Lecture 15-16 : Riemann Integration

Integration is concerned with the problem of finding the area of a region under a curve.

Let us start with a simple problem : *Find the area A of the region enclosed by a circle of radius r.* For an arbitrary n, consider the n equal inscribed and superscibed triangles as shown in Figure 1.

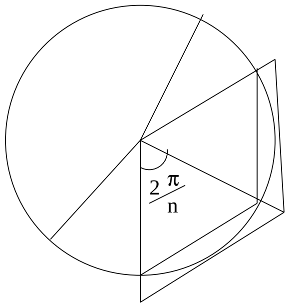

Figure 1

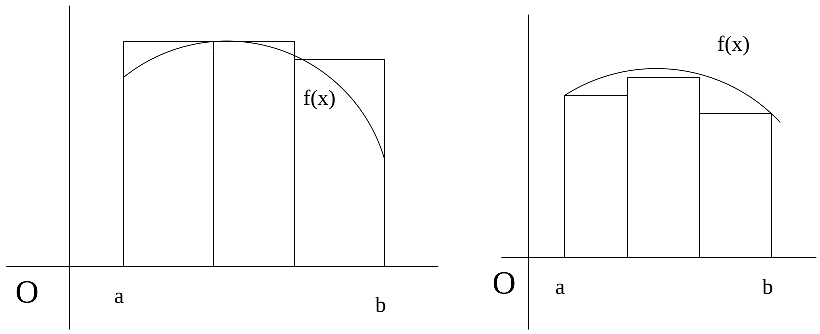

Figure 2

Since A is between the total areas of the inscribed and superscribed triangles, we have

$$nr^2 sin(\pi/n)cos(\pi/n) \leq A \leq nr^2 tan(\pi/n).$$

By sandwich theorem, $A = \pi r^2$. **We will use this idea to define and evaluate the area of the region under a graph of a function.**

Suppose f is a non-negative function defined on the interval $[a, b]$. We first subdivide the interval into a finite number of subintervals. Then we squeeze the area of the region under the graph of f between the areas of the inscribed and superscribed rectangles constructed over the subintervals as shown in Figure 2. **If** the total areas of the inscribed and superscribed rectangles converge to the same limit as we make the partition of $[a, b]$ finer and finer then the area of the region under the graph of f can be defined as this limit and f is said to be integrable.

Let us define whatever has been explained above formally.

The Riemann Integral

Let $[a, b]$ be a given interval. A *partition* P of $[a, b]$ is a finite set of points $x_0, x_1, x_2, \ldots, x_n$ such that $a = x_0 \leq x_1 \leq \cdots \leq x_{n-1} \leq x_n = b$. We write $P = \{x_0, x_1, x_2, \ldots, x_n\}$.

If P is a partition of $[a, b]$ we write $\Delta x_i = x_i - x_{i-1}$ for $1 \leq i \leq n$. Let f be a bounded real valued function on $[a, b]$. For a partition P of $[a, b]$, we define

$$M_i = \sup\{f(x) : x_{i-1} \leq x \leq x_i\} \ \ and \ \ m_i = \inf\{f(x) : x_{i-1} \leq x \leq x_i\}.$$

$$U(P, f) = \sum_1^n M_i \Delta x_i \ \ and \ \ L(P, f) = \sum_1^n m_i \Delta x_i.$$

The numbers $U(P, f)$ and $L(P, f)$ are called *upper and lower Riemann sums* for the partition P (see Figure 2).

Since f is bounded, there exist real numbers m and M such that $m \leq f(x) \leq M$, for all $x \in [a, b]$. Thus for every partition P,

$$m(b - a) \leq L(P, f) \leq U(P, f) \leq M(b - a).$$

We define

$$\overline{\int_a^b} f dx = \inf U(P, f) \tag{1}$$

and

$$\underline{\int_a^b} f dx = \sup L(P, f). \tag{2}$$

(1) and (2) are called *upper and lower Riemann integrals* of f over $[a, b]$ respectively.

If the upper and lower integrals are equal, we say that f *is Riemann integrable* or *integrable*. In this case the common value of (1) and (2) is called the Riemann integral of f and is denoted by $\int_a^b f dx$ or $\int_a^b f(x)dx$.

Examples : 1. Consider the function $f : [0, 1] \to \mathbb{R}$ defined by

$$f(\frac{1}{2}) = 1 \text{ and } f(x) = 0 \text{ for all } x \in [0, 1] \backslash \{\frac{1}{2}\}.$$

Then f is integrable. We show this using the definition as follows. For any partition P of $[0, 1]$, $L(P, f)$ is always 0 and hence the lower integral is 0. Let us evaluate the upper integral. Let $P = \{x_1, x_2, .., x_n\}$ be any partition of $[0, 1]$ and $\frac{1}{2} \in [x_i, x_{i+1}]$ for some i. Then $U(P, f) \le 2max\Delta x_j$. Since we can always choose a partition P such that $max\Delta x_j$ is as small as possible, the upper integral, which is the infimum of $U(P, f)'s$, is 0. Hence, f is integrable and $\int_0^1 f(x)dx = 0$.

2. Not every bounded function is integrable. For example the function

$$f(x) = 1 \;\; if\ x\ is\ rational\ and\ 0\ otherwise$$

is not integrable over any interval $[a, b]$ (Check this).

In general, determining whether a bounded function on $[a, b]$ is integrable, using the definition, is difficult. For the purpose of checking the integrability, we give a criterion for integrability, called Riemann criterion, which is analogous to the Cauchy criterion for the convergence of a sequence.

Let us define some concepts and results before presenting the criterion. Throughout, we will assume that f is a bounded real function on $[a, b]$.

Definition: A partition P_2 of $[a, b]$ is said to be finer than a partition P_1 if $P_2 \supset P_1$. In this case we say that P_2 is a *refinement* of P_1. Given two partition P_1 and P_2, the partition $P_1 \cup P_2 = P$ is called their common refinement.

The following theorem illustrates that refining partition increases lower terms and decreases upper terms.

Theorem 1 : *Let P_2 be a refinement of P_1 then $L(P_1, f) \le L(P_2, f)$ and $U(P_2, f) \le U(P_1, f)$.*

Proof (*): First we assume that P_2 contains just one more point than P_1. Let this extra point be $x^\star$. Suppose $x_{i-1} < x^\star < x_i$, where x_{i-1} and x_i are consecutive points of P_1. Let

$$\begin{aligned} w_1 &= \inf\{f(x) : \; x_{i-1} \le x \le x^\star\} \;\text{ and} \\ w_2 &= \inf\{f(x) : \; x^\star \le x \le x_i.\} \end{aligned}$$

Then $w_1 \ge m_i$ and $w_2 \ge m_i$ where $m_i = \inf\{f(x) : \; x_{i-1} \le x \le x_i\}$. Then

$$L(P_2, f) - L(P_1, f) = w_1(x^\star - x_{i-1}) + w_2(x_i - x^\star) - m_i(x_i - x_{i-1}) \ge 0.$$

If P_2 contains k more points then we repeat this process $k-$times. The other inequality is analogously proved. (Prove it). □

The geometric interpretation suggests that the lower integral is less than or equal to the upper integral. So the next result is also anticipated.

Corollary 2 : $\overline{\int_a^b} f dx \geq \underline{\int_a^b} f dx$.

Proof (*): Let P_1, P_2 be two partitions and let P be their common refinement. Then

$$L(P_1, f) \leq L(P, f) \leq U(P, f) \leq U(P_2, f).$$

Thus for any two partitions P_1 and P_2, we have $L(P_1, f) \leq U(P_2, f)$.
Fix P_2 and take sup over all P_1. Then $\underline{\int_a^b} f dx \leq U(P_2, f)$. Now take inf over all P_2 to get the desired result. □

In the following result we present the Reimann criterion (a necessary and sufficient condition for the existence of the integral of a bounded function).

Theorem 3 : (Riemann's criterion for integrability): *f is integrable on $[a, b] \Leftrightarrow$ for every $\epsilon > 0$ there exists a partition P such that*

$$U(P, f) - L(P, f) < \epsilon. \tag{3}$$

Proof (*): For any P, we have

$$L(P, f) \leq \underline{\int_a^b} f dx \leq \overline{\int_a^b} f dx \leq U(P, f).$$

Therefore (3) implies that

$$\overline{\int_a^b} f dx - \underline{\int_a^b} f dx < \epsilon, \quad \forall \ \epsilon \ > \ 0.$$

Hence $\underline{\int_a^b} f dx = \overline{\int_a^b} f dx$ i.e. f is integrable. Conversely, suppose f is integrable and $\epsilon > 0$. Then there exist partitions P_1 and P_2 such that

$$U(P_2, f) - \int_a^b f dx < \epsilon/2 \ \text{ and } \ \int_a^b f dx - L(P_1, f) < \epsilon/2$$

Let P be the common refinement of P_1 and P_2. Then $U(P, f) - L(P, f) < \epsilon$. □

The proof of the following corollary is immediate from the previous theorem.

Corollary 3 : Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Suppose (P_n) is a sequence of partitions of $[a, b]$ such that $U(P_n, f) - L(P_n, f) \to 0$, then f is integrable.

Problem : Let $f : [0, \ 1] \to \mathbb{R}$ such that $f(x) = \begin{cases} \frac{1}{n} & \text{if } x = \frac{1}{n} \\ 0 & \text{otherwise.} \end{cases}$ Show that f is integrable and find $\int_0^1 f(x) dx$.

Solution : We will use the Riemann criterion to show that f is integrable on $[0, 1]$. Let $\varepsilon > 0$ be given. We will choose a partition P such that $UP, f) - L(P, f) < \varepsilon$. Since $1/n \to 0$, there exists N such that $1/n \in [0, \varepsilon]$ for all $n > N$. So only finite number of $\frac{1}{n}'s$ lie in the interval $[\epsilon, 1]$. Cover these finite number of $\frac{1}{n}'s$ by the intervals $[x_1, x_2], [x_3, x_4], ..., [x_{m-1}, x_m]$ such that $x_i \in [\varepsilon, 1]$ for all

$i = 1, 2, .., m$ and the sum of the length of these m intervals is less than ε. Consider the partition $P = \{0, \varepsilon, x_1, x_2, ..., x_m\}$. It is clear that $U(P, f) - L(P, f) < 2\varepsilon$. Hence by the Reimann criterion the function is integrable. Since the lower integral is 0 and the function is integrable, $\int_0^1 f(x)dx = 0$.

We will apply the Riemann criterion for integrability to prove the following two existence theorems.

Theorem 4: *If f is continuous on $[a, b]$ then f is integrable.*

Proof : Let $\epsilon > 0$. Since f is uniformly continuous, choose $\delta > 0$ such that $|s - t| < \delta \Rightarrow |f(s) - f(t)| < \epsilon$ for $s, t \in [a, b]$.

Let P be a partition of $[a, b]$ such that $\Delta x_i < \delta \;\forall\, i = 1, 2, \ldots, n$. Then

$$M_i - m_i \leq \epsilon \;\forall\, i = 1, 2, \ldots, n.$$

Hence

$$U(P, f) - L(P, f) = \sum_1^n (M_i - m_i)\Delta x_i \leq \epsilon(b - a).$$

This implies that f is integrable. □

Theorem 5: *If f is a monotone function then f integrable.*

Proof : Suppose f is monotonically increasing (the proof is similar in the other case.) Choose a partition P such that $\Delta x_i = \frac{b-a}{n}$. Then $M_i = f(x_i)$ and $m_i = f(x_{i-1})$. Therefore

$$\begin{aligned} U(P, f) - L(P, f) &= \frac{b-a}{n}\sum_1^n [f(x_i) - f(x_{i-1})] \\ &= \frac{b-a}{n}[f(b) - f(a)] \\ &< \epsilon \qquad \text{for large } n. \end{aligned}$$

Hence f is integrable. □

In the following problem we will see that limit and integral cannot be interchanged.

Problem : *Let* $g_n(y) = \begin{cases} \frac{ny^{n-1}}{1+y} & \text{if } 0 \leq y < 1 \\ 0 & \text{if } y = 1 \end{cases}$. *Then prove that* $\lim_{n\to\infty} \int_0^1 g_n(y)dy = \frac{1}{2}$ *whereas* $\int_0^1 \lim_{n\to\infty} g_n(y)dy = 0$.

Solution : From the ratio test for sequences we can show that $\lim_{n\to\infty} \frac{ny^{n-1}}{1+y} = 0$, for each $0 < y < 1$. Therefore $\int_0^1 \lim_{n\to\infty} g_n(y)dy = 0$.

For the other part, use integration by parts to see that $\int_0^1 \frac{ny^{n-1}}{1+y}dy = \frac{1}{2} + \int_0^1 \frac{y^n}{(1+y)^2}dy$. Note that $\int_0^1 \frac{y^n}{(1+y)^2}dy \leq \int_0^1 y^n = \frac{1}{n+1} \to 0$ as $n \to \infty$. Therefore, $\lim_{n\to\infty} \int_0^1 g_n(y)dy = \frac{1}{2}$.