# Lecture 30 : Maxima, Minima, Second Derivative Test

In calculus of single variable we applied the Bolzano-Weierstrass theorem to prove the existence of maxima and minima of a continuous function on a closed bounded interval. Moreover, we developed first and second derivative tests for local maxima and minima. In this lecture we will see a similar theory for functions of several variables.

**Definition :** A non-empty subset $D$ of $\mathbb{R}^n$ is said to be closed if a sequence in $D$ converges then its limit point lies in $D$.

For example, the sets $B_1 = \{X_0 \in \mathbb{R}^2 : \| X \| \leq 1\}$ and $H = \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\}$ are closed subsets of $\mathbb{R}^2$. However, the sets $S_1 = \{X \in \mathbb{R}^2 : \| X \| < 1\}$ and $H^+ = \{(x, y) \in \mathbb{R}^2 : x > 0, y > 0\}$ are not closed.

**Definition :** Let $D \subseteq \mathbb{R}^n$ and $X_0 \in D$. We say that $X_0$ is an interior point of $D$ if there exists $r > 0$ such that the neighborhood $N_r(X_0) = \{X \in \mathbb{R}^n : \| X_0 - X \| < r\}$ is contained in $D$.

For example, all the points of $S_1$ are interior points of $B_1$. Similarly, all the points of $H^+$ are interior points of $H$.

The notions of maxima, minima, local maxima and local minima are similar to the ones defined for the functions of one variable. The proof of the following theorem is similar to the proof of the existence of maximum and minimum of a continuous function on a closed bounded interval.

**Theorem 30.1(Existence of Maxima and Minima):** *Let $D$ be a closed and bounded subset of $\mathbb{R}^2$ and $f : D \to \mathbb{R}$ be continuous. Then $f$ has a maximum and a minimum in $D$.*

**Theorem 30.2(Necessary Condition for Local Maximum and Minimum):** *Suppose $D \subseteq \mathbb{R}^2$, $f : D \to \mathbb{R}$ and $(x_0, y_0)$ is an interior point of $D$. Let $f_x$ and $f_y$ exist at the point $(x_0, y_0)$. If $f$ has a local maximum or local minimum at $(x_0, y_0)$ then $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$.*

**Proof :** Note that (the one variable) functions $f(x, y_0)$ and $f(x_0, y)$ have local maximum or minimum at $x_0$ and $y_0$ respectively. Therefore, the derivatives of these functions are zero at $x_0$ and $y_0$ respectively. That is, $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$. □

Note that the conditions given in the previous results are not sufficient. For example, consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = xy$. Note that $f_x(0, 0) = f_y(0, 0) = 0$ but $(0, 0)$ is neither a local minimum nor a local maximum for $f$.

**Second Derivative Test for Local Maximum and Local Minimum :** Suppose $D \subseteq \mathbb{R}^2$ and $f : D \to \mathbb{R}$. Suppose $f_x$ and $f_y$ are continuous and they have continuous partial derivatives on $D$. With these assumptions we prove the following result.

**Theorem 30.3:** *Let $(x_0, y_0)$ be an interior point of $D$ and $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$. Suppose $(f_{xx}f_{yy} - f_{xy}^2)(x_0, y_0) > 0$. Then*

*(i) if $f_{xx}(x_0, y_0) > 0$ then $f$ has a local minimum at $(x_0, y_0)$.*

*(ii) if $f_{xx}(x_0, y_0) < 0$ then $f$ has a local maximum at $(x_0, y_0)$.*

**Proof (*):** We prove (i) and the proof of (ii) is similar. Suppose $f_{xx}(x_0, y_0) > 0$ and $(f_{xx}f_{yy} - f_{xy}^2)(x_0, y_0) > 0$. Then there exists a neighborhood N of $(x_0, y_0)$, such that

$$f_{xx}(x, y) > 0 \text{ and } (f_{xx}f_{yy} - f_{xy}^2)(x, y) > 0 \quad \text{for all} \quad (x, y) \in N.$$

Let $(x_0 + h, y_0 + k) \in N$. Them by the Extended MVT (applying over N, which is possible), there exists some $C$ lying in the line joining $(x_0+h, y_0+k)$ and $(x_0, y_0)$ such that

$$f(x_0+h, y_0+k) - f(x_0, y_0) = Q(C) = \frac{1}{2}(h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy})(C).$$

Note that $2f_{xx}(C)Q(C) = \{(hf_{xx} + kf_{xy})(C)\}^2 + k^2(f_{xx}f_{yy} - f_{xy}^2)(C) \ > \ 0.$
Since $f_{xx}(C) > 0$ we have $Q(C) > 0$ and hence $f(x_0+h, y_0+k) > f(x_0, y_0)$. Therefore, $f$ has a local minimum at $(x_0, y_0)$. □

**Remarks :** 1. If $(x_0, y_0)$ is an interior point of $D$, $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$ and $(f_{xx}f_{yy} - f_{xy}^2)(x_0, y_0) < 0$, then one can show that in every neighborhood of $(x_0, y_0)$ we can find two points $(x_1, y_1)$ and $(x_2, y_2)$ such that $f(x_1, y_1) > f(x_0, y_0)$ and $f(x_2, y_2) < f(x_0, y_0)$, that is $(x_0, y_0)$ is a saddle point.

2. The above test is inconclusive when $f_x(x_0, y_0) = f_y(x_0, y_0) = (f_{xx}f_{yy} - f_{xy}^2)(x_0, y_0) = 0$.

**Examples :** 1. The functions $f_1(x, y) = -(x^4 + y^4)$ and $f_2(x, y) = x^4 + y^4$ satisfy the above equation for $(x_0, y_0) = (0, 0)$ but $f_1$ has a local maximum at $(0, 0)$ and $f_2$ has a local minimum at $(0, 0)$.

2. Consider the function $f(x, y) = (x+y)^2 - x^4$. This function satisfies the above equation for $(x_0, y_0) = (0, 0)$ but it has neither a local maximum nor a local minimum at $(0, 0)$. In fact, $(0, 0)$ is a saddle point. This can be verified as follows. Note that for $0 < x < 1, f(x, x) > 0$ and $f(x, -x) < 0$.

3. Let $f(x, y) = x \sin y$. Here $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$ for $(x_0, y_0) = (0, n\pi), n \in \mathbb{N}$. Note that $(f_{xx}f_{yy} - f_{xy}^2)(x_0, y_0) < 0$. Therefore, the points $(0, n\pi), n \in \mathbb{N}$ are saddle points.

**Problem 1:** Let $f(x, y) = 3x^4 - 4x^2y + y^2$. Show that $f$ has a local minimum at $(0, 0)$ along every line through $(0, 0)$. Does $f$ have a minimum at $(0, 0)$? Is $(0, 0)$ a saddle point for $f$?

*Solution :* Let $f(x, y) = 3x^4 - 4x^2y + y^2$. Along, the $x$-axis, the local minimum of the function is at $(0, 0)$. Let $x = r\cos\theta$ and $y = r\sin\theta$, for a fixed $\theta \neq 0, \pi$ (or let $y = mx$). Then, $f(r\cos\theta, r\sin\theta) = 3r^4\sin^4\theta - 4r^3\cos^2\theta\sin\theta + r^2\sin^2\theta$ which is a function of one variable. By the second derivative test (for functions of one variable), we see that $(0, 0)$ is a local minima. Since, $f(x, y) = (3x^2 - y)(x^2 - y)$, we see that in the region between the parabolas $3x^2 = y$ and $y = x^2$, the function takes negative values and is positive everywhere else. Thus, $(0, 0)$ is a saddle point for $f$.

**Problem 2:** Let $D = [-2, 2] \times [-2, 2]$ and $f : D \to \mathbb{R}$ be defined as $f(x, y) = 4xy - 2x^2 - y^4$. Find absolute maxima and absolute minima of $f$ in $D$.

**Solution (Hints) :** Note that $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$ for $(x_0, y_0) = (0, 0), (1, 1)$ or $(-1, -1)$. Since these points lie in the interior of $D$, these are the candidates for maxima and minima for $f$ on the set of interiors of $D$.

Now we have to check the behavior of the function over the boundary of $D$. Note that $(x, y) \in D$ is a boundary point if and only if $x = \pm 2$ or $y = \pm 2$. So we have to consider the functions $f(2, y), f(-2, y), f(x, 2)$ and $f(x, -2)$ over the interval $[-2, 2]$. For example, $f(2, y) = 8y - 8 - y^4, y \in [-2, 2]$, has absolute maximum at $y = \sqrt[3]{2}$ and absolute minimum at $y = -2$. So, $(2, \sqrt[3]{2})$ and $(2, -2)$ are the candidates for maxima and minima on the boundary line $\{(2, y) : y \in [-2, 2]\}$. Find all possible candidates for maxima and minima and choose the maxima and minima from these candidates.

The absolute maximum value of $f$ on $D$ is 1 which is obtained at $(1, 1)$ and $(-1, -1)$. The absolute minimum value of $f$ on $D$ is -40 and which is obtained at $(2, -2)$ and $(-2, 2)$.