$ lies in the space described in Theorem 4.5 (with $N = k - j$ and $0 < j \le k$) and $i_1, \dots, i_j$ are the indices which do not appear in $\lambda_{i_1,...i_j}$, is called a *multilinear concomitant*. When $j = 0$, by a multilinear concomitant we just mean a multilinear invariant.

Theorem 5.5. *The space of multilinear $Q_\epsilon(U)$-invariant maps from $\mathfrak{q}_\epsilon(U)^k \to \mathfrak{q}_\epsilon(U)$ is spanned by the multilinear concomitants defined above.*

Proof. Let f be a multilinear invariant map from $\mathfrak{q}_\epsilon(U)^k \to \mathfrak{q}_\epsilon(U)$. Now following the same argument as in the above theorem and then using Theorem 4.5, we obtain the desired result. $\square$

Remark 5.6. Theorems 5.4 and 5.5 may be viewed as analogues of [3, Theorem 7.4, Corollary 7.5] where Berele used the identification of the superalgebra $\mathfrak{q}(n)$ with the matrix ring $M_n(E)$ over the infinite grassmann algebra E, in order to describe the notion of polynomials on $\mathfrak{q}(n)$.

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