Lecture 1: The Real Number System

In this note we will give some idea about the real number system and its properties.

We start with the set of integers. We know that given any two integers, these can be added, one can be subtracted from the other and they can be multiplied. The result of each of these operations is again an integer. Further, if p and q are integers and $p - q > 0$, then we say that $p > q$ (this defines an order relation '$<$' on the set of integers.)

Next we consider numbers of the form m/n, where m and n are integers and $n \neq 0$, called rational numbers. (We may assume that m and n have no common factor.) The operations of addition (and subtraction), multiplication and '$<$' extend to this set in a natural way. We shall see, from a very simple situation that numbers other than rational numbers are needed. Consider a square whose side has unit length. Then by Pythagoras Theorem, the length l of the diagonal must satisfy $l^2 = 2$ (we write $l = \sqrt{2}$). What is l? Suppose $l = m/n$, where m and n are integers, which are not both even. Then $l^2n^2 = 2n^2 = m^2$. Thus m^2 is even. Since the square of an odd integer is odd, we conclude that m is even, so that n^2 is even. Hence n is divisible by 2. This contradicts our assumption.

The above discussion shows that $\sqrt{2}$ is not a rational number and we need numbers such as $\sqrt{2}$. Such numbers will be called irrational numbers. The problem now is to define these numbers from $\mathbb{Q}$.

We will not present the mathematical definition of real numbers here, *since it is bit involved.* Instead we will give a rough idea about real numbers.

On a straight line, if we mark off segments ..., $[-1, 0], [0, 1], [1, 2], ...$ then all the rational numbers can be represented by points on this straight line. The set of points representing rational numbers seems to fill up this line (rational number $\frac{r+s}{2}$ lies in between the rational numbers r and s). But we have seen above that the rationals do not cover the entire straight line. Intuitively we feel that there should be a larger set of numbers, say $\mathbb{R}$ such that there is a correspondence between $\mathbb{R}$ and the points of this straight line. Indeed, one can construct such a set of numbers from the rational number system $\mathbb{Q}$, called set of real numbers, which contains the set of rationals and also numbers such as $\sqrt{2}, \sqrt{3}, \sqrt{5}$ and more. Moreover, on this set we can define operations of addition and multiplication, and an order in such a way that when these operations are restricted to the set of rationals, they coincide with the usual operations and the usual order. The set $\mathbb{R}$ with these operations is called the real number system.

An important property of $\mathbb{R}$, which is missing in $\mathbb{Q}$ is the following.

Completeness property of real number system:

A subset A of $\mathbb{R}$ is said to be *bounded above* if there is an element $x_0 \in \mathbb{R}$ such that $x \le x_0$ for all $x \in A$. Such an element x_0 is called an *upper bound* of A . Similarly A is said to be *bounded below* if there exists $y_0 \in \mathbb{R}$ such that $y_0 \le x$ for all $x \in \mathbb{R}$.

An upper bound x_0 of A is said to be a *least upper bound* (l.u.b.) or supremum (sup) of A if whenever z is an upper bound of A, $x_0 \le z$. A *greatest lower bound* (g.l.b.) or infimum (inf) is defined similarly.

Remark. The least upper bound or the greatest lower bound may not belong to the set A. For example, 1 is the l.u.b of the sets $\{x : 0 < x < 1\}, \{x : 0 \le x \le 1\}$ and $\{1 - \frac{1}{n} : n \in \mathbb{N}\}$.

Real number system has the property that every non-empty subset of $\mathbb{R}$ which is bounded above has a least upper bound. This property is called *least upper bound property.* The greatest lower bound property is defined similarly. By *completeness property* we mean either l.u.b. property or g.l.b. property.

We will now show that the following properties which will be used later are important consequences of the completeness property of $\mathbb{R}$.

Proposition 1.1 (Archimedean property): *If $x, y \in \mathbb{R}$ and $x > 0$, then there is a positive integer n such that $nx > y$.*

Proof (*): Suppose that $nx \le y$ for every positive integer n. Then y is an upper bound of the set $A = \{nx : n \in \mathbb{N}\}$. By the least upper bound property, let α be a l.u.b. of A. Then $(n+1)x \le \alpha$ for all n and so $nx \le \alpha - x < \alpha$ for all n i.e. $\alpha - x$ is also an upper bound which is smaller than α. This is a contradiction. □

Remark : Let $A = \{r \in \mathbb{Q} : r > 0, r^2 < 2\}$; this is a non-empty and bounded subset of $\mathbb{Q}$. The set A does not have l.u.b. in $\mathbb{Q}$ (see Problem 10 in Practice Problems 1). This shows that $\mathbb{Q}$ does not have the least upper bound property.

The Archimedean property leads to the *"density of rationals in $\mathbb{R}$"* and *"density of irrationals in $\mathbb{R}$"*.

Proposition 1.2: *Between any two distinct real numbers there is a rational number.*

Proof : Suppose $x, y \in \mathbb{R}$, $y - x > 0$. We have to find two integers m and n, $n \neq 0$ such that

$$x < \frac{m}{n} < y \quad \text{i.e.,} \quad x < \frac{m}{n} < x + (y - x).$$

Now by the Archimedean property there exists a positive integer n such that $n(y-x) > 1$. Then we can find an integer m lying between nx and $ny = nx + n(y-x)$ (see Problem 8 of Practice Problems 1). This proves the result. □

Problem 1 : *Between any two distinct real numbers there is an irrational number.*

Solution : Suppose $x, y \ge 0$, $y - x > 0$. Then $\frac{1}{\sqrt{2}}x < \frac{1}{\sqrt{2}}y$. By Proposition 1.2, there exists a rational number r such that $x < r\sqrt{2} < y$. □

Problem 2 : *Find the supremum and the infimum of the set $\left\{\frac{m}{m+n} : m, n \in \mathbb{N}\right\}$.*

Solution : First note that $0 < \frac{m}{m+n} < 1$. We guess that $\inf = 0$ because $\frac{1}{1+n}$ is in the set and it approaches 0 when n is very large. Formally to show that 0 is the infimum, we have to show that 0 is a lower bound and it is the least among all the lower bounds of the set. It is clear that 0 is a lower bound. It remains to show that a number $\alpha > 0$ cannot be a lower bound of the given set. This is true because we can find an n such that $\frac{1}{1+n} < \alpha$ using the Archimedean property. Note that $\frac{1}{1+n}$ is in the given set! Similarly we can show that $\sup = 1$. □

The following problem will be used later.

Problem 3 : *Let A be a nonempty subset of $\mathbb{R}$ and α a real number. If $\alpha = supA$ then $a \le \alpha$ for all $a \in A$ and for any $\varepsilon > 0$, there is some $a_0 \in A$ such that $\alpha - \varepsilon < a_0$.*

Solution : Suppose $\alpha = \sup A$. Since it is an upper bound we have $a \le \alpha$ for all $a \in A$. Suppose $\varepsilon > 0$. If there is no $a \in A$ such that $\alpha - \varepsilon < a$, then we have $a \le \alpha - \varepsilon < \alpha$ for all $a \in A$. This implies that $\alpha - \varepsilon$ is an upper bound. This contradicts the fact that α is the least upper bound. □