Lecture 17 : Fundamental Theorems of Calculus, Riemann Sum

By looking at the definitions of differentiation and integration, one may feel that these notions are totally different. Even the geometric interpretations do not give any idea that these two notions are related. In this lecture we will discuss two results, called fundamental theorems of calculus, which say that differentiation and integration are, in a sense, inverse operations.

Theorem 17.1: ***(First Fundamental Theorem of Calculus)*** Let f be integrable on $[a, b]$. For $a \leq x \leq b$, let $F(x) = \int_a^x f(t)dt$. Then F is continuous on $[a, b]$ and if f is continuous at x_0 then F is differentiable at x_0 and $F'(x_0) = f(x_0)$.

Proof (*): Suppose $M = \sup\{|f(x)| : x \in [a, b]\}$. Let $a \leq x < y \leq b$. Then

$$|F(y) - F(x)| = |\int_x^y f(t)dt| \leq M(y - x).$$

Thus $|F(x) - F(y)| \leq M|x - y|$ for $x, y \in [a, b]$. Hence F is continuous, in fact uniformly continuous.

Now suppose f is continuous at x_0. Given $\epsilon > 0$ choose $\delta > 0$ such that

$$|t - x_0| < \delta \ \Rightarrow |f(t) - f(x_0)| < \epsilon.$$

Let x be such that $0 \leq |x - x_0| < \delta$. Then

$$\left|\frac{F(x) - F(x_0)}{x - x_0} - f(x_0)\right| = \left|\frac{1}{x - x_0}\int_{x_0}^x [f(t) - f(x_0)]dt\right| < \epsilon.$$

This implies that $F'(x_0) = f(x_0)$. $\square$

In the previous theorem, in a sense, we obtained f by differentiating integral of f when f is continuous on $[a, b]$. A function F such that $F'(x) = f(x) \ \forall \ x \in [a, b]$ is called an antiderivative of f on $[a, b]$. The existence of an antiderivative for a continuous function on $[a, b]$ follows from the first F.T.C.

If an integrable function f has an antiderivative (and if we can find it), then calculating its integral is very simple. The second F.T.C. explains this.

Theorem 17.2 : ***(Second Fundamental Theorem of Calculus)*** Let f be integrable on $[a, b]$. If there is a differentiable function F on $[a, b]$ such that $F' = f$ then $\int_a^b f(x)dx = F(b) - F(a)$.

Proof (*): Let $\epsilon > 0$. Since f is integrable we can find a partition $P := \{x_0, x_1, ..., x_n\}$ of $[a, b]$ such that $U(P, f) - L(P, f) < \epsilon$. By the mean value theorem there exists $c_i \in [x_{i-1}, x_i]$ such that $F(x_i) - F(x_{i-1}) = f(c_i)\Delta x_i$. Hence

$$\sum f(c_i)\Delta x_i = F(b) - F(a).$$

we know that

$$L(P, f) \leq \int_a^b f dx \leq U(P, f)$$

and

$$L(P, f) \leq \sum f(c_i)\Delta x_i \leq U(P, f).$$

Therefore $|F(b) - F(a) - \int_a^b f dx| < \epsilon$. This completes the proof. $\square$

The second F.T.C. explains why the indefinite integral of F' is defined to be F.

Remark : The proof of Theorem 17.2 becomes simpler if, instead of assuming f to be integrable, we make stonger assumption that f is continuous on $[a, b]$. In fact, the proof follows from Theorem 17.1 (prove !).

Problem 1: *Let p be a fixed number and let f be a continuous function on $\mathbb{R}$ that satisfies the equation $f(x+p) = f(x)$ for every $x \in \mathbb{R}$. Show that the integral $\int_a^{a+p} f(t)dt$ has the same value for every real number a.*

Solution : Suppose $a, p > 0$. Then by the first F.T.C., we have

$$\frac{d}{da}(\int_a^{a+p} f(t)dt) = \frac{d}{da}(\int_0^{a+p} f(t)dt - \int_0^a f(t)dt) = f(a+p) - f(a) = 0.$$

Problem 2: *Let f be a continuous function on $[0, \pi/2]$ and $\int_0^{\pi/2} f(t)dt = 0$. Show that there exists a $c \in (0, \pi/2)$ such that $f(c) = 2cos2c$.*

Solution : Define F on $[0, \frac{\pi}{2}]$ such that $F(x) = \int\limits_0^x f(t)dt - \sin 2x$. Apply the first F.T.C. and Rolle's theorem.

Problem 3: *Show that $lim_{x\to 0} \frac{1}{x^3} \int\limits_0^x \frac{t^2}{1+t^4} dt = \frac{1}{3}$.*

Solution : Apply the first F.T.C. and the L'Hospital rule.

Riemann Sum

We now see an important property of integrable functions.

Definition: Let $f : [a, b] \to \mathbb{R}$ and let $P = \{x_0, x_1, ..., x_n\}$ be a partition of $[a, b]$. Let $c_k \in [x_{k-1}, x_k], k = 1, 2, ..., n$. Then a *Riemann sum* for f (corresponding to the partition P and the intermediate points c_k) is $S(P, f) = \sum_{k=1}^n f(c_k)\Delta x_k$.

The norm of P is defined by $\| P \| = max_{1\le i\le n} \Delta x_i$.

Theorem 4: *Let $f : [a, b] \to \mathbb{R}$ be integrable. Then $\lim\limits_{\|P\|\to 0} S(P, f) = \int_a^b f(x)dx$.*

We will not present the proof of this theorem, however, we will use it later when we discuss the applications of integration. This result also has some other applications. For example, we can use this to approximate the integral of f when we cannot evaluate it exactly. Moreover, this result can also be used to find limit of certain type of sequences.

Example : Let us evaluate $\lim_{n\to\infty} x_n$ where $x_n = \frac{1}{n} + \frac{1}{n+1} + ... + \frac{1}{2n-1}$ using the above theorem. Basically we have to write x_n as a Riemann sum of some function on some interval. Note that

$$x_n = \frac{1}{n}(\frac{1}{1} + \frac{1}{1+\frac{1}{n}} + \frac{1}{1+\frac{2}{n}} + ... + \frac{1}{1+\frac{n-1}{n}}) = S(P_n, f)$$

where $f(x) = \frac{1}{x}$ for $x \in [1, 2]$ and $P_n = \{1, 1+\frac{1}{n}, 1+\frac{2}{n}, ..., 1+\frac{n}{n}\}$. Note that here $c_{k+1} = \frac{1}{1+\frac{1}{k}}$. Therefore, by the previous theorem we have $\lim_{n\to\infty} x_n = \int_1^2 \frac{1}{x}dx$.