GRADED PICTURE INVARIANTS AND POLYNOMIAL INVARIANTS FOR MIXED TENSOR SUPERSPACES

SANTOSHA PATTANAYAK, PREENA SAMUEL

Abstract. In this paper we consider the mixed tensor space of a $\mathbb{Z}_2$-graded vector space. We obtain a spanning set of invariants of the associated symmetric algebra under the action of the general linear supergroup as well as the queer supergroup over the Grassmann algebra. As a consequence, we give a generating set of polynomial invariants for the simultaneous adjoint action of the general linear supergroup on several copies of its Lie superalgebra. We show that in this special case, this set turns out to be the set of supertrace monomials, which is analogous to the result of Procesi for the general linear group. A queer supergroup analogue of these results is also obtained.

MSC subject classification 2020: 17B10, 17B35, 17B65, 14M30.
Keywords: Picture invariants, Invariant theory, Lie supergroups, Lie superalgebras.

1. Introduction

The theory of Lie superalgebras has aroused much interest both in mathematics and physics. Lie superalgebras and their representations play a fundamental role in theoretical physics since they are used to describe supersymmetry in a mathematical framework. A comprehensive description of the mathematical theory of Lie superalgebras is given in [8], containing the complete classification of all finite-dimensional simple Lie superalgebras over an algebraically closed field of characteristic zero. Since then, these superalgebras have found applications in various areas including quantum mechanics, nuclear physics, particle physics, and string theory. In the last few years, the theory of Lie superalgebras has experienced a remarkable evolution with many results obtained on their representation theory and classification, most of them extending well-known facts from the theory of Lie algebras.

In [13], Procesi studied tuples $(A_1, \cdots, A_k)$ of endomorphisms of a finite-dimensional vector space up to simultaneous conjugation by studying the corresponding ring of invariants. He

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showed that for an algebraically closed field F of characteristic zero the algebra of invariants $F[(A_i)_{jl}]^{GL_r(F)}$ can be generated by traces of monomials in $A_1, \cdots A_k$. The main tool used in the work of Procesi, in order to describe the invariants, is the Schur–Weyl duality. This tool was used in a similar way also to study more complicated algebraic structures than just a vector space equipped with endomorphisms. In [6], Datt, Kodiyalam and Sunder applied this machinery to the study of finite-dimensional complex semisimple Hopf algebras. They were able to obtain a complete set of invariants for separating isomorphism classes of complex semi-simple Hopf algebras of a given dimension. This they accomplished by giving an explicit spanning set for the invariant ring of the mixed tensor space. These are called "picture invariants". The picture invariants obtained in the case of comlex semisimple Hopf algebras were also obtained by using techniques from Geometric Invariant Theory in [11]. In [12] the author generalized the above results to any type of algebraic structure based on a finite-dimensional vector space over an algebraically closed field of characteristic zero and gave an uniform approach for studying these rings of invariants, which does not depend on the dimension of the vector space. In [9] these picture invariants were used to describe a finite collection of rational functions in the structure constants of a Lie algebra, which form a complete set of invariants for the isomorphism classes of complex semisimple Lie algebras of a given dimension. More recently, the invariant ring of the mixed tensor spaces was used to extend results of [6] to separate isomorphism classes of complex Hopf algebras of small dimensions [14]. The invariant ring of the mixed tensor space arises naturally in obtaining invariants of any such isomorphism classes. Here, we look at the analogue of this invariant ring in the supersetting. We define "graded picture invariants" for the invariant ring of the mixed tensor superspace in the supersetting and show that these invariants Λ-linearly span the invariant ring.

The invariant theory of superalgebras has seen remarkable progress in the recent years. In [15], Sergeev proved fundamental theorems of invariant theory for classical supergroups and their Lie superalgebras (see [17] and [18] for a detailed exposition). A super analogue of the Schur–Weyl duality between the general linear superalgebra and the symmetric group was established independently in [3] and [15]. In [19], Shader and Moon gave a super Schur–Weyl duality for the general linear superalgebra acting on the mixed tensor space.

The general linear supergroup of a super vector space V over $\mathbb{C}$ is defined as a super group scheme. We, however, take the popular view of considering this group scheme as an actual group by taking the Λ-valued points of the general linear group scheme. We denote this group as $\mathrm{GL}(U)$ where $U := V \otimes_{\mathbb{C}} \Lambda$. The queer supergroup $Q(U)$ is defined as a subgroup of $\mathrm{GL}(U)$. By working with $\mathrm{GL}(U)$ and $Q(U)$ we are able to take a more elementary approach to the theory of invariants for these supergroups. Allowing scalars to come from Λ is crucial in obtaining a group-theoretic version of the well-known Schur-Weyl-Sergeev duality for the complex general linear superalgebra and queer lie superalgebra (See Theorem 2.2, Theorem 5.1). The fact that Λ has infinite degree ensures that there no more invariants than those given by the Schur-Weyl-Sergeev duality for the complex lie superalgebras of interest.

In [1], [2], Berele successfully extended results of [13] to the supersetting by showing that the invariants for the $\mathrm{GL}(U)$-action on the super ring of polynomials in the entries of d-tuples of degree-0 matrices over Λ are generated by supertrace monomials in the d matrices. As in the classical case, the main ingredient for obtaining the invariants comes from the super analogue of the Schur-Weyl duality. In this paper, we extend Berele's results to the action of the general linear supergroup and the queer supergroup on the mixed tensor superspace $\oplus_{i=1}^{s} U_{b_i}^{t_i}$ where t_i, b_i are in $\mathbb{N} \cup \{0\}$ for all $i = 1, \dots s$. The case of invariants of d copies of matrices, as in [2] may be seen as a special case of the above by taking $t_i = 1 = b_i$ for all $i = 1 \dots, s$. We show in Theorem 4.2 that the ring of invariants of the supersymmetric algebra $S(\oplus_{i=1}^{s} U_{b_i}^{t_i})^*$, is generated by certain special invariants which are analogues of the "picture invariants" in [6]. We then show in Theorem 4.11 that in the special case of $t_i = 1 = b_i$ for all $i = 1 \dots, s$, this agrees with the results of [2]. This result is covered under the case of the s-looped quiver in [4] where the authors describe the polynomial (semi)-invariants of super-representations of quivers over a field of characteristic zero.

In the supersetting, we use the notion of Λ-valued polynomials over U, as introduced in [10]. This is done via the supersymmetric algebra of the dual space U^*. We consider the Λ-module of maps from the degree-0 component of U to Λ, denoted as $\mathcal{F}(U_0, \Lambda)$. Then using an analogue of the restitution map from the r-fold tensor space of U^* to $\mathcal{F}(U_0, \Lambda)$, as defined in [10], we call the image under this map to be the space of homogeneous polynomials of degree r. The polynomial ring on U is then taken to be the direct sum of these spaces. It is also shown in *loc. cit* that this polynomial ring is isomorphic to the symmetric algebra of U^*, under the restitution map. We view this description of polynomials over U as a suitable alternative to Berele's notion of a polynomial over matrices with scalars in Λ, as defined in [2], since we are interested in mixed tensor superspaces which do not have any natural identification with matrices. In this paper, we use the above notion of Λ-valued polynomials on $\oplus_{i=1}^{s} U_{b_i}^{t_i}$ and obtain a generating set for the polynomial ring of invariants for a mixed tensor superspace. We obtain 'graded picture invariants' as invariant polynomial functions on the mixed tensor superspace. We then show that in the special case when the mixed tensor superspace corresponds to several copies of the endomorphism space of U, the graded picture invariants are just supertrace monomials as given in [2]. This may be regarded as the superanalogue of Procesi's result in [13].

The queer Lie superalgebras form a family of "strange" Lie superalgebras since they do not possess non-degenerate invariant even bilinear forms. The representation theory and invariant theory of the queer superalgebras $\mathfrak{q}(n)$ have been intensively studied in the last two decades. The analogous statement of Schur-Weyl duality for the queer superalgebra was given by Sergeev [16] during the study of their polynomial representations. The duality is often called the Schur–Weyl–Sergeev duality which gives an isomorphism between a super-extension of the symmetric group called the Sergeev algebra, denoted as Ser_k, and $End_{\mathfrak{q}(n)}(V^{\otimes k})$ for $n \geq k$. This duality also lifts to the queer supergroup $Q(U)$ and the Sergeev superalgebra (see [2]). An extension of this duality to mixed tensor spaces can be found in [7]. Using the Schur-Weyl-Sergeev duality, we show in Theorem 6.1 that the polynomial ring of invariants of the mixed tensor superspace $\oplus_{i=1}^{s} U_{b_i}^{t_i}$ under the action of the queer supergroup is generated by a larger collection of "graded picture invariants". As a corollary, we show that the polynomial invariants in the case of $t_i = 1 = b_i$ for all $i = 1, \ldots, s$ are generated by queer trace monomials, which is also proved in [2].

We now give an outline of the paper. In section 2 we review preliminaries of super vector spaces and Lie superalgebras and we also recall the super analogue of the classical Schur-Weyl duality. In section 3 we include a proof of the first fundamental theorem of invariant theory for tensors. In section 4 we introduce the notion of graded picture invariants and prove that these graded picture invariants span the invariants of the supersymmetric algebra of the dual of the mixed tensor superspace. Using this result we give a spanning set for the polynomial invariants of the mixed tensor superspace and thereby show that the supertrace monomials span the polynomial invariants for the simultaneous adjoint action of the general linear supergroup on several copies of its Lie superalgebra. In section 5 we recall the definition of Queer supergroup and state the Schur-Weyl-Sergeev duality for it. In section 6 we extend the results of section 4 to the Queer supergroup.

2. Preliminaries

2.1. **Grassmann algebra.** Given a vector space M we denote by $T(M)$ the tensor algebra on M. Let I be the 2-sided ideal of $T(M)$ generated by the elements $v \otimes w + w \otimes v$, $v, w \in M$. Then the exterior algebra over M is the $\mathbb{Z}_+$-graded algebra defined by $\Lambda(M) = T(M)/I$. As a $\mathbb{C}$-vector space, $\Lambda(M)$ has dimension 2^n, where $n = dim(M)$. Note that $\Lambda(M)$ is $\mathbb{Z}_2$-graded in which Λ_0 (resp Λ_1) is the direct sum of the homogeneous subspaces of even (resp. odd) degrees. This superalgebra is called the Grassmann algebra of degree n. Choosing a basis $v_1, v_2, \cdots, v_n$ of M, the set $\{v_1^{\alpha_1} v_2^{\alpha_2}, \cdots, v_n^{\alpha_n} : \alpha_i = 0, 1\}$ forms a basis of $\Lambda(M)$. Since $\Lambda(M)$ depends only on $n = dim(M)$ up to isomorphism, we write $\Lambda(n)$ for $\Lambda(M)$.

For any flag $0 \subset M_1 \subset M_2 \subset \cdots \subset M_{n-1} \subset M_n = M$, where $dim(M_i) = i$, we have $\mathbb{C} \subset \Lambda(1) \subset \Lambda(2) \subset \cdots \subset \Lambda(n-1) \subset \Lambda(n)$. Then the infinite dimensional Grassmann algebra is defined by the direct limit of this direct system, $\Lambda = \varinjlim \Lambda(n)$ (see §2.2 of [10]). Apart from the $\mathbb{Z}_2$-grading, Λ has a $\mathbb{Z}_+$-grading $\Lambda = \oplus_{i=0}^{\infty} \Lambda_i$, where $\Lambda_0 = \mathbb{C}$. The Grassmann algebra Λ is super commutative means that if $x, y \in \Lambda$ are homogeneous, then $xy = (-1)^{|x||y|} yx$, where $|x| = i$ if $x \in \Lambda_i$. This implies that any right or left module over Λ is automatically a bimodule. For example, a right Λ-module V_Λ can be made a left Λ-module via the composition of the right action map with the braiding $\Lambda \otimes_{\mathbb{C}} V_\Lambda \to V_\Lambda \otimes_{\mathbb{C}} \Lambda$ given by $\lambda \otimes v \mapsto (-1)^{|v||\lambda|} v \otimes \lambda$.

Throughout this paper all the Λ-modules that we consider are free over the infinite dimensional Grassmann algebra Λ. The projection map $\Lambda \to \mathbb{C}$ will be used to specialize the scalars to $\mathbb{C}$.

2.2. **Super vector spaces.** A $\mathbb{Z}_2$-graded vector space V over $\mathbb{C}$ is a direct sum $V_0 \oplus V_1$ of two subspaces, the even subspace V_0 and the odd subspace V_1. If $\dim(V_0) = k$ and $\dim(V_1) = l$, we write $\mathrm{sdim}(V) = (k|l)$, and call it the **superdimension** of V. Define the parity $|v|$ of any homogeneous element $v \in V_i$ by $|v| = i$.

For any $\mathbb{Z}_2$-graded vector space V over $\mathbb{C}$, we let $U := V \otimes \Lambda$ be the corresponding free Λ-module. We have the natural embedding $V \hookrightarrow U, v \mapsto v \otimes 1$. If W is another $\mathbb{Z}_2$-graded $\mathbb{C}$-vector space and $U' := W \otimes \Lambda$, then $\mathrm{Hom}_\Lambda(U, U') \cong \mathrm{Hom}_\mathbb{C}(V, W) \otimes \Lambda$.

Let $M_{k,l}(\Lambda)$ be the set of all $(k+l) \times (k+l)$ matrices over Λ. We have $M_{k,l}(\Lambda) \cong M_{k,l}(\mathbb{C}) \otimes \Lambda$ as $\mathbb{Z}_2$-graded vector spaces. We write a matrix T in block form as

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{2.1}$$

If T is even (resp. odd), then the entries of A and D belong to Λ_0 (resp. Λ_1), and the entries of B and C belong to Λ_1 (resp. Λ_0) . If $sdim(V) = (k|l)$ and $U := V \otimes \Lambda$, we have a $\mathbb{Z}_2$-graded vector space isomorphism $\mathrm{End}_\Lambda(U) \cong M_{k,l}(\Lambda)$.

We define the supertrace function $str : M_{k,l}(\Lambda) \to \Lambda$ given by $str(\begin{pmatrix} A & B \\ C & D \end{pmatrix}) = tr(A) - tr(D)$, where for a matrix X, $tr(X)$ denotes the sum of the diagonal entries of X. We note that the supertrace function is Λ_0-linear and satisfies $str(ST) = str(TS)$ for $S, T \in M_{k,l}(\Lambda)$.

The general linear lie superalgebra $\mathfrak{gl}(V)$ is the vector space $\mathrm{End}_\mathbb{C}(V)$ endowed with the lie superbracket defined for $X, Y \in \mathfrak{gl}(V)$ by

$$[X, Y] = XY - (-1)^{|X||Y|}YX.$$

When $U := V \otimes_\mathbb{C} \Lambda$, we define the general linear superalgebra $\mathfrak{gl}(U)$ as the degree 0 part of $\mathrm{End}_\mathbb{C}(V) \otimes \Lambda$.

The general linear supergroup $\mathrm{GL}(U)$ is $\{g \in \mathrm{End}_\Lambda(U)_0 : g \textit{ is invertible}\}$. (See [10, Section 2.4]).

2.3. Let V be a super vector space over $\mathbb{C}$ and let $V^{\otimes k} = V \otimes V \otimes \cdots \otimes V$ be the k-fold tensor product of V. Then $V^{\otimes k}$ is a $\mathfrak{gl}(V)$-module by defining

$$\begin{aligned} x.(v_1 \otimes v_2 \otimes \cdots \otimes v_k) = x.v_1 \otimes v_2 \otimes \cdots \otimes v_k + (-1)^{|x||v_1|} v_1 \otimes x.v_2 \otimes \cdots \otimes v_k + \\ \cdots + (-1)^{|x|(|v_1|+|v_2|+\cdots+|v_{k-1}|)} v_1 \otimes v_2 \otimes \cdots \otimes x.v_k, \end{aligned} \tag{2.2}$$

where $x \in \mathfrak{gl}(V)$ and $v_j \in V$ are $\mathbb{Z}_2$-homogeneous elements of degrees $|x|$ and $|v_j|$ respectively.

When $U := V \otimes_{\mathbb{C}} \Lambda$, the general linear superalgebra $\mathfrak{gl}(U)$ acts on $U^{\otimes k}$ by the ordinary derivation, i.e.,

$$x.(u_1 \otimes u_2 \otimes \cdots \otimes u_k) = \sum_{i=1}^{k} u_1 \otimes u_2 \otimes \cdots \otimes x.u_i \otimes \cdots \otimes u_k.$$

On the other hand the symmetric group S_k acts on $V^{\otimes k}$ by

$$(i,i+1).v_1 \otimes v_2 \otimes \cdots \otimes v_i \otimes v_{i+1} \otimes \cdots \otimes v_k = (-1)^{|v_i||v_{i+1}|} v_1 \otimes v_2 \otimes \cdots \otimes v_{i+1} \otimes v_i \otimes \cdots \otimes v_k, \quad (2.3)$$

where (i,j) denotes a transposition in S_k and v_i, v_{i+1} are $\mathbb{Z}_2$-homogeneous. More generally the action of S_k on $V^{\otimes k}$ is defined as follows: For $\sigma \in S_k$ and $\underline{v} = v_1 \otimes v_2 \otimes \cdots \otimes v_k$,

$$\sigma.\underline{v} = \gamma(\underline{v}, \sigma^{-1})(v_{\sigma^{-1}(1)} \otimes v_{\sigma^{-1}(2)} \otimes \cdots \otimes v_{\sigma^{-1}(k)}),$$

where $\gamma(\underline{v}, \sigma) = \prod_{(i,j) \in Inv(\sigma)} (-1)^{|v_i||v_j|}$, with $Inv(\sigma) = \{(i,j) : i < j \text{ and } \sigma(i) > \sigma(j)\}$.

It is easy to see that $\gamma(\underline{v}, \sigma\tau) = \gamma(\sigma^{-1}\underline{v}, \tau)\gamma(\underline{v}, \sigma)$ for two permutations σ and τ.

We denote the representations of S_k and $\mathfrak{gl}(V)$ on $V^{\otimes k}$ by η_k and ρ_k respectively.

2.4. **Isomorphisms.** In this subsection we list out all the isomorphisms we require in this paper. Let V be a super vector space over $\mathbb{C}$.

1. $V \otimes V^* \cong \mathrm{End}(V)$ defined by $(v, \phi) \mapsto T_{(v,\phi)}$ where $T_{(v,\phi)}(w) = v\phi(w)$.

This implies that $\mathrm{End}(V)^{\otimes k} \cong (V \otimes V^*)^{\otimes k}$ for $k \in \mathbb{Z}_{\geq 0}$.

2. For two super vector spaces V and W over $\mathbb{C}$, we have an isomorphism $V \otimes W \to W \otimes V$ defined by $v \otimes w \mapsto (-1)^{|v||w|} w \otimes v$, where v and w are homogeneous elements of V and W respectively.

3. We have an evaluation map $ev : (V^*)^{\otimes k} \times V^{\otimes k} \to \mathbb{C}$ defined by

$$(f_1^* \otimes f_2^* \otimes \cdots \otimes f_k^*, v_1 \otimes v_2 \otimes \cdots \otimes v_k) \mapsto (-1)^{p(\underline{f}^*, \underline{v})} f_1^*(v_1) \cdots f_k^*(v_k),$$

where $\underline{f}^* = f_1^* \otimes f_2^* \otimes \cdots \otimes f_k^*$, $\underline{v} = v_1 \otimes v_2 \otimes \cdots \otimes v_k$ for homogeneous elements f_i^* and v_i, and $p(\underline{f}^*, \underline{v})$ is defined as follows. For homogeneous elements f_i^* and v_i, let $d_j = |v_j|(|f_{j+1}^*| + \cdots + |f_k^*|)$ with $d_k = 0$. Then $p(\underline{f}^*, \underline{v}) = \sum_{j=1}^{k} d_j$ and extend this definition by linearity.

4. The above evaluation map is a non-degenerate bilinear form and hence we get a $\mathrm{GL}(V)$ isomorphism between $(V^*)^{\otimes k}$ and $(V^{\otimes k})^*$.

5. Let τ be the permutation which takes $(1, 2, \cdots, k, k+1, \cdots, 2k)$ to $(\tau(1), \tau(2), \cdots, \tau(k), \tau(k+1), \cdots, \tau(2k)) = (1, 3, 5, \cdots, 2k-1, 2, 4, 6, \cdots, 2k)$. The symmetric group S_{2k} acts on $(V \oplus V^*)^{\otimes k}$. Under this action, the permutation τ takes $V^{\otimes k} \otimes (V^*)^{\otimes k}$ to $(V \otimes V^*)^{\otimes k} = (V \otimes V^*) \otimes (V \otimes V^*) \otimes \cdots \otimes (V \otimes V^*)$. Thus we get a $\mathrm{GL}(V)$-isomorphism between $V^{\otimes k} \otimes (V^*)^{\otimes k}$ and $(V \otimes V^*)^{\otimes k}$. We then have

$$V^{\otimes k} \otimes (V^*)^{\otimes k} \cong (V \otimes V^*)^{\otimes k} \cong \mathrm{End}(V)^{\otimes k}.$$

Let $\underline{v} = v_1 \otimes v_2 \otimes \cdots \otimes v_k$ and $\underline{f}^* = f_1^* \otimes f_2^* \otimes \cdots \otimes f_k^*$. The map in the reverse direction is defined by $\underline{v} \otimes \underline{f}^* \mapsto (-1)^{p(\underline{v},\underline{f}^*)}(v_1 \otimes f_1^*) \otimes (v_2 \otimes f_2^*) \otimes \cdots \otimes (v_k \otimes f_k^*)$. Note that the sign $(-1)^{p(\underline{v},\underline{f}^*)}$ is exactly $\gamma(\underline{v} \otimes \underline{f}^*, \tau^{-1})$.

6. Let $W = W_1 \oplus \cdots \oplus W_k$, where W_i are super vector spaces over $\mathbb{C}$. Then the group S_{2k} acts on $(W \oplus W^*)^{\otimes 2k}$. The permutation τ defined above then induces an isomorphism between $W_1^* \otimes \cdots \otimes W_k^* \otimes W_1 \otimes \cdots \otimes W_k$ and $W_1^* \otimes W_1 \otimes \cdots \otimes W_k^* \otimes W_k$. This isomorphism followed by the evaluation map to $\mathbb{C}$ gives a non-degenerate pairing between $W_1^* \otimes \cdots \otimes W_k^*$ and $(W_1 \otimes \cdots \otimes W_k)^*$. As the S_{2k} action commutes with the $\mathrm{GL}(W)$-action, so the isomorphism $W_1^* \otimes \cdots \otimes W_k^* \cong (W_1 \otimes \cdots \otimes W_k)^*$ is a $\mathrm{GL}(W)$ isomorphism.

It may be noted here that under the above isomorphism, the dual basis element $T_{w_1 \otimes \cdots \otimes w_n} \in (W_1 \otimes \cdots \otimes W_k)^*$ corresponds to $(-1)^{p(\underline{w},\underline{w})} T_{w_1} \otimes \cdots \otimes T_{w_k} \in W_1^* \otimes \cdots \otimes W_k^*$ where $\underline{w} = w_1 \otimes \ldots \otimes w_k$ is a basis vector of $W_1 \otimes \cdots \otimes W_k$.

7. Let $\nu \in S_{2k}$ be the permutation $(1\,2)(3\,4)\cdots(2k-1\,2k)$ and $\mathrm{ev}^{\otimes k} : (V^* \otimes V)^{\otimes k} \to \mathbb{C}$ denote the evaluation map $(f_1^* \otimes v_1) \otimes (f_2^* \otimes v_2) \otimes \cdots \otimes (f_k^* \otimes v_k) \mapsto f_1^*(v_1) \cdots f_k^*(v_k)$. We have a non-degenerate bilinear form $\mathrm{End}(V^{\otimes k}) \otimes V^{\otimes k} \otimes (V^*)^{\otimes k} \to \mathbb{C}$ given by $\langle A, \underline{v} \otimes \underline{f}^* \rangle := \mathrm{ev}^{\otimes k}[(\nu.\tau).(A(\underline{v}) \otimes \underline{f}^*)]$. This gives a $\mathrm{GL}(V)$ isomorphism

$$\Theta : \mathrm{End}(V^{\otimes k}) \cong (V^{\otimes k} \otimes (V^*)^{\otimes k})^*$$

Remark 2.1. *All the isomorphisms given above hold* mutatis mutandis *even with V replaced by $U := V \otimes_{\mathbb{C}} \Lambda$.*

2.5. **Supertrace.** For $U = V \otimes_{\mathbb{C}} \Lambda$, we have $U^* = \mathrm{Hom}_\Lambda(U, \Lambda)$ and $U \otimes U^* \cong \mathrm{End}_\Lambda(U)$ as graded Λ-algebras where the multiplication on $U \otimes U^*$ is defined by $(v \otimes \phi)(w \otimes \psi) = v \otimes \phi(w)\psi$.

We define the supertrace map $str : \mathrm{End}_\Lambda(U) \to \Lambda$ by

$$str(v \otimes \phi) = (-1)^{|v||\phi|}\phi(v)$$

and with the above identification this definition agrees with the definition given on $\mathrm{End}_\Lambda(U)$. It satisfies $str(AB) = (-1)^{|A||B|} str(BA)$, for $A, B \in \mathrm{End}_\Lambda(U)$. When restricted to $\mathrm{End}_\Lambda(U)_0$, the str map behaves like the ordinary trace map in the sense that $str(AB) = str(BA)$ for $A, B \in \mathrm{End}_\Lambda(U)_0$ and $str(CDC^{-1}) = str(D)$ for $D \in \mathrm{End}_\Lambda(U)$ and $C \in GL(U)$.

2.6. **Schur-Weyl duality.** For a super vector space V over $\mathbb{C}$, the actions of $\mathfrak{gl}(V)$ on $V^{\otimes k}$ and the right action of S_k on $V^{\otimes k}$ defined in §2.3 commute with each other and the super analogue of Schur-Weyl duality says that the span of the images of S_k and $\mathfrak{gl}(V)$ in $\mathrm{End}(V^{\otimes k})$ are centralizers of each other (see [3]). In particular we have a surjective algebra homomorphism:

$$\eta_k : \mathbb{C}[S_k] \to \mathrm{End}_{\mathfrak{gl}(V)}(V^{\otimes k}),$$

where $\mathrm{End}_{\mathfrak{gl}(V)}(V^{\otimes k})$ is the space of $\mathfrak{gl}(V)$-module homomorphisms of $V^{\otimes k}$. This is equivalent to taking the right action of S_k on $V^{\otimes k}$ given by $\underline{v}.\sigma := \sigma^{-1}.\underline{v}$ and the homomorphism corresponding to it to be

$$\Psi_k : \mathbb{C}[S_k] \to \mathrm{End}_{\mathfrak{gl}(V)}(V^{\otimes k})^{op}.$$

2.7. **Schur-Weyl duality over Λ.** Let V be a super vector space over $\mathbb{C}$ and let $U = V \otimes_{\mathbb{C}} \Lambda$. Let $U^{\otimes r} = U \otimes \cdots \otimes U$ (r-times) be the r-fold tensor product of U. The general linear supergroup $\mathrm{GL}(U)$ acts on $U^{\otimes r}$ by

$$g.(w_1 \otimes w_2 \otimes \cdots \otimes w_r) = g.w_1 \otimes g.w_2 \otimes \cdots \otimes g.w_r$$

for $g \in \mathrm{GL}(U)$. The dual superspace U^* of U has a natural $GL(U)$-module structure given by $g.f(v) = f(g^{-1}.v)$, for all $v \in U$.

On the other hand, the symmetric group S_r acts on $U^{\otimes r}$ by

$$(i, i+1).v_1 \otimes v_2 \otimes \cdots \otimes v_i \otimes v_{i+1} \otimes \cdots \otimes v_r = (-1)^{|v_i||v_{i+1}|} v_1 \otimes v_2 \otimes \cdots \otimes v_{i+1} \otimes v_i \otimes \cdots \otimes v_r, \quad (2.4)$$

where (i, j) denotes a transposition in S_r and v_i, v_{i+1} are $\mathbb{Z}_2$-homogeneous. We denote the representations of S_r and $\mathrm{GL}(U)$ on $U^{\otimes r}$ by η_r and ρ_r respectively.

The group ring $\Lambda S_r = \mathbb{C}S_r \otimes \Lambda$ is an associative superalgebra, with $\mathbb{C}S_r$ regarded as purely even. By extending the action of S_r on $U^{\otimes r}$ Λ-linearly we get an action of the super algebra ΛS_r on $U^{\otimes r}$. It is easy to see that the actions of ΛS_r and $GL(U)$ on $U^{\otimes r}$ commute with each other.

We then have the following super analogue of Schur-Weyl duality (for a proof see [10]).

Theorem 2.2. *Let $End_{(GL(U)}(U^{\otimes r}) = \{\phi \in End_\Lambda(U^{\otimes r}) : g\phi = \phi g, \forall g \in GL(U)\}$. Then $End_{GL(U)}(U)^{\otimes r} = \eta(\Lambda S_r)$.* ☐

3. Tensor FFT for $\mathfrak{gl}(V)$

3.1. **Tensor FFT over $\mathbb{C}$.** Let $V = V_0 \oplus V_1$ be a super vector space over $\mathbb{C}$. The tensor version of the first fundamental theorem of invariant theory for $\mathfrak{gl}(V)$ describes a spanning set for $(V^{\otimes k} \otimes (V^*)^{\otimes k})^{\mathfrak{gl}(V)}$ which can be derived from the super analogue of the Schur-Weyl duality described above. Here we provide a proof for the convenience of the reader.

Let $e_1, \cdots, e_{m+n}$ be a basis for V with the first m vectors are of degree 0 and the rest are of degree 1. Let $e_1^*, \cdots, e_{m+n}^*$ be the dual basis for V^*.

We set $S = S_0 \cup S_1$, where $S_0 = \{1, 2, \cdots, m\}$ and $S_1 = \{m+1, m+2, \cdots, m+n\}$. We define parity for I as $|i| = 0$ if $i \in S_0$ and $|i| = 1$ if $i \in S_1$. We also set $|E_{ij}| = |i| + |j|$.

For $\sigma \in S_k$, we define

$$\theta_\sigma = \sum_I (-1)^{p(I,I)} \gamma(I, \sigma^{-1})(e_{i_{\sigma^{-1}(1)}} \otimes \cdots \otimes e_{i_{\sigma^{-1}(k)}}) \otimes (e_{i_1}^* \otimes \cdots \otimes e_{i_k}^*),$$

where the sum is over all the k-element ordered subsets $I = \{i_1, i_2, \cdots, i_k\}$ of S and $\gamma(I, \sigma) = \gamma(e_I, \sigma)$ and $p(I, I) = p(e_I, e_I)$ with $e_I = e_{i_1} \otimes \cdots \otimes e_{i_k}$.

In what follows we use the notation $p(I,J) = p(e_I, e_J)$, where $e_I = e_{i_1} \otimes \cdots \otimes e_{i_k}$ and $e_J = e_{j_1} \otimes \cdots \otimes e_{j_k}$ for ordered subsets $I = \{i_1, i_2, \cdots, i_k\}$ and $J = \{j_1, j_2, \cdots, j_k\}$.

Note that the elements $(\theta_\sigma)_{\sigma \in S_k}$ are in $(V^{\otimes k} \otimes (V^*)^{\otimes k})^{\mathfrak{gl}(V)}$.

Theorem 3.1. *(Tensor FFT for $\mathfrak{gl}(V)$). The space $(V^{\otimes k} \otimes (V^*)^{\otimes k})^{\mathfrak{gl}(V)}$ is spanned by the set $\{\theta_\sigma : \sigma \in S_k\}$.*

Proof. Since $V \otimes V^* \cong \mathrm{End}(V) \cong \mathfrak{gl}(V)$, we have an isomorphism

$$\Phi : (V^{\otimes k} \otimes (V^*)^{\otimes k}) \to \mathrm{End}(V^{\otimes k}).$$

Since the tensor product action of $\mathfrak{gl}(V)$ on $V \otimes V^*$ corresponds to the adjoint action on $\mathfrak{gl}(V)$, we have an isomorphism of $\mathfrak{gl}(V)$-invariants:

$$\Phi : (V^{\otimes k} \otimes (V^*)^{\otimes k})^{\mathfrak{gl}(V)} \to (\mathrm{End}(V^{\otimes k}))^{\mathfrak{gl}(V)} \cong \mathrm{End}_{\mathfrak{gl}(V)}(V^{\otimes k}),$$

where $\mathrm{End}_{\mathfrak{gl}(V)}(V^{\otimes k})$ denotes the space of $\mathfrak{gl}(V)$-module endomorphisms of $V^{\otimes k}$.

By Schur-Weyl duality, we have $\eta_k(\mathbb{C}[S_k]) = \mathrm{End}_{\mathfrak{gl}(V)}(V^{\otimes k})$. So we have an isomorphism (we still denote it by Φ):

$$\Phi : (V^{\otimes k} \otimes (V^*)^{\otimes k})^{\mathfrak{gl}(V)} \to \eta_k(\mathbb{C}[S_k]).$$

We claim that the map Φ maps θ_σ to $\eta_k(\sigma)$. Let $J = \{j_1, j_2, \cdots, j_k\}$ be an ordered k-subset of S. Then

$$\Phi(\theta_\sigma)(e_{j_1} \otimes \cdots \otimes e_{j_k})$$

$$= \sum_I (-1)^{p(I,I)} \gamma(I, \sigma^{-1})(e_{i_{\sigma^{-1}(1)}} \otimes \cdots \otimes e_{i_{\sigma^{-1}(k)}}) \otimes (e^*_{i_1} \otimes \cdots \otimes e^*_{i_k})(e_{j_1} \otimes \cdots \otimes e_{j_k})$$

$$= \sum_I (-1)^{(p(I,J)+p(I,I))} \gamma(I, \sigma^{-1})(e^*_{i_1}(e_{j_1}) \cdots e^*_{i_k}(e_{j_k}))(e_{i_{\sigma^{-1}(1)}} \otimes \cdots \otimes e_{i_{\sigma^{-1}(k)}}).$$

$$= \gamma(J, \sigma^{-1})(e_{j_{\sigma^{-1}(1)}} \otimes \cdots \otimes e_{j_{\sigma^{-1}(k)}}) = \eta_k(\sigma)(e_{j_1} \otimes \cdots \otimes e_{j_k}).$$

So the set $\{\theta_\sigma : \sigma \in S_k\}$ spans $(V^{\otimes k} \otimes V^{*\otimes k})^{\mathfrak{gl}(V)}$.

∎

3.2. **Tensor FFT over Λ.** Let V be a super vector space over $\mathbb{C}$ and let $U = V \otimes_{\mathbb{C}} \Lambda$. Then $U^{\otimes r} \otimes_\Lambda U^{*\otimes s}$ is a $GL(U)$ module. For $\alpha \in \mathbb{C}^*$, the element αI belongs to the centre of $GL(U)$ and it acts on $U^{\otimes r} \otimes U^{*\otimes s}$ by $\alpha I.x = \alpha^{r-s}x$, where I is the identity matrix of size $(m+n) \times (m+n)$. Hence there are no invariants if $r \neq s$, so we only need to consider the $GL(U)$-invariants of $U^{\otimes r} \otimes_\Lambda U^{*\otimes r}$. Then we have a $GL(U)$-module isomorphism

$$U^{\otimes r} \otimes_\Lambda U^{*\otimes r} \cong \mathrm{End}_\Lambda(U^{\otimes r}).$$

From the Schur-Weyl duality it follows that

$$(U^{\otimes r} \otimes_\Lambda U^{*\otimes r})^{GL(U)} \cong \mathrm{End}_{GL(U)}(U^{\otimes r}) \cong \eta(\Lambda S_r).$$

We denote the image of $\sigma \in S_r$ in $(U^{\otimes r} \otimes (U^*)^{\otimes r})^{GL(U)}$ also by θ_σ as in the above theorem.

4. Graded invariants for mixed tensor superspaces

For a complex super vector space V of super dimension $(m|n)$ and non-negative integers t, b, we denote by V_b^t the tensor space $V^{\otimes b} \otimes (V^*)^{\otimes t}$. Recall that the tensor space has a natural $\mathbb{Z}_2$-grading where the degree of a homogeneous element $v_1 \otimes \cdots \otimes v_b \otimes \phi_1 \otimes \cdots \otimes \phi_t$ is the sum of the degrees of the individual elements taken modulo 2. By a *mixed tensor superspace* we mean the direct sum of finitely many tensor spaces, *i.e.,* of the form $\oplus_{i=1}^s (V)_{b_i}^{t_i}$, for an $s \in \mathbb{N}$ and non-negative integers t_i, b_i, $i = 1, \ldots s$. The mixed tensor superspace inherits its $\mathbb{Z}_2$-grading from its graded direct summands. Let U denote the free Λ-module $V \otimes_{\mathbb{C}} \Lambda$, introduced earlier. Recall that there is a natural identification $V \hookrightarrow U$, such that $v \mapsto v \otimes 1$. As was also noted there, the supergroup $\mathrm{GL}(U)$ acts on U, hence it acts naturally on the tensor space U_b^t and acts diagonally on the mixed tensor superspace $\oplus_{i=1}^s (U)_{b_i}^{t_i} \cong (\oplus_{i=1}^s (V)_{b_i}^{t_i}) \otimes_{\mathbb{C}} \Lambda$.

4.1. **Supersymmetric algebra of the dual of the mixed tensor superspace.** For the $\mathbb{Z}_2$-graded vector space V of superdimension $(m|n)$ over $\mathbb{C}$, let $\{e_1, \ldots, e_{n+m}\}$ be a basis with the first m vectors from the even part and the next n from the odd part. For notational convenience, we rename the dual basis $e_1^*, \ldots, e_{m+n}^*$ by $x_1, \ldots, x_m, y_1 \ldots, y_n$ respectively. Under the identification $V \hookrightarrow U$ and $V^* \hookrightarrow V^* \otimes_{\mathbb{C}} \Lambda \cong U^*$, we denote the Λ-basis of U and U^* corresponding to $\{e_1, \ldots, e_{n+m}\}$ and $\{x_1, \ldots, x_m, y_1 \ldots, y_n\}$ respectively, also by the same notation.

The tensor algebra on U^* is $T(U^*) = \oplus_{n \in \mathbb{Z}_{\geq 0}} T^n(U^*)$ where $T^n(U^*) := (U^*)^{\otimes_\Lambda n}$ for $n > 0$ and $T^0(U^*) := \Lambda$. This is naturally a $\mathbb{Z}$-graded superalgebra. Let $I(U^*)$ be the ideal generated by elements of the form $u \otimes v - (-1)^{|u||v|} v \otimes u$ where $u, v \in U^*$. Note that $I(U^*)$ is both a $\mathbb{Z}$-graded and a $\mathbb{Z}_2$-graded ideal in $T(U^*)$. Hence the supersymmetric algebra $S(U^*) := T(U^*)/I(U^*)$ inherits a $\mathbb{Z}$-grading as well as a $\mathbb{Z}_2$-grading from $T(U^*)$. We denote the image of $T^r(U^*)$ in $S(U^*)$ by $S^r(U^*)$ and the natural map from $T^r(U^*) \to S^r(U^*)$ is denoted as ϖ_r. In $S(U^*)$, we have the relations $x_i \otimes x_j = x_j \otimes x_i$, $x_i \otimes y_j = y_j \otimes x_i$ and $y_i \otimes y_j = -y_j \otimes y_i$. Hence we can identify $S(V^*)$ with $(\mathbb{C}[x_1, \ldots, x_m] \otimes_{\mathbb{C}} \Lambda) \otimes_\Lambda$

$(\wedge[y_1,\dots,y_n]\otimes_{\mathbb{C}}\Lambda)$ where $\mathbb{C}[x_1,\dots,x_m]$ denotes the polynomial algebra over m commuting variables and $\wedge[y_1\dots,y_n]$ denotes the exterior algebra generated by $y_1,\dots,y_n$ satisfying the anti-commutativity relations $y_iy_j=-y_jy_i$. In other words, $S(U^*)$ has a Λ-basis given by monomials of the form $x_1^{r_1}\cdots x_m^{r_m}y_1^{s_1}\cdots y_m^{s_m}$ where $r_i\in\mathbb{N}\cup\{0\}$ and $s_i\in\{0,1\}$. One may identify $S^r(U^*)$ with the $GL(U)$-stable subspace $e(r)T^r(U^*)$ where $e(r):=\frac{1}{r!}\sum_{\sigma\in S_r}\sigma$. Since the $GL(U)$-action on $T^r(U^*)$ commutes with the S_r-action, so the action of $GL(U)$ on $T^r(U^*)$ descends to $S^r(U^*)$, thereby making ϖ_r a $\mathrm{GL}(U)$-equivariant map.

In the case of V_b^t, let $T_{i_1\cdots i_b}^{j_1\cdots j_t}$ denote the linear functional on V_b^t that takes value 1 only on the basis vector $e_{i_1}\otimes\cdots\otimes e_{i_b}\otimes e_{j_1}^*\otimes\cdots\otimes e_{j_t}^*$ and 0 on others. Under the natural identification, $(U_b^t)^*\cong(V_b^t)^*\otimes\Lambda$, we denote the Λ-linear map corresponding to $T_{i_1\cdots i_b}^{j_1\cdots j_t}$ also by the same notation. With this notation, following the same reasoning as in the case of $S(U^*)$, the super ring $S((U_b^t)^*)$ can be identified with the super ring of polynomials in the variables $T_{i_1\cdots i_b}^{j_1\cdots j_t}$ with the supercommutativity relation $T_{i_1\cdots i_b}^{j_1\cdots j_t}T_{i'_1\cdots i'_b}^{j'_1\cdots j'_t}=(-1)^{(\sum_k a_{i_k}+\sum_l a_{j_l})(\sum_k a_{i'_k}+\sum_l a_{j'_l})}T_{i'_1\cdots i'_b}^{j'_1\cdots j'_t}T_{i_1\cdots i_b}^{j_1\cdots j_t}$ where a_i denotes the parity of the basis element e_i (as well as of e_i^*).

More generally, for $W=\oplus_{i=1}^sU_{b_i}^{t_i}$ the mixed tensor superspace as defined above we use the natural identification of W^* with $\oplus_{i=1}^s(U_{b_i}^{t_i})^*$ to obtain a Λ-basis for W^* given by the linear maps $T(i)_{l_1\dots l_{b_i}}^{u_1\dots,u_{t_1}}\in(U_{b_i}^{t_i})^*$, for $i=1,\dots,s$, $1\le l_j\le m+n$ and $1\le u_j\le m+n$. Under this identification the linear map $T(i)_{l_1\dots l_{b_i}}^{u_1\dots,u_{t_i}}\in W^*$ evaluates on $w=(w_1,\dots,w_s)$ to $T(i)_{l_1\dots l_{b_i}}^{u_1\dots u_{t_i}}(w_i)$. With this notation, the supersymmetric algebra $S(W^*)$ can be identified with the superalgebra of polynomials in the variables $T(l)_{i_1\cdots i_b}^{j_1\cdots j_t}$, $l=1,\dots,s$ with the supercommutativity relation $T(l)_{i_1\cdots i_{b_l}}^{j_1\cdots j_{t_l}}T(l')_{i'_1\cdots i'_{b_l}}^{j'_1\cdots j'_{t_l}}=(-1)^{(\sum_k a_{i_k}+\sum_l a_{i_l})(\sum_k a_{i'_k}+\sum_l a_{i'_l})}T(l')_{i'_1\cdots i'_{b_l}}^{j'_1\cdots j'_{t_l}}T(l)_{i_1\cdots i_{b_l}}^{j_1\cdots j_{t_l}}$ where a_i denotes the parity of the basis element e_i. From the above identifications of $S(W^*)$ and $S((U_{b_i}^{t_i})^*)$ as polynomial algebras, it follows that there is a natural isomorphism from $\otimes_{i=1}^sS((U_{b_i}^{t_i})^*)$ to $S(W^*)$ given by $(\phi_1\otimes\cdots\otimes\phi_s)\mapsto\phi_1\cdots\phi_s$.

Remark: For purposes of book-keeping, we fix an ordering of basis elements of $U_{b_i}^{t_i}$ for each $i=1\dots s$ such that the degree 0 elements appear before the degree 1 elements.

4.2. **Graded picture invariants on W.** Let $W=\oplus_{i=1}^sU_{b_i}^{t_i}$ for an $s\in\mathbb{N}$ and non-negative integers t_i,b_i, $i=1,\dots s$. The $\mathrm{GL}(U)$ action on W induces an action on $T^r(W^*)$ for each $r\in\mathbb{Z}^{\ge0}$. This action commutes with action of the symmetric group S_r for each $r>0$ hence the $\mathrm{GL}(U)$-action descends to the subspace $e(r)T^r(W^*)$ which in turn can be identified with $S^r(W^*)$. Thus, we note that the $\mathrm{GL}(U)$ action on W descends to the associated supersymmetric algebra $S(W^*)$ of "polynomials" on W. With respect to this action of $\mathrm{GL}(U)$ on $S(W^*)$ we are interested in finding a generating set for the invariant subalgebra $S(W^*)^{\mathrm{GL}(U)}$. For this, we define the notion of a *graded picture invariant* following [6]. In the supersymmetric setting, to an element of the symmetric group, we associate a polynomial in $S(W^*)$ which we shall call as a graded picture invariant. A *graded picture invariant* is defined as follows: given an s-tuple $(m_1,\dots,m_s)$ of non-negative integers such that $\sum_{k=1}^s m_k t_k = \sum_{k=1}^s m_k b_k = N$ and a $\sigma \in S_N$ we associate the polynomial $\psi_\sigma \in S(W^*)$ given by

$$\sum_{r_1,\dots,r_N \in \{1,\dots,n\}} (-1)^{p_\sigma(r_1,\dots,r_N,m_1,\dots,m_s)} \prod_{i=1}^{s}\prod_{j=1}^{m_i} T(i)^{r_{(\sum_{p<i} m_p t_p+(j-1)t_i+1)}\cdots r_{(\sum_{p<i} m_p t_p+jt_i)}}_{r_{\sigma(\sum_{p<i} m_p b_p+(j-1)b_i+1)}\cdots r_{\sigma(\sum_{p<i} m_p b_p+jb_i)}} \tag{4.1}$$

where $p_\sigma(I,M)$ for an N-tuple $I=(r_1,\dots,r_N)$ of N elements from $\{1,\dots n\}$ and $M=(m_1,\dots,m_s)\in(\mathbb{N}\cup\{0\})^s$ such that $\sum_k m_k t_k = \sum_k m_k b_k = N$ takes the value ± 1 determined by the formula $\gamma(I,\sigma^{-1})\times(-1)^{q(I,M,\sigma)}$ where $q(I,M,\sigma)$ is

$$\sum_{i=1}^{s}\sum_{j=1}^{m_i}\left[\left(\sum_{l=1}^{b_i} a_{\sigma(\sum_{p<i} m_p b_p+(j-1)b_i+l)}\right)\left(\sum_{k=1}^{\sum_{p<i} m_p t_p+(j-1)t_i} a_k\right)\right]+$$
$$\sum_{i=1}^{N} a_i + \sum_{i=1}^{N-1} a_i\left(\sum_{j=i+1}^{N} a_j\right) + p(\underline{w},\underline{w}).$$

In the above formula[1], a_i is the parity of e_{r_i} (and also of its dual vector $e^*_{r_i}$) and $\underline{w} := w_1\otimes w_2\otimes\cdots\otimes w_{\sum_{i=1}^s m_i}$ where each $w_{\sum_{p<i} m_p+j} := e_{r_{\sum_{p<i} m_p b_p+(j-1)b_i+1}}\otimes\cdots\otimes e_{r_{\sum_{p<i} m_p b_p+jb_i}}\otimes e^*_{r_{\sigma^{-1}(\sum_{p<i} m_p t_p+(j-1)t_i+1)}}\otimes\cdots\otimes e^*_{r_{\sigma^{-1}(\sum_{p<i} m_p t_p+jt_i)}}$ for $i=1,\dots,s;\ j=1,\dots,m_i$. It can be seen that the degree of $w_{\sum_{p<i} m_p+j}$ is

$$\sum_{k=1}^{b_i} a_{(\sum_{p<i} m_p b_p+(j-1)b_i+k)} + \sum_{k=1}^{t_i} a_{\sigma^{-1}(\sum_{p<i} m_p t_p+(j-1)t_i+k)}.$$

Following the convention that for $\sigma\in S_N$, $\sigma.(r_1,\dots,r_N) := (r_{\sigma^{-1}(1)},\dots,r_{\sigma^{-1}(N)})$, it may be seen that $\gamma(\sigma.I,\tau^{-1})\gamma(I,\sigma^{-1}) = \gamma(I,(\tau\sigma)^{-1})$ (see §2.3). Using this, we note that the above sign $(-1)^{q(I,M,\sigma)}\gamma(I,\sigma^{-1})$ may be realised as $\gamma(\mu^{-1}.(I,\sigma I),(\nu\tau\hat{\sigma}\mu)^{-1})$ where for a $\sigma\in S_N$ we define $\hat{\sigma}\in S_{2N}$ as

$$\hat{\sigma}(i)=\sigma(i) \quad \text{for } i\le N$$
$$\hat{\sigma}(i)=i \quad \text{for } i>N$$

and for an $M=(m_1,\dots,m_s)\in(\mathbb{N}\cup\{0\})^s$ such that $\sum_{k=1}^s m_k t_k = \sum_{k=1}^s m_k b_k = N$ the permutation μ takes $(1,\dots,2N)$ to $(1,\dots b_1, N+1,\dots,N+t_1, b_1+1,\dots 2b_1,\cdots)$.

Remark 4.1. *In the above formula the monomials with repeated odd degree variables are 0. More explicitly, those indices $r_1,\dots,r_N$ are to be dropped from (4.1) for which the variable*

$$T(i)^{r_{(\sum_{p<i} m_p t_p+(j-1)t_i+1)}\cdots r_{(\sum_{p<i} m_p t_p+jt_i)}}_{r_{\sigma(\sum_{p<i} m_p b_p+(j-1)b_i+1)}\cdots r_{\sigma(\sum_{p<i} m_p b_p+jb_i)}} \in S(W^*)$$

is of degree 1 and,

$$(r_{\sigma(\sum_{p<i} m_p b_p+(j-1)b_i+1)},\dots,r_{\sigma(\sum_{p<i} m_p b_p+jb_i)}) = (r_{\sigma(\sum_{p<i} m_p b_p+(j'-1)b_i+1)},\dots,r_{\sigma(\sum_{p<i} m_p b_p+j'b_i)}),$$

[1] In (4.1), the monomials should be considered modulo the anti-commutativity relations in $S(W^*)$. See Remark 4.1 above.

$$(r_{\sum_{p<i} m_p t_p+(j-1)t_i+1},\dots,r_{\sum_{p<i} m_p t_p+jt_i}) = (r_{\sum_{p<i} m_p t_p+(j'-1)t_i+1},\dots,r_{\sum_{p<i} m_p t_p+j't_i})$$

for some $1 \le i \le s$, $1 \le j < j' \le m_i$.

The next proposition establishes that the above graded picture invariants generate the invariant super ring $S(W^*)^{\mathrm{GL}(U)}$. This is analogous to the statement of [6, Proposition 7]. Further, it may be remarked that the proof follows the same line of argument as in [6, Proposition 7]. However, the identifications involved in the proof there will have to be modified to the braided category of $\mathbb{Z}_2$-graded vector spaces. This leads to polynomials which are different from that of *loc.cit*.

Theorem 4.2. *Given a mixed tensor superspace* $W = \oplus_{i=1}^{s} U_{b_i}^{t_i}$, *the graded picture invariants as defined above* Λ*-linearly span* $S(W^*)^{\mathrm{GL}(U)}$.

Proof. The degree n component of the ring $S(W^*)$, denoted as $S^n(W^*)$, is the image of $T^n(W^*)$ under ϖ_n. The basis for $S^n(W^*)$ is then given by the degree n monomials in the super-commuting variables $T(i)_{l_1\cdots l_{b_i}}^{u_1\cdots u_{t_i}}$ arranged in order from $i=1$ to $i=s$ and the fixed ordering of the standard basis of $U_{b_i}^{t_i}$ for each i indicated in Remark 4.1 and such that the odd variables appear atmost once in such a monomial. Hence it can be seen that

$$S^n(W^*) \cong \oplus_{\substack{m_1,\dots,m_s \\ m_1+\cdots+m_s=n}} (\otimes_{i=1}^{s} S^{m_i}((U_{b_i}^{t_i})^*)).$$

This isomorphism may be structurally described as coming from the degree n component of $S(W^*)$ being expressed as a direct sum of its $(m_1,\dots,m_s)$-multidegree summands obtained by setting the multidegree of the basis elements of the form $T(i)_{l_1\dots l_{b_i}}^{u_1,\dots,u_{t_i}}$ to be the s-tuple $(0,\dots,0,1,0,\dots)$ with 1 only in the i^{th}-coordinate. We have the natural surjection from $\varpi_{m_i} : ((U_{b_i}^{t_i})^*)^{\otimes m_i} \to S^{m_i}((U_{b_i}^{t_i})^*)$ for each i. Since the $\mathrm{GL}(U)$-action on $((U_{b_i}^{t_i})^*)^{\otimes m_i}$ commutes with the S_{m_i}-action and $S^{m_i}((U_{b_i}^{t_i})^*)$ can be identified under ϖ_{m_i} with the subspace $e(m_i)\left(((U_{b_i}^{t_i})^*)^{\otimes m_i}\right)$ where $e(m_i) := \frac{1}{m_i!}\sum_{\sigma\in S_{m_i}} \sigma$, so the $\mathrm{GL}(U)$-action descends to $S^{m_i}((U_{b_i}^{t_i})^*)$ making ϖ_{m_i} a $\mathrm{GL}(U)$-equivariant map. Thereby, we get a $\mathrm{GL}(U)$-equivariant surjection $\varpi_{m_1}\otimes\ldots\otimes\varpi_{m_s} : \otimes_{i=1}^{s}((U_{b_i}^{t_i})^*)^{\otimes m_i} \twoheadrightarrow \otimes_{i=1}^{s} S^{m_i}((U_{b_i}^{t_i})^*)$. Set $W_{i+\sum_{j<r} m_j} := U_{b_r}^{t_r}$ for $i = 1,\dots,m_r$ and $r = 1\dots s$. Then, by the isomorphism listed in (6) of §2.4 we have,

$$(\otimes_{i=1}^{k} W_i)^* \cong \otimes_{i=1}^{k}(W_i^*) \tag{4.2}$$

where $k = \sum_i m_i$, given by sending the basis element $T_{w_1\otimes\cdots\otimes w_{\sum_i m_i}} \mapsto (-1)^{p(\underline{w},\underline{w})} T(1)_{w_1} \otimes \cdots \otimes T(s)_{w_{\sum_i m_i}}$ where $\underline{w} := w_1\otimes\cdots\otimes w_{\sum_i m_i}$. Here $w_i \in W_i$ denotes a basis vector of W_i and if $W_i = U_{b_r}^{t_r}$ for some $r = 1,\dots,s$ then the dual basis vector corresponding to w_i is denoted as $T(r)_{w_i}$. Thus we have identified $(\otimes_{i=1}^{s}(U_{b_i}^{t_i})^{\otimes m_i})^*$ with $\otimes_{i=1}^{s}((U_{b_i}^{t_i})^*)^{\otimes m_i}$ by the above isomorphism. More explicitly, for index sets $L = (l_1,\dots,l_{\sum_i m_i b_i})$, $U = (u_1,\dots,u_{\sum_i m_i t_i})$ the above isomorphism takes $T_{l_1,\dots,l_{b_1},u_1\dots,u_{t_1},l_{b_1+1},\dots,l_{2b_1},\dots}$ to $(-1)^{p(\underline{w},\underline{w})} T(1)_{l_1\dots l_{b_1}}^{u_1\dots u_{t_1}} \otimes T(1)_{l_{b_1+1}\dots l_{2b_1}}^{u_{t_1+1}\dots u_{2t_1}} \otimes\cdots \in \otimes_{i=1}^{s}((U_{b_i}^{t_i})^*)^{\otimes m_i}$ where $\underline{w}$ is as defined above and each $w_{\sum_{p<i} m_p + j} := e_{l_{\sum_{p<i} m_p b_p+(j-1)b_i+1}} \otimes$

$\cdots \otimes e_{l_{\sum_{p<i} m_p b_p + j b_i}} \otimes e^*_{u_{(\sum_{p<i} m_p t_p + (j-1)t_i + 1)}} \otimes \cdots \otimes e^*_{u_{(\sum_{p<i} m_p t_p + j t_i)}}$ for $i = 1, \ldots, s$; $j = 1, \ldots, m_i$. We then apply the permutation μ as defined earlier, which induces an isomorphism from $\otimes_{i=1}^s (U_{b_i}^{t_i})^{\otimes m_i} \to U^{\otimes(\sum_i m_i b_i)} \otimes (U^*)^{\otimes(\sum_i m_i t_i)}$ and hence also on their duals.

For index sets $L = (l_1, \ldots, l_{\sum_i m_i b_i})$, $U = (u_1, \ldots, u_{\sum_i m_i t_i})$ we note that $\mu^{-1}.(L, U) = (l_1, \ldots, l_{b_1}, u_1 \ldots, u_{t_1}, l_{b_1+1}, \ldots, l_{2b_1}, \ldots)$. So the isomorphism induced by μ on the duals is given by

$$\mu^* : \left(U^{\otimes(\sum_i m_i b_i)} \otimes (U^*)^{\otimes(\sum_i m_i t_i)} \right)^* \to (\otimes_{i=1}^s (U_{b_i}^{t_i})^{\otimes m_i})^* \tag{4.3}$$

such that $T_{L,U}$ maps to $\gamma(\mu^{-1}.(L,U), \mu^{-1}) T_{l_1, \ldots, l_{b_1}, u_1 \ldots, u_{t_1}, l_{b_1+1}, \ldots, l_{2b_1}}, \cdots \in (\otimes_{i=1}^s (U_{b_i}^{t_i})^{\otimes m_i})^*$. A standard argument as outlined at the end of §3 gives us that $U^{\otimes(\sum_i m_i b_i)} \otimes (U^*)^{\otimes(\sum_i m_i t_i)}$ has invariants if and only if $\sum_i m_i b_i = \sum_i m_i t_i$. From Theorem 2.2 we get a spanning set of $\mathrm{End}(U^{\otimes N})^{\mathrm{GL}(U)}$. So, in the case when $\sum_i m_i b_i = \sum_i m_i t_i = N$, going by the isomorphism Θ defined in (7) of §2.4, we get invariants in $\left(U^{\otimes N} \otimes (U^*)^{\otimes N}\right)^*$ given by $\Theta(\sigma)$, where $\sigma \in S_N$ is regarded as an element of $\mathrm{End}_\Lambda(U^{\otimes N})$. As noted in §2.4 above, $\Theta(\sigma)(\underline{v} \otimes \underline{f}) = ev[\nu.\tau.(\sigma(\underline{v}) \otimes \underline{f^*})]$. It can be easily seen that $\Theta(\sigma)$ can be expressed in terms of the basis of $(U^{\otimes N} \otimes (U^*)^{\otimes N})^*$ as

$$\sum_{L=(l_1, \ldots, l_N)} \gamma(L, \sigma^{-1}) \gamma(\sigma L, \sigma L, \tau^{-1}) \gamma(\tau.(\sigma L, \sigma L), \nu^{-1}) T_{L, \sigma L}. \tag{4.4}$$

Now going through the isomorphisms given in (4.2), (4.3) and the map $\varpi_{m_1} \otimes \ldots \otimes \varpi_{m_s}$, it can be seen that this element corresponds to the graded picture invariants defined above. $\square$

Remark 4.3. *In the above proof the invariants $\Theta(\sigma)$ are the images of θ_σ in §3.2 under the isomorphism from $(U^{\otimes N} \otimes (U^*)^{\otimes N})$ to its dual $(U^{\otimes N} \otimes (U^*)^{\otimes N})^*$ (via their identification to $\mathrm{End}(U^{\otimes N})$ using (1), (4) of §2.4).*

Remark 4.4. *The linear map on $\otimes_{i=1}^s (U_{b_i}^{t_i})^{\otimes m_i}$ got by evaluation the isomorphism in (4.3) on $\Theta(\sigma)$ in the above proposition is obtained by applying a sequence of permutations, namely $\nu.\tau.\hat{\sigma}.\mu$, followed by the evaluation map ev. Further, in the case that $W = \oplus_{i=1}^s U_1^1$, we note that μ^{-1} is just the permutation τ.*

4.3. **Restitution map and realizing linear maps as multilinear polynomials on W_0.** Following [10, §3], we consider the Λ-module of functions from $W \to \Lambda$ denoted by $\mathcal{F}(W, \Lambda)$. Let $F^r : T^r(W^*) \to \mathcal{F}(W, \Lambda)$ be given by $F^r(\boldsymbol{\alpha})(v) = ev(\boldsymbol{\alpha} \otimes v \otimes \cdots \otimes v)$ where $ev : T^r(W^*) \otimes T^r(W) \to \Lambda$ is as in (3) of §2.4. It is noted in [10, Lemma 3.11] that F^r restricts to a map $F^r : S^r(W^*) \to \mathcal{F}(W_0)$; further, it is shown in [10, Prop 3.14] that this restriction is an isomorphism from $S^r(W^*) \to \mathcal{P}^r(W_0)$ where $\mathcal{P}^r(W_0)$ is the image of F^r in $\mathcal{F}(W_0, \Lambda)$. The space $\mathcal{P}^r(W_0)$ is called the space of *polynomials of degree r on* W_0. For $f \in S^r(W^*)$ and $w \in W_0$, we note that $F^r(f)(w) = ev(\boldsymbol{f} \otimes w^{\otimes r})$ where $\boldsymbol{f} \in T^r(W^*)$ such that $\varpi_r(\boldsymbol{f}) = f$ and $w^{\otimes r} := w \otimes w \otimes \cdots$ (r-times); (recall, $\varpi_r : T^r(W^*) \to S^r(W^*)$ is the natural quotient map).

Let $W = \oplus_{i=1}^s U_{b_i}^{t_i}$. Since $W^* \cong \oplus_{i=1}^s (U_{b_i}^{t_i})^*$, the set of all linear maps $T(i)_{l_1 \ldots l_{b_i}}^{u_1 \ldots, u_{t_1}} \in (U_{b_i}^{t_i})^*$, for each $i = 1, \ldots, s$, $1 \leq l_j \leq m+n$ and $1 \leq u_j \leq m+n$ gives a Λ-basis of W^*, which shall also be denoted by the same notation; regarded as a linear map on W^*, $T(i)^{u_1\dots,u_{t_1}}_{l_1\dots l_{b_i}}(u_1,\dots,u_s) := T(i)^{u_1\dots,u_{t_1}}_{l_1\dots l_{b_i}}(u_1)$. As was described in the proof of Theorem 4.2, setting the multidegree of $T(i)^{u_1\dots u_{t_1}}_{l_1\dots l_{b_i}} \in W^*$ to be $(0,\dots,0,1,0\dots,0)$ with 1 in the i^{th}-position and 0 elsewhere, denote by $S^{(m_1,\dots,m_s)}(W^*)$ the $(m_1,\dots,m_s)$-multidegree summand of $S^r(W^*)$. For the same ordering of the bases of $U^{t_i}_{b_i}$ as fixed in the proof of Theorem 4.2, it may be noted that the monomials $\prod_{i=1}^s T(i)^{u_1\dots u_{t_i}}_{l_1\dots l_{b_i}} T(i)^{u_{t_i+1}\dots u_{2t_i}}_{l_{b_i+1}\dots l_{2b_i}}\cdots T(i)^{u_{(m_i-1)t_i+1}\dots u_{m_it_i}}_{l_{(m_i-1)b_i+1}\dots l_{m_ib_i}}$ where the l_j's and u_j's vary in $\{1,\dots,m+n\}$ forms a basis of $S^{(m_1,\dots,m_s)}(W^*)$. Hence the natural homomorphism from $\otimes_{i=1}^s S^{m_i}((U^{t_i}_{b_i})^*)$ to $S^{(m_1,\dots,m_s)}(W^*)$ given by $f_1\otimes\cdots\otimes f_s \mapsto f_1\cdots f_s$, is an isomorphism. Also, note that $\otimes_{i=1}^s((U^{t_i}_{b_i})^*)^{\otimes m_i}$ is a subspace of $T^r(W^*)$ where $r=\sum_{i=1}^s m_i$. We have,

Lemma 4.5. *With notations as above, the following diagram commutes*

$$\begin{tikzcd}
\otimes_{i=1}^s((U^{t_i}_{b_i})^*)^{\otimes m_i} \arrow[r, two heads, "\varpi_{m_1}\otimes\ldots\otimes\varpi_{m_s}"] \arrow[dd, hook] & \otimes_{i=1}^s S^{m_i}((U^{t_i}_{b_i})^*) \arrow[d, "\cong"] \\
 & S^{(m_1,\dots,m_s)}(\oplus_{i=1}^s(U^{t_i}_{b_i})^*) \arrow[d, hook] \\
T^r(W^*) \arrow[r, two heads, "\varpi_r"] & S^r(W^*)
\end{tikzcd}$$

Proof. It is an easy checking to observe that the basis element

$$\otimes_{i=1}^s \otimes_{j=1}^{m_i} T(i)^{u_{\sum_{j<i} m_jt_j+(kt_i)+1}\dots u_{\sum_{j<i} m_jt_j+(k+1)t_i)}}_{l_{\sum_{j<i} m_jb_j+(kb_i)+1}\dots l_{\sum_{j<i} m_jb_j+(k+1)b_i)}}$$

of $\otimes_{i=1}^s((U^{t_i}_{b_i})^*)^{\otimes m_i}$ goes to the same monomial via both ϖ_r and $\varpi_{m_1}\otimes\ldots\otimes\varpi_{m_s}$, as required. □

Lemma 4.6. *With notations as above, for each s-tuple of non-negative integers $(m_1,\dots,m_s)$ the map F^r restricts to an isomorphism from $S^{(m_1,\dots,m_s)}(W^*) \to \mathcal{P}^{(m_1,\dots,m_s)}(W_0)$ where $\mathcal{P}^{(m_1,\dots,m_s)}(W_0)$ denotes the image of $S^{(m_1,\dots,m_s)}(W^*)$ in $\mathcal{F}(W_0,\Lambda)$. Further under the natural identification of $S^{(m_1,\dots,m_s)}(W^*)$ with $\otimes_{i=1}^s S^{m_i}((U^{t_i}_{b_i})^*)$, the map $F^{(m_1,\dots,m_s)} := F^r\circ(\varpi_{m_1}\otimes\ldots\otimes\varpi_{m_s}) : T^{m_1}((U^{t_1}_{b_1})^*)\otimes\ldots\otimes T^{m_s}((U^{t_s}_{b_s})^*) \to \mathcal{P}^{(m_1,\dots,m_s)}(W_0)$ is given by $F^{(m_1,\dots,m_s)}(\boldsymbol{f_1}\otimes\ldots\otimes\boldsymbol{f_s})(u_1,\dots,u_s) = ev(\boldsymbol{f_1}\otimes u_1^{\otimes m_1})\cdots ev(\boldsymbol{f_s}\otimes u_s^{\otimes m_s})$. In other words, the following diagram commutes*

$$\begin{tikzcd}
 & \otimes_{i=1}^s S^{m_i}((U^{t_i}_{b_i})^*) \arrow[r, "F^{m_1}\otimes\ldots\otimes F^{m_s}"] \arrow[d, "\cong"] & \otimes_{i=1}^s\mathcal{F}((U^{t_i}_{b_i})_0,\Lambda) \arrow[dd] \\
\otimes_{i=1}^s((U^{t_i}_{b_i})^*)^{\otimes m_i} \arrow[ur, two heads, "\varpi_{m_1}\otimes\ldots\otimes\varpi_{m_s}"] \arrow[dr] & S^{(m_1,\dots,m_s)}(\oplus_{i=1}^s(U^{t_i}_{b_i})^*) \arrow[d, hook] & \\
 & S^r(W^*) \arrow[r, "F^r"] & \mathcal{F}(W_0,\Lambda)
\end{tikzcd}$$

where the right-most arrow takes $f_1 \otimes \cdots \otimes f_s \in \otimes_{i=1}^s \mathcal{F}((U_{b_i}^{t_i})_0, \Lambda)$ *to the map* $(u_1, \ldots, u_s) \mapsto f_1(u_1) \cdots f_s(u_s)$.

Proof. The first part is a consequence of the fact noted earlier that $F^r : S^r(W^*) \to \mathcal{F}(W_0)$ is injective. For the second part, note that under the natural isomorphism between $\otimes_{i=1}^s S^{m_i}((U_{b_i}^{t_i})^*)$ and $S^{(m_1,\ldots,m_s)}(W^*)$, the element $f_1 \otimes \cdots \otimes f_s \in \otimes_{i=1}^s S^{m_i}((U_{b_i}^{t_i})^*)$ where $f_i \in S^{m_i}((U_{b_i}^{t_i})^*)$ can be identified with the monomial $f_1 \cdots f_s \in S^{(m_1,\ldots,m_s)}(W^*)$. Then the commutativity of the diagram in Lemma 4.5 implies that $F^r(f_1 \cdots f_s)(\boldsymbol{u}) = F^r(f_1 \otimes \cdots \otimes f_s)(\boldsymbol{u}) = ev(\boldsymbol{f_1} \otimes \cdots \otimes \boldsymbol{f_s} \otimes \boldsymbol{u}^{\otimes(\sum_i m_i)}) = ev(\boldsymbol{f_1} \otimes \boldsymbol{u}^{\otimes m_1} \otimes \cdots \otimes \boldsymbol{f_s} \otimes \boldsymbol{u}^{m_s})$, where $\boldsymbol{u} = (u_1, \ldots, u_s) \in W_0$ and $\boldsymbol{f_i} \in T^{m_i}((U_{b_i}^{t_i})^*)$ such that $\varpi_{m_i}(\boldsymbol{f_i}) = f_i$. Note that, since $\boldsymbol{u}$ is a degree 0 element there is no sign change in the above equality. The right side of the above equality evaluates to $\mathcal{F}^{m_1}(\boldsymbol{f_1})(u_1) \cdots \mathcal{F}^{m_1}(\boldsymbol{f_s})(u_s)$. ☐

Remark 4.7. *We note from the commutativity of the above diagram and with the notations as in the above lemma,* $F^r(\boldsymbol{T})(\boldsymbol{u}) = ev(\boldsymbol{T} \otimes u_1^{\otimes m_1} \otimes \cdots \otimes u_s^{\otimes m_s})$ *where* $\boldsymbol{T} \in \otimes_{i=1}^s T^{m_i}((U_{b_i}^{t_i})^*)$ *and* $\boldsymbol{u} = (u_1, \ldots, u_s) \in W$.

In view of the above Lemma 4.6, we call the image of $S^{(m_1,\ldots,m_s)}(W^*)$ in $\mathcal{F}(W_0, \Lambda)$ as the *space of multihomogeneous polynomials on* W_0, denoted as $\mathcal{P}^{(m_1,\ldots,m_s)}(W_0)$. Further, the map F^r restricted to $S^{(m_1,\ldots,m_s)}(W^*)$ is the same as the restitution map $F_1^{m_1} \otimes \cdots \otimes F_s^{m_s} : \otimes_{i=1}^s S^{m_i}((U_{b_i}^{t_i})^*) \to \mathcal{P}^{(m_1,\ldots,m_s)}(W_0)$. When $m_i = 1$ for all $i = 1, \ldots, s$, this is the *space of multilinear polynomials on* W_0.

4.3.1. *Polynomial invariants and 'pictures'.* By a polynomial invariant on W_0, we shall mean an element of $\mathcal{F}(W_0, \Lambda)$ which is in the image of $S(W^*)^{\mathrm{GL}(U)}$. By Theorem 4.2, this means that the subalgebra of invariant polynomials on W_0 is spanned by $F^r(\varphi_\sigma)$ where φ_σ are the picture invariants as defined above. Let T_σ be the linear map described in Remark 4.4. Then we have,

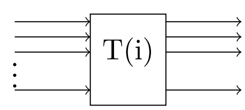
Proposition 4.8. *The graded picture invariant* $\varphi_\sigma \in S^{(m_1,\ldots,m_s)}(W^*)$ *maps under restitution* F^r *to* $F^r(\varphi_\sigma)$ *given by* $F^r(\varphi_\sigma) = T_\sigma(u_1^{\otimes m_1} \otimes \cdots \otimes u_s^{\otimes m_s})$ *where* $\boldsymbol{u} = (u_1 \ldots, u_s) \in W_0$.

Proof. The isomorphism $\iota : T^r(W^*) \to (T^r(W))^*$ used in Theorem 4.2 is obtained from the non-degenerate pairing $T^r(W^*) \otimes T^r(W) \to \Lambda$, as in (4) of §2.4. Therefore, for any $\phi \in T^r(W^*)$ we get a linear map $\iota(\phi)$ on $T^r(W)$ and the evaluation $\iota(\phi)(w_1 \otimes \ldots \otimes w_r)$ is given by $ev(\phi \otimes w_1 \otimes \ldots \otimes w_r)$. Hence we have the following commuting diagram:

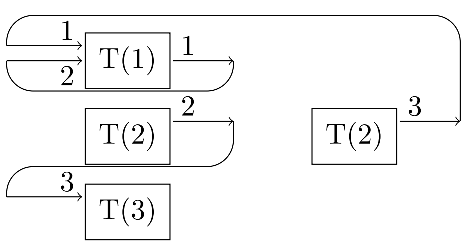
$$\begin{tikzcd}
\otimes_{i=1}^s((U_{b_i}^{t_i})^*)^{\otimes m_i} \arrow[r, hook] & T^r(W^*) \arrow[r] & S^r(W^*) \arrow[r, "F^r"] & \mathcal{F}(W_0, \Lambda) \\
\left(\otimes_{i=1}^s((U_{b_i}^{t_i})^{\otimes m_i}\right)^* \arrow[u] \arrow[r, hook] & (T^r(W))^* \arrow[u, "\cong"'] \arrow[urr, dashed] & &
\end{tikzcd}$$

where the inclusion $\otimes_{i=1}^{s}((U_{b_i}^{t_i})^*)^{\otimes m_i} \hookrightarrow T^r(W^*)$ is as earlier while the map $\left(\otimes_{i=1}^{s}((U_{b_i}^{t_i})^{\otimes m_i}\right)^* \hookrightarrow (T^r(W))^*$ is induced by the projection $\mathrm{pr} : T^r(W) \twoheadrightarrow \left(\otimes_{i=1}^{s}((U_{b_i}^{t_i})^{\otimes m_i}\right)$; for $\psi \in \left(\otimes_{i=1}^{s}((U_{b_i}^{t_i})^{\otimes m_i}\right)^*$ and $w_1 \otimes \ldots \otimes w_r \in T^r(W)$, $\psi(w_1 \otimes \ldots \otimes w_r) := \psi(\mathrm{pr}(w_1 \otimes \ldots \otimes w_r))$. By the construction of φ_σ in Theorem 4.2, we know that $T_\sigma \in \left(\otimes_{i=1}^{s}((U_{b_i}^{t_i})^{\otimes m_i}\right)^*$ maps to $\varphi_\sigma \in S^r(W^*)$. Let $\boldsymbol{\varphi_\sigma} \in T^r(W^*)$ such that $\iota(\boldsymbol{\varphi_\sigma}) = T_\sigma$ and $\varpi_r(\boldsymbol{\varphi_\sigma}) = \varphi_\sigma$. By Remark 4.7 we have $F^r(\varphi_\sigma)(\boldsymbol{u}) = ev(\boldsymbol{\varphi_\sigma} \otimes u_1^{\otimes m_1} \otimes \cdots \otimes u_s^{\otimes m_s})$ where $\boldsymbol{u} = (u_1 \ldots, u_s) \in W_0$. Under the isomorphism ι, as seen above, the latter evaluates to $T_\sigma(u_1^{\otimes m_1} \otimes \cdots \otimes u_s^{m_s})$, as required. $\square$

By the above proposition, the invariant polynomials on W_0 are given by the evaluations of the linear maps $T_\sigma := ev \circ \eta(\nu\tau\hat{\sigma}\mu)$ where $\eta := \eta_{2N}$ is as in §2.3. In the spirit of [6], we associate certain pictures to these invariants. To each variable $T(i)_{l_1 \ldots l_{b_i}}^{u_1 \ldots u_{t_i}} \in W^*$ we associate a picture



with t_i in-arrows and b_i out-arrows. Given an s-tuple $(m_1, \ldots, m_s)$ of non-negative integers such that $\sum_{i=1}^{s} m_i t_i = \sum_{i=1}^{s} m_i b_i = N$, we take for each i, m_i many copies of the pictures associated to $T(i)_{l_1 \ldots l_{b_i}}^{u_1 \ldots u_{t_i}}$ and arrange them in order from $i = 1$ to $i = s$ and then label all the in-arrows and all the out-arrows from 1 to N in the order that they appear. Now given a $\sigma \in S_N$ we associate a closed picture obtained by joining the in-arrow labelled i with the out-arrow labelled $\sigma^{-1}(i)$. For example, let $W = V_2^1 \oplus V_1^0 \oplus V_0^1$ and $m_1 = 1; m_2 = 2; m_3 = 1$. Then corresponding to $\sigma = (1\ 2\ 3)$ we get the closed picture below:



This picture corresponds to the graded picture invariant φ_σ as defined earlier in (4.1).

4.3.2. *Graded picture invariants in terms of supertraces.* We now restrict to the case when $W = (U_1^1)^s$. By (1) of §2.4, we have $U_1^1 \cong End_\Lambda(U)$. We can then consider the supertrace function on U_1^1 given by $str(v \otimes \alpha) = (-1)^{|v||\alpha|}\alpha(v)$ and $(v \otimes \alpha).(w \otimes \beta) = v\alpha(w) \otimes \beta$. With this notation, for a permutation $\sigma \in S_N$ the trace monomial tr_σ is defined on $(U_1^1)^{\otimes N}$ as $tr_\sigma(v_1 \otimes \phi_1 \otimes \cdots \otimes v_N \otimes \phi_N) = str(v_{i_1} \otimes \phi_{i_1}.v_{i_2} \otimes \phi_{i_2}. \cdots .v_{i_r} \otimes \phi_{i_r}) str(v_{j_1} \otimes \phi_{j_1}.v_{j_2} \otimes \phi_{j_2}. \cdots \otimes v_{j_t} \otimes \phi_{j_t})$, where $\sigma^{-1} = (i_1\ i_2\ \ldots i_r)(j_1\ j_2 \cdots j_t) \cdots$. This definition is dependent on the permutation σ

and its cycle decomposition as well. However, it is well-defined when restricted to $(U_1^1)_0^{\otimes N}$. The following is similar to [1, Lemma 2.2]),

Lemma 4.9. *For a $\sigma \in S_N$, the $\mathrm{GL}(U)$-invariant linear map $ev \circ \eta(\nu.\tau.\hat{\sigma}.\tau^{-1})$ (as in Remark 4.4) and tr_σ as defined above, agree on $v_1 \otimes \phi_1 \otimes \cdots \otimes v_N \otimes \phi_N$, upto a sign. Further, when evaluated on $v_1 \otimes \phi_1 \otimes \cdots \otimes v_N \otimes \phi_N \in ((U_1^1)_0)^{\otimes N}$, they are equal.*

Proof. For the first part, by Λ-linearity it suffices to prove the above for the basis elements $(e_{l_1} \otimes e^*_{u_1} \otimes \cdots e_{l_N} \otimes e^*_{u_N})$. The effect of applying $\nu.\tau.\hat{\sigma}.\tau^{-1}$ to $e_{l_1} \otimes e^*_{u_1} \otimes \cdots e_{l_N} \otimes e^*_{u_N}$ from the left is to get $\pm(e^*_{u_1} \otimes e_{l_{\sigma^{-1}(1)}} \otimes \cdots e^*_{u_N} \otimes e_{u_{\sigma^{-1}(N)}})$. Now applying the evaluation map ev on this vector gives, $\pm e^*_{u_1}(e_{l_{\sigma^{-1}(1)}}) \cdots e^*_{u_r}(v_{l_{\sigma^{-1}(r)}}) e^*_{u_{r+1}}(e_{l_{\sigma^{-1}(r+1)}}) \cdots$. After a rearrangement, the latter is equal to $e^*_{u_{i_1}}(e_{l_{i_2}}) \cdots e^*_{u_{i_r}}(e_{l_{i_1}}) e^*_{u_{j_1}}(e_{l_{j_2}}) \cdots$, upto a sign. This in turn is the trace monomial $tr_\sigma(e_{l_1} \otimes e^*_{u_1} \otimes \cdots e_{l_N} \otimes e^*_{u_N})$.

For the second part, we try to determine the sign change explicitly. First, we consider the special case of $\sigma^{-1} = (1\,2\, \cdots r)(r+1 \cdots t) \cdots$. Let $v_i \otimes \phi_i \in V \otimes V^*$. Let v_i have parity a_i then ϕ_i have parity a'_i. The effect of applying $\nu.\tau.\hat{\sigma}.\tau^{-1}$ to $v_1 \otimes \phi_1 \otimes \cdots \otimes v_N \otimes \phi_N$ on the left is to get $\pm(\phi_1 \otimes v_{\sigma^{-1}(1)} \otimes \phi_N \otimes v_{\sigma^{-1}(N)})$. We consider the special case of $\sigma^{-1} = (1\,2\, \cdots r)(r+1 \cdots t) \cdots$. In this case, acting $\nu.\tau.\hat{\sigma}.\tau^{-1}$ from the left on $(v_1 \otimes \phi_1 \otimes \cdots v_N \otimes \phi_N)$ we get $\pm(\phi_1 \otimes v_2 \otimes \phi_2 \otimes v_3 \cdots \otimes \phi_r \otimes v_1) \otimes (\phi_{r+1} \otimes v_{r+2} \otimes \cdots \otimes \phi_t \otimes v_{r+1}) \otimes \cdots$. Now applying the evaluation map ev on this vector gives, $\pm \phi_1(v_2) \cdots \phi_r(v_1) \phi_{r+1}(v_{r+2}) \cdots$, leaving the sign unchanged. The total sign change is then given by $a_1(a'_1 + a_2 + a'_2 + \cdots + a_r + a'_r) + a_{r+1}(a'_{r+1} + a_{r+2} + \cdots + a'_t) + \cdots$. If $v_i \otimes \phi_i$ are of degree 0 then $a_i = a'_i$ for each i. So, for these values we see that the sign is $(-1)^{a_1 + a_{r+1} + \cdots}$. This agrees with the value of the trace monomial tr_σ evaluated on $(v_{l_1} \otimes \phi_{u_1} \otimes \cdots v_{l_N} \otimes \phi_{u_N})$. Thus, in the case that $\sigma^{-1} = (1\ 2\ \cdots r)(r+1 \cdots t) \cdots$ the linear maps $ev \circ \eta(\nu.\tau.\hat{\sigma}.\tau^{-1})$ and tr_σ agree on $\left((U_1^1)_0\right)^{\otimes N}$. The proof of the general case is same as in [1, Lemma 2.2], which we recall for completeness: Let $\sigma^{-1} = (i_1\ i_2\ \ldots i_r)(i_{r+1}\ i_{r+2} \cdots i_t) \cdots \in S_N$. Let $\pi \in S_N$ such that $\pi(j) = i_j$ then $\pi\sigma^{-1}\pi^{-1} = (1\ 2\ \cdots r)(r+1 \cdots t) \cdots$. We have, $tr_{\pi\sigma\pi^{-1}}(v_{\pi(1)} \otimes \phi_{\pi(1)} \otimes \cdots \otimes v_{\pi(N)} \otimes \phi_{\pi(N)}) := str(v_{\pi(1)} \otimes \phi_{\pi(1)} \otimes \cdots \otimes v_{\pi(r)} \otimes \phi_{\pi(r)}) str(v_{\pi(r+1)} \otimes \phi_{\pi(r+1)} \otimes \cdots \otimes v_{\pi(t)} \otimes \phi_{\pi(t)}) \cdots = tr_\sigma(v_1 \otimes \phi_1 \otimes \cdots v_N \otimes \phi_N)$. By the special case proved earlier, we have $tr_{\pi\sigma\pi^{-1}}(v_{\pi(1)} \otimes \phi_{\pi(1)} \otimes \cdots \otimes v_{\pi(N)} \otimes \phi_{\pi(N)}) = (ev \circ \eta(\nu\tau(\widehat{\pi\sigma\pi^{-1}})\tau^{-1}))(v_{\pi(1)} \otimes \phi_{\pi(1)} \otimes \cdots \otimes v_{\pi(N)} \otimes \phi_{\pi(N)}) = ev((v_{\pi(1)} \otimes \phi_{\pi(1)} \otimes \cdots \otimes v_{\pi(N)} \otimes \phi_{\pi(N)})\tau.(\widehat{\pi\sigma\pi^{-1}})^{-1}.\tau^{-1}.\nu) = ev((v_1 \otimes \phi_1 \otimes \cdots \otimes v_N \otimes \phi_N).\pi^*.\tau(\widehat{\pi\sigma\pi^{-1}})^{-1}.\tau^{-1}.\nu)$ where $\pi^* = \tau(\pi^{-1}, \pi^{-1})\tau^{-1}$. We have

$$\begin{aligned}
\tau(\pi^{-1}, \pi^{-1})\tau^{-1}\tau(\widehat{\pi\sigma\pi^{-1}})^{-1}\tau^{-1}.\nu &= \tau(\pi^{-1}, \pi^{-1})(\pi\sigma^{-1}\pi^{-1}, 1)\tau^{-1}.\nu \\
&= \tau(\sigma^{-1}\pi^{-1}, \pi^{-1})\tau^{-1}\nu \\
&= \tau\hat{\sigma}^{-1}(\pi^{-1}, \pi^{-1})\tau^{-1}\nu \\
&= \tau\hat{\sigma}^{-1}\tau^{-1}\tau(\pi^{-1}, \pi^{-1})\tau^{-1}\nu
\end{aligned}$$

$$= \tau\hat{\sigma}^{-1}\tau^{-1}\pi^*\nu$$
$$= \tau\hat{\sigma}^{-1}\tau^{-1}\nu\pi^*.$$

The last equality comes from noting that $\pi^*\nu$ and $\nu\pi^*$ agree on vectors of the form $v_1 \otimes \phi_1 \otimes \cdots \otimes v_N \otimes \phi_N$. Also, $ev([v_1 \otimes \phi_1 \otimes \cdots \otimes v_N \otimes \phi_N]\pi^*) = ev(v_1 \otimes \phi_1 \otimes \cdots \otimes v_N \otimes \phi_N)$ on $(V \otimes V^*)^{\otimes N}$. Putting all the above observations together, we have $tr_\sigma(v_1 \otimes \phi_1 \otimes \cdots \otimes v_N \otimes \phi_N) = (ev \circ \eta(\tau\hat{\sigma}\tau^{-1}\nu))(v_1 \otimes \phi_1 \otimes \cdots \otimes v_N \otimes \phi_N))$ for all $\sigma \in S_N$, proving the second part. □

It is shown in [2, Lemma 6.6] that in the special case of $W = (U_1^1)^d$ and when $m = n$, that $\mathcal{P}^{(1,\ldots,1)}(W_0)$ is actually the space of multilinear polynomials in the co-ordinates of $M_{n,n}(\Lambda)_0$. The following corollary may be viewed as a generalization of this result.

Corollary 4.10. *The picture invariant $\varphi_\sigma \in S^{(m_1,\ldots,m_s)}(W^*)$ maps under restitution F^r to the trace monomial given by $F^r(\varphi_\sigma)(\boldsymbol{u}) = tr_\sigma(u_1^{\otimes m_1} \otimes \cdots \otimes u_s^{\otimes m_s})$ where $\boldsymbol{u} = (u_1 \ldots, u_s) \in W_0$.*

Proof. The proof follows from Proposition 4.8 and the above lemma. □

The invariants in $P(W_0)$ for induced action of $\mathrm{GL}(U)$ such that the isomorphism is $\mathrm{GL}(U)$-equivariant are called the invariant polynomials on W_0. The following is Theorem 6.7 of [2],

Theorem 4.11. *The invariant polynomials for the simultaneous action of $\mathrm{GL}(U)$ on $\oplus_{i=1}^s (U_1^1)_0$ are spanned by the trace monomials tr_σ given by*

$$\mathrm{tr}_\sigma(A_1, \ldots, A_s) := \mathrm{str}(A_{f(i_1)} \cdots A_{f(i_r)})\mathrm{str}(A_{f(i_{r+1})} \cdots A_{f(i_t)}) \cdots$$

where $A_1, \ldots, A_s \in \oplus_{i=1}^s U_1^1$, $\sigma = (i_1 \ldots i_r)(i_{r+1} \ldots i_t) \cdots \in S_n$ and a map $f : \{1, \ldots, N\} \to \{1, \ldots, s\}$ for $n \in N$.

Proof. The invariants in $\mathcal{P}(W_0)$ are the image of $S(W^*)^{\mathrm{GL}(U)}$ which in turn is spanned by φ_σ, by Theorem 4.2. By the above Corollary 4.10 we then get the required result. □

5. Schur-Weyl-Sergeev Duality for queer superalgebra

5.1. **Queer superalgebra.** Let $V = V_0 \oplus V_1$ be the $\mathbb{Z}_2$-graded superspace for which $V_0 = \mathbb{C}^n$ and $V_1 = \mathbb{C}^n$. Let $\{v_1, \cdots, v_n\}$ and $\{v_{-1}, \cdots, v_{-n}\}$ be bases for V_0 and V_1 respectively.

Define a map $P : V \to V$ by $v_i \mapsto (-1)^{|v_i|} v_{-i}$ for all i.

Recall that the superbracket in $End_{\mathbb{C}}(V)$ is given by

$$[f, g] = fg - (-1)^{|f||g|} gf.$$

for homogeneous elements $f, g \in End_{\mathbb{C}}(V)$.

Let $U := V \otimes_{\mathbb{C}} \Lambda$. The map $P \in End_{\mathbb{C}}(V)$ extends to a map in $End_\Lambda(U)$ which we still denote by P. We have $P^2 = -I$.

We define $\mathfrak{q}(U) := \{f \in End_\Lambda(U)_0 : [f, P] = 0\}$. It is closed under the superbracket and we call it the queer superalgebra.

With respect to the basis $B := B_0 \cup B_1$, the operator P can be written in matrix form as

$$P = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}.$$

The queer Lie superalgebra $\mathfrak{q}(U)$ (or $\mathfrak{q}(n)$) can be expressed in the matrix form

$$\mathfrak{q}(n) = \{\begin{pmatrix} A & B \\ B & A \end{pmatrix} : A \in M_n(\Lambda_0), B \in M_n(\Lambda_1)\}.$$

We define the queer trace of an element of $\mathfrak{q}(n)$ to be

$$qtr(\begin{pmatrix} A & B \\ B & A \end{pmatrix}) = [tr(A+B)]_1,$$

where $[tr(A+B)]_1$ is the degree 1 component of the sum of the diagonal entries of $A+B$. We can see that qtr satisfy $qtr(MN) = qtr(NM)$ and $qtr(RMR^{-1}) = qtr(M)$ for all $M, N, R \in \mathfrak{q}(n)$, where R is invertible.

The queer superalgebra $\mathfrak{q}(U)$ is a subalgebra of $\mathfrak{gl}(U)$ and it acts on $U^{\otimes k}$ by ordinary derivation (see §2.3).

The queer supergroup $Q(U)$ is the subsupergroup of $GL(U)$ which commute with P. The diagonal action of $GL(U)$ on $U^{\otimes k}$ restricts to an action of $Q(U)$ on $U^{\otimes k}$.

5.2. **The Sergeev superalgebras.** Let S_k be the symmetric group on k letters. It is generated by the transpositions $s_1, \cdots, s_{k-1}$.

The Sergeev superalgebra Ser_k is the associative superalgebra generated by $s_1, \cdots, s_{k-1}$ and $c_1, \cdots, c_{k-1}, c_k$ with the following defining relations:

$s_i^2 = 1, s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, s_i s_j = s_j s_i \quad (|i-j| > 1),$

$c_i^2 = -1, c_i c_j = -c_j c_i, \ (i \neq j)$

$s_i c_i s_i = c_{i+1}, s_i c_j = c_j s_i, \ j \neq i, i+1.$

The generators $s_1, \cdots, s_{k-1}$ are regarded as even and the subalgebra generated by them is isomorphic to the group algebra $\mathbb{C}[S_k]$ of S_k; the generators $c_1, \cdots, c_{k-1}, c_k$ are called odd and the $\mathbb{C}$-subalgebra generated by them is isomorphic to the Clifford superalgebra $\mathbf{C}l_k$.

The Sergeev superalgebra Ser_k is isomorphic to the superalgebra $\mathbb{C}[S_k] \ltimes \mathbf{C}l_k$, which is $\mathbb{C}[S_k] \otimes \mathbf{C}l_k$ as a superspace with the multiplication given by

$$(\sigma \otimes c_{i_1} \cdots c_{i_l})(\tau \otimes c_{j_1} \cdots c_{j_m}) = \sigma\tau \otimes c_{\tau^{-1}(i_1)} \cdots c_{\tau^{-1}(i_l)} c_{j_1} \cdots c_{j_m},$$

where $1 \leq i_s, j_t \leq k$.

Let $\Lambda Ser_k := Ser_k \otimes_{\mathbb{C}} \Lambda$. There is a left action of the superalgebra ΛSer_k on $U^{\otimes k}$ which is given as follows (see §3.4.1 of [5]):

The generators s_j act on $U^{\otimes k}$ by

$$s_j.(u_1 \otimes \cdots \otimes u_k) = (-1)^{|u_j||u_{j+1}|} u_1 \otimes \cdots \otimes u_{j-1} \otimes u_{j+1} \otimes u_j \otimes u_{j+2} \otimes \cdots \otimes u_k.$$

The generators c_j act on $U^{\otimes k}$ by

$$c_j.(u_1 \otimes \cdots \otimes u_k) = (-1)^{|u_1|+\cdots+|u_{j-1}|} u_1 \otimes \cdots \otimes u_{j-1} \otimes P(u_j) \otimes u_{j+1} \otimes u_{j+2} \otimes \cdots \otimes u_k.$$

The actions of s_i and c_j give rise to an action of the superalgebra ΛSer_k on $U^{\otimes k}$. So by Λ-linearity, we get an algebra homomorphism

$$\eta_k : \Lambda Ser_k \to \mathrm{End}_\Lambda(U^{\otimes k}).$$

From the above action, we may note that for $\varepsilon_i \in \mathbb{Z}_2$, for $i = 1, \dots, k$,

$$c_1^{\varepsilon_1} \cdots c_k^{\varepsilon_k}.(u_1 \otimes \cdots \otimes u_k) = (-1)^{\sum_{i>j} \varepsilon_i \alpha_j} P^{\varepsilon_1} u_1 \otimes \cdots \otimes P^{\varepsilon_k} u_k$$

where $\alpha_j = |u_j|$.

Let $\mathrm{End}_{Q(U)}(U^{\otimes k})$ be the centralizer algebra of the $Q(U)$-action on $U^{\otimes k}$. That is

$$\mathrm{End}_{Q(U)}(U^{\otimes k}) := \{f \in \mathrm{End}_\Lambda(U^{\otimes k}) : gf = fg, \text{ for all } g \in Q(V)\}.$$

Sergeev [16] proved the double centralizer theorem along the lines of the Schur-Weyl duality for the queer superalgebra and it was extended to the Queer supergroups by Berele (see Theorem 4.2 of [2]).

Theorem 5.1. *Let $\rho : Q(U) \to \mathrm{End}_\Lambda(U^{\otimes k})$ and $\eta_k : \Lambda Ser_k \to \mathrm{End}_\Lambda(U^{\otimes k})$ be the maps induced from the actions defined above. Let A be the image of η_k and let B be the Λ-subalgebra of $\mathrm{End}_\Lambda(U^{\otimes k})$ spanned by the image of ρ. Then A and B are centralizers of each other.*

Remark 5.2. *It may be noted that in the statement of the above theorem we consider the left action of the Sergeev algebra whereas Berele considers the action from the right and uses ΛSer_k^{op} instead.*

5.3. In this subsection we list out some facts that are needed to prove the main results in §6. We give an outline of the proofs here, for more details refer to [2].

1. There is a $GL(U)$ equivariant isomorphism $\mathrm{End}_\Lambda(U^{\otimes k}) \cong (\mathrm{End}_\Lambda(U)^{\otimes k})^*$ (see (7) of §2.4). Then from Theorem 5.1, it follows that, under this isomorphism the $Q(U)$-invariant elements of $(\mathrm{End}_\Lambda(U)^{\otimes k})^*$ correspond to $\Lambda Ser_k \subseteq \mathrm{End}_\Lambda(U^{\otimes k})$.

2. Let $T \in \mathrm{End}_\Lambda(U^{\otimes k})$. Then, again it follows from Theorem 5.1 that T commutes with $\rho(g)$ for all $g \in Q(U)$ if and only if T is in the image of ΛSer_k.

3. Recall that we defined the supertarce map $str : \mathrm{End}_\Lambda(U) \to \Lambda$ by

$$str(v \otimes \phi) = (-1)^{|v||\phi|}\phi(v).$$

For $A \in \mathrm{End}_\Lambda(U)_0$, we define the queer trace as $qtr(A) = str(PA)$.

4. Let $\mathcal{I}$ be the set of elements of $\mathrm{End}_\Lambda(U)_0$ which commute with P. Note that $Q(V)$ equals the set of invertible elements of $\mathcal{I}$. We have $str(A) = 0$ for $A \in \mathcal{I}$ as $P^2 = -I$ and $str(PAP) = str(A)$.

5. Let $A_1, \cdots A_d \in \mathcal{I}$ and $\epsilon_1, \cdots \epsilon_d \in \mathbb{Z}_2$. Then it follows from (4) that if $\sum_i \epsilon_i$ is even then $str(P^{\epsilon_1}A_1 \cdots P^{\epsilon_d}A_d) = 0$ and if $\sum_i \epsilon_i$ is odd then $str(P^{\epsilon_1}A_1 \cdots P^{\epsilon_d}A_d) = qtr(A_1 \cdots A_k)$.

6. Extending results of §4 to $Q(U)$-invariants

Given a mixed tensor superspace $W = \oplus_{i=1}^s (U_{b_i}^{t_i})$ as in §4, the restriction to $Q(U)$ of the $\mathrm{GL}(U)$-action on W and on $T^r(W^*)$ gives a natural $Q(U)$-action on these superspaces. As the $\mathrm{GL}(U)$-action commutes with the natural S_r-action on $T^r(W^*)$, so also does the $Q(U)$-action. Hence, as seen in §3, the subspace $e(r)T^r(W^*)$ is a $Q(U)$-invariant subspace which is isomorphic to $S^r(W^*)$. Thus $S^r(W^*)$ has a $Q(U)$-action defined on it and the map $\varpi_r : T^r(W^*) \to S^r(W^*)$ is $Q(U)$-equivariant with respect to this action.

With this notation, to obtain an analogue of Theorem 4.2 we will need to associate further invariants in addition to the graded picture invariants given by (4.1). Denoting the Λ-basis of U corresponding to $\{v_1, \ldots, v_n, v_{-1}, \ldots, v_{-n}\}$ also by the same symbols, we note as in §4.2 that $S(W^*)$ is a polynomial algebra with supercommuting variables $T(i)_{l_1 \ldots l_{b_i}}^{u_1 \ldots u_{t_i}}$ where l_j and u_j take values in $\{1, \ldots, n, -1, \ldots, -n\}$. Let $(m_1, \ldots, m_s)$ be an s-tuple such that $\sum_{k=1}^s m_k t_k = \sum_{k=1}^s m_k b_k = N$, then associated to the elements $c_h \in \Lambda Ser_N$ the picture invariant φ_{c_h} is defined by

$$\sum_{r_1,\ldots,r_N \in \{1,\ldots,n,-1,\cdots,-n\}} (-1)^{\hat{q}(h,r_1,\ldots,r_N,m_1,\ldots,m_s)} \prod_{i=1}^{s} \prod_{j=1}^{m_i} T(i)_{r_{(\sum_{p<i} m_p b_p + (j-1)b_i+1)} \cdots r_{(\sum_{p<i} m_p b_p + j b_i)}}^{r'_{(\sum_{p<i} m_p t_p + (j-1)t_i+1)} \cdots r'_{(\sum_{p<i} m_p t_p + j t_i)}}$$

where $r'_j = r_j$ for $j \neq h$ and $r'_h = -r_h$. The value of $\hat{q}(h, r_1, \ldots, r_N, m_1, \ldots, m_s)$ is given by $\gamma(\mu^{-1}.(I, I^h), \mu^{-1})\gamma((I, I^h), (\tau.\nu)^{-1})\hat{\gamma}(I, h)(-1)^{p(I,I^h)}$, where $I = (r_1, \ldots, r_N)$ and $\hat{\gamma}(I, h) = (-1)^{|v_{r_1}| + \cdots + |v_{r_{h-1}}| + |v_{r_h}|}$ and $I^h = (r_1, \ldots, -r_h, \ldots, r_N)$.

The above formula is obtained by going through the isomorphisms in the proof of Theorem 4.2, as was done there for the case of $\sigma \in S_N$. These isomorphisms are $\mathrm{GL}(U)$-equivariant and hence also $Q(U)$-equivariant. Replacing by $-1, \ldots, -n$ the indices $n+1, \ldots, n+n$ wherever they appear in the isomorphisms listed in the proof of Theorem 4.2 we get the generators of $\left(\otimes_{i=1}^s S^{m_i}((U_{b_i}^{t_i})^*)\right)^{Q(U)}$. Using the same argument as given there, it may be first noted that there are invariants in $\otimes_{i=1}^s S^{m_i}((U_{b_i}^{t_i})^*)$ if and only if $\sum_{k=1}^s m_k t_k = \sum_{k=1}^s m_k t_k = N$. For such an s-tuple $(m_1, \dots, m_s)$, we start with the generators of $(\mathrm{End}_\Lambda(U^{\otimes N}))^{Q(U)}$ prescribed by Theorem 5.1. Then as done in Theorem 4.2, going via the isomorphisms given there, the $\sigma \in S_N$ give rise to the polynomials φ_σ as defined in §4.2. Now by the Schur-Weyl-Sergeev duality, the elements c_h, $h = 1, \dots, N$ give $Q(U)$-equivariant endomorphisms on $U^{\otimes N}$ which correspond under the natural isomorphism listed in (7) of §2.4, $\mathrm{End}_\Lambda(U^{\otimes N}) \cong \left(U^{\otimes N} \otimes_\Lambda (U^*)^{\otimes N}\right)^*$, to the linear map given by $\underline{v} \otimes \underline{f}^* \mapsto ev[\nu.\tau.(c_h.\underline{v} \otimes \underline{f}^*)]$. This may be expressed in term of the standard dual basis of $U^{\otimes N} \otimes_\Lambda (U^*)^{\otimes N}$ by

$$\sum_{L=(l_1,\dots,l_N)} \hat{\gamma}(L,h)\gamma((L^h, L^h), \tau^{-1})\gamma(\tau.(L^h, L^h), \nu)T_{L,L^h}.$$

where $l_i \in \{-n, \dots, -1, 1, \dots, n\}$ for all $i = 1, \dots, N$, $(l_1, \dots, l_N)^h := (l'_1, \dots, l'_N)$ given by $l'_i = l_i$ unless $i = h$ and $l'_h = -l_h$. Now going through the isomorphisms in (4.2) and (4.3), it can be seen that this element corresponds to φ_{c_h} with the sign $(-1)^{\hat{q}(h,l_1,\dots,l_N,m_1,\dots,m_s)}$ given by $\gamma(\mu^{-1}.(L, L^h), \mu^{-1})\gamma((L, L^h), (\nu\tau)^{-1})\hat{\gamma}(I, h)$.

More generally, for $\sigma \in S_N$ and $c_1^{\varepsilon_1} \dots c_N^{\varepsilon_N} \in \mathbf{C}l_N$ which we also denote in short-hand by $c^{\boldsymbol{\varepsilon}}$, the linear map associated to the element $\sigma \otimes c_1^{\varepsilon_1} \dots c_N^{\varepsilon_N} \in \Lambda Ser_N$ is given by

$$\sum_{L=(l_1,\dots,l_N)} \hat{\gamma}(L,\boldsymbol{\varepsilon})\gamma((\sigma(L^{\boldsymbol{\varepsilon}}), \sigma(L^{\boldsymbol{\varepsilon}})), \tau^{-1})\gamma(\tau.(\sigma(L^{\boldsymbol{\varepsilon}}), \sigma(L^{\boldsymbol{\varepsilon}})), \nu^{-1})T_{L,\sigma(L^{\boldsymbol{\varepsilon}})}$$

where $\boldsymbol{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_N)$, $(l_1, \dots, l_N)^{\boldsymbol{\varepsilon}} := ((-1)^{\varepsilon_1} l_1, \dots, (-1)^{\varepsilon_N} l_N)$ and $\hat{\gamma}((l_1, \dots, l_N), \boldsymbol{\varepsilon}) := (-1)^{\sum_{i=1}^N \sum_{j:i\geq j} \varepsilon_i |v_{l_j}|}$.

Going through the isomorphisms (4.2), (4.3) and the map $\varpi_{m_1} \otimes \cdots \otimes \varpi_{m_s}$ the above linear map corresponds to the following element of $S(W^*)$:

$$\sum_{L=(l_1,\dots,l_N)} (-1)^{\hat{q}_{\sigma,\boldsymbol{\varepsilon}}(L,m_1,\dots,m_s)} \prod_{i=1}^{s} \prod_{j=1}^{m_i} T(i)^{l_{(\sum_{p<i} m_p t_p + (j-1)t_i + 1)} \cdots l_{(\sum_{p<i} m_p t_p + j t_i)}}_{l'_{(\sum_{p<i} m_p b_p + (j-1)b_i + 1)} \cdots l'_{(\sum_{p<i} m_p b_p + j b_i)}} \tag{6.1}$$

where $l_i \in \{1, \dots, n, -1, \cdots, -n\}$ for all $i = 1, \dots, N$, $l'_j = (-1)^{\varepsilon_j} l_{\sigma(j)}$ for $j = 1 \dots, N$. The value of $\hat{q}_{\sigma,\boldsymbol{\varepsilon}}(L, m_1, \dots, m_s)$ is given by the formula

$$(-1)^{p(L,\sigma(L^{\boldsymbol{\varepsilon}}))}\gamma(\mu^{-1}.(L, \sigma(L^{\boldsymbol{\varepsilon}})), \mu^{-1})\hat{\gamma}(L,\boldsymbol{\varepsilon})\gamma((\sigma(L^{\boldsymbol{\varepsilon}}), \sigma(L^{\boldsymbol{\varepsilon}})), \tau^{-1})\gamma(\tau.(\sigma(L^{\boldsymbol{\varepsilon}}), \sigma(L^{\boldsymbol{\varepsilon}})), \nu^{-1}).$$

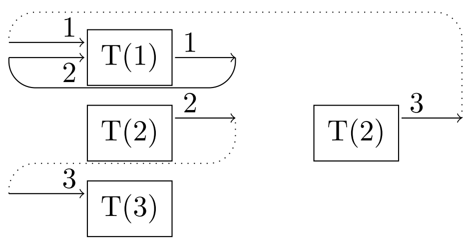
This element in $S(W^*)$ is denoted as $\varphi_{\sigma\otimes c^{\boldsymbol{\varepsilon}}}$. For s-tuples $(m_1, \dots, m_s)$ of non-negative integers, such that $\sum_{k=1}^s m_k t_k = \sum_{k=1}^s m_k t_k = N$, as σ varies in S_N and $c^{\boldsymbol{\varepsilon}}$ varies in Cl_N, the elements $\varphi_{\sigma\otimes c^{\boldsymbol{\varepsilon}}}$ will be refered to as 'coloured picture invariants' [2].

We have thus obtained the following analogue of Theorem 4.2:

Theorem 6.1. *Given a mixed tensor superspace $W = \oplus_{i=1}^s U_{b_i}^{t_i}$, the coloured picture invariants as in (6.1) span $S(W^*)^{Q(U)}$.* □

[2]In (6.1), the monomials should be considered modulo the anti-commutativity relations in $S(W^*)$.

6.0.1. *Coloured Picture invariants.* As was done in Section 4.2, one may associate a "coloured picture" to the above invariants. Given an s-tuple $(m_1, \ldots, m_s)$ of non-negative integers such that $\sum_{i=1}^{s} m_i t_i = \sum_{i=1}^{s} m_i b_i = N$, we take for each i, m_i many copies of the pictures associated to $T(i)^{u_1 \ldots u_{t_i}}_{l_1 \ldots l_{b_i}}$ and arrange them in order from $i = 1$ to $i = s$ and then label all the in-arrows and all the out-arrows from 1 to N in the order that they appear. To the element $\sigma \otimes c_1^{\varepsilon_1} \ldots c_N^{\varepsilon_N} \in Ser_N$ we associate a closed picture obtained by joining the in-arrow labelled i with the out-arrow labelled $\sigma^{-1}(i)$ along with the colour prescribed by ε_i, which we denote by dotted arrows if ε_i is 1 and solid arrow if it is 0. For example, let $W = V_2^1 \oplus V_1^0 \oplus V_0^1$ and $m_1 = 1; m_2 = 2; m_3 = 1$. Then corresponding to $\sigma \otimes c^{\boldsymbol{\varepsilon}} = (1\ 2\ 3) \otimes c_1 c_3$ we associate the closed picture below:



This picture corresponds to the coloured picture invariant $\varphi_{\sigma \otimes c^{\boldsymbol{\varepsilon}}}$ as defined earlier in (6.1).

6.0.2. *Coloured picture invariants in terms of queer traces.* For an element $\alpha \in \Lambda Ser_N \subset \mathrm{End}_\Lambda(U^{\otimes N})$, let $\hat{\alpha}$ denote the endomorphism $\alpha \otimes id$ on $U^{\otimes N} \otimes_\Lambda (U^*)^{\otimes N}$. In order to avoid confusion with the above notation, we denote the element $\sigma \otimes c_1^{\varepsilon_1} \ldots c_N^{\varepsilon_N} \in \Lambda Ser_N$ by $\sigma c_1^{\varepsilon_1} \ldots c_N^{\varepsilon_N}$. The following lemma is the analogue of Lemma 4.9 in the case of $Q(U)$, (see also [2, Lemma 5.8])

Lemma 6.2. *For a $\sigma \in S_N$ and $c_1^{\varepsilon_1} \ldots c_N^{\varepsilon_N} \in \mathbf{Cl}_N$, we have*

$$ev(\nu\tau.(\widehat{\sigma c_1^{\varepsilon_1} \ldots c_N^{\varepsilon_N}}).\tau^{-1})(v_1 \otimes \phi_1 \otimes \cdots \otimes v_N \otimes \phi_N) = \pm tr_\sigma(P^{\varepsilon_1} v_1 \otimes \phi_1, \ldots, P^{\varepsilon_N} v_N \otimes \phi_N),$$

where $(v_1 \otimes \phi_1 \otimes \cdots \otimes v_N \otimes \phi_N) \in ((U_1^1)_0)^{\otimes N}$.

Proof. Consider the action of $c_1^{\varepsilon_1} \ldots c_N^{\varepsilon_N}$ followed by σ from the left on $\tau^{-1}.(v_1 \otimes \phi_1 \otimes \cdots \otimes v_N \otimes \phi_N)$ where $|v_i| = \alpha_i = |\phi_i|$. This gives the vector $(-1)^{p(\underline{v},\underline{\phi}) + \sum_{i>j} \varepsilon_i \alpha_j} \gamma((\varepsilon_1 + \alpha_1, \ldots, \varepsilon_N + \alpha_N), \sigma^{-1})(P^{\varepsilon_{\sigma^{-1}(1)}} v_{\sigma^{-1}(1)} \otimes \ldots \otimes P^{\varepsilon_{\sigma^{-1}(N)}} v_{\sigma^{-1}(N)} \otimes \phi_1 \otimes \ldots \otimes \phi_N)$. Applying $\nu\tau$ to the above vector followed by the evaluation we get $\prod_{i=1}^{N} \phi_i(P^{\varepsilon_{\sigma^{-1}(i)}} v_{\sigma^{-1}(i)})$, up to a sign. Here the sign is given by

$$(-1)^{\sum_{i>j}(\alpha_i+\varepsilon_i)\alpha_j} \gamma((\varepsilon_1+\alpha_1, \ldots, \varepsilon_N+\alpha_N), \sigma^{-1}) \gamma((\sigma.(\varepsilon_1+\alpha_1, \ldots, \varepsilon_N+\alpha_N), \alpha_1, \ldots, \alpha_N), (\nu\tau)^{-1}).$$

Hence we note that

$$ev(\nu\tau.(\widehat{\sigma \otimes c_1^{\varepsilon_1} \ldots c_N^{\varepsilon_N}}).\tau^{-1})(v_1 \otimes \phi_1 \otimes \cdots \otimes v_N \otimes \phi_N)) = ev((\nu\tau.\widehat{\sigma}.\tau^{-1}).(P^{\varepsilon_1} v_1 \otimes \phi_1 \otimes \cdots \otimes P^{\varepsilon_N} v_N \otimes \phi_N)).$$

From Lemma 4.9 we get that this is equal to $\pm tr_\sigma(P^{\varepsilon_1} v_1 \otimes \phi_1, \ldots, P^{\varepsilon_N} v_N \otimes \phi_N)$. $\square$

We also have the following analogue of Proposition 4.8. The proof is similar to the proof given there. Let $T_{\sigma\otimes c^{\varepsilon}}$ be the linear map $ev(\nu\tau.(\sigma\widehat{c_1^{\varepsilon_1}\dots c_N^{\varepsilon_N}}).\tau^{-1})$ of the above lemma.

Proposition 6.3. *The coloured picture invariant $\varphi_{\sigma\otimes c^{\varepsilon}} \in S^{(m_1,\dots,m_s)}(W^*)$ maps under restitution F^r to $F^r(\varphi_{\sigma\otimes c^{\varepsilon}})$ given by $F^r(\varphi_{\sigma\otimes c^{\varepsilon}})(\boldsymbol{u}) = T_{\sigma\otimes c^{\varepsilon}}(u_1^{\otimes m_1} \otimes \cdots \otimes u_s^{\otimes m_s})$ where $\boldsymbol{u} = (u_1 \dots, u_s) \in W_0$.* □

Identifying $U \otimes U^*$ with $\mathrm{End}(U)$, we say that a vector in $U \otimes U^*$ *commutes* with P if it does so as an endomorphism of U. We denote by $\mathcal{S}$ the subspace of vectors in $(U \otimes U^*)_0$ which commute with P. We then have the following result which agrees with Theorem 5.13 of [2].

Corollary 6.4. *The invariant polynomials for the simultaneous action of $Q(U)$ on vectors in $\oplus_{i=1}^s (U_1^1)_0$ which commute with P, are spanned by the trace monomials tr_σ given by*

$$\mathrm{qtr}(A_{f(i_1)} \cdots A_{f(i_r)})\mathrm{qtr}(A_{f(i_{r+1})} \cdots A_{f(i_t)}) \cdots$$

where $A_1, \dots, A_s \in \mathcal{S}$, $\sigma = (i_1 \dots i_r)(i_{r+1} \dots i_t) \cdots \in S_n$ and a map $f : \{1, \dots, n\} \to \{1, \dots, s\}$ for $n \in \mathbb{N}$.

Proof. Theorem 6.1 shows that the elements $\varphi_{\sigma\otimes c^{\varepsilon}}$ span $S(W^*)^{Q(U)}$. The linear functional on $\oplus_{i=1}^s (U_1^1)_0$ (and hence also on the vectors in $\oplus_{i=1}^s (U_1^1)_0$ which commute with P) corresponding to $\varphi_{\sigma\otimes c^{\varepsilon}}$ is given by $ev(\nu\tau.(\sigma \otimes \widehat{c_1^{\varepsilon_1}\dots c_N^{\varepsilon_N}}).\tau^{-1})$. By Proposition 6.3 and Lemma 6.2 we get that $\varphi_{\sigma\otimes c^{\varepsilon}}$ is $tr_\sigma(P^{\varepsilon_1}v_1 \otimes \phi_1, \dots, P^{\varepsilon_N}v_N \otimes \phi_N)$. Then (5) of §5.3, gives the required result. □

Acknowledgements The authors would like to thank the referee for very careful reading and many helpful comments.

Declaration of competing interest The authors have no competing interests to declare that are relevant to the content of this article.

Data availability No data was used for the research described in the article.

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