ASSIGNMENT 1
SOLUTIONS
MTH102A

(1) Show that matrix multiplication is associative i.e. $A(BC) = (AB)C$ whenever the multiplication is defined.
Solution. Let $A = (a_{ij})_{m\times n}$,$B = (b_{ij})_{n\times p}$,$C = (a_{ij})_{p\times q}$ be matrices.
We first consider $A(BC)$
Let $R = (r_{ij})_{n\times q} = BC$, $S = (s_{ij})_{m\times q} = A(BC)$, Then

$$s_{ij} = \sum_{l=1}^{n} a_{il} r_{lj}$$

$$r_{lj} = \sum_{k=1}^{p} b_{lk} c_{kj}$$

$$s_{ij} = \sum_{l=1}^{n} a_{il} (\sum_{k=1}^{p} b_{lk} c_{kj})$$

$$\implies s_{ij} = \sum_{l=1}^{n} \sum_{k=1}^{p} a_{il} b_{lk} c_{kj}.$$

Similarly for $(AB)C$, taking $R = (r_{ij})_{m\times p} = AB$ and $S = (s_{ij})_{m\times q} = (AB)C$ we get

$$s_{ij} = \sum_{l=1}^{n} \sum_{k=1}^{p} a_{il} b_{lk} c_{kj}$$

Therefore $A(BC) = (AB)C$.

(2) Suppose A and B are matrices of order $m \times n$ such that $A\bar{x} = B\bar{x}$ for all $\bar{x} \in \mathbb{R}^n$. Prove that $A = B$.
Solution. Let $A = (a_{ij})_{m\times n}$ and $B = (a_{ij})_{m\times n}$ be matrices. Take $\bar{x} = e_j$ where $e_j = (0, .., 1, .., 0) \in \mathbb{R}^n$, (1 at the jth position).
So $Ae_j = Be_j \implies j$th coloumn of A and B is same for $j = 1 \textit{ to } n$
Hence $A = B$.

(3) Let $A = (a_{ij})$ be a matrix. Transpose of A, denoted by A^T, is defined to be $A^T = (b_{ij})$ where $b_{ij} = a_{ji}$.
(i) Show that $(A + B)^T = A^T + B^T$, whenever $A + B$ is defined
(ii) Show that $(AB)^T = B^T A^T$, whenever AB is defined.
Solution. (i) Let $A = (a_{ij})_{n\times n}$, $A^T = (a'_{ij})_{n\times n}$, $B = (b_{ij})_{n\times n}$, $B^T = (b'_{ij})_{n\times n}$, $(A+B) = (c_{ij})$ and $(A+B)^T = (d_{ij})$. Then $d_{ij} = c_{ji} = a_{ji}+b_{ji} = (a'_{ij}+b'_{ij}$.
Hence $(A + B)^T = A^T + B^T$.

(ii) Let $AB = (c_{ij})_{n\times n}$ and $(AB)^T = (d_{ij})_{n\times n}$. Then $d_{ij} = c_{ji} = \sum_{k=1}^{n} a_{jk}b_{ki} = \sum_{k=1}^{n} a'_{kj}b'_{ik} = \sum_{k=1}^{n} b'_{ik}a'_{kj}$. Hence $(AB)^T = B^T A^T$.

(4) A square matrix A is said to be *symmetric* if $A = A^T$ and a square matrix A is said to be *skew symmetric* if $A = -A^T$.
Prove that a square matrix can be written as a sum of symmetric and skew symmetric matrix.
Solution. $A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$.
Now $\left(\frac{1}{2}(A + A^T)\right)^T = \frac{1}{2}(A^T + (A^T)^T) = \frac{1}{2}(A + A^T)$.
$\left(\frac{1}{2}(A - A^T)\right)^T = \frac{1}{2}(A^T - (A^T)^T) = -\frac{1}{2}(A + A^T)$.

(5) A square matrix A is said to be *nilpotent* if $A^n = 0$ for some natural number n.
(i) Give examples of non-zero nilpotent matrices,
(ii) Prove that if A is nilpotent then $A + I$ is an invertible matrix, where I is identity matrix.
Solution. (i) Strictly upper triangular matrices (i.e. $a_{ij} = 0$ for all $i \leq j$) are nilpotent.

(ii) $(I + A)(I - A + ... + (-1)^{n-1}A^{n-1}) = I + (-1)^{n-1}A^n = I$

(6) Trace of a square matrix A, denoted by $Tr(A)$, is defined to be the sum of all diagonal entries.
(i) Suppose A, B are two square matrices of same order. Prove that $Tr(AB) = Tr(BA)$.
(ii) Show that if A is invertible then $Tr(ABA^{-1}) = Tr(B)$.
Solution. (i) Let $A = (a_{ij})_{n\times n}$, $B = (b_{ij})_{n\times n}$, $AB = (c_{ij})_{n\times n}$ and $BA = (d_{ij})_{n\times n}$. Then $c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$ and $d_{ij} = \sum_{k=1}^{n} b_{ik}a_{kj}$. So $Tr(AB) = \sum_{i=1}^{n} c_{ii} = \sum_{i=1}^{n}\sum_{k=1}^{n} a_{ik}b_{ki} = \sum_{k=1}^{n}\sum_{i=1}^{n} b_{ki}a_{ik} = \sum_{k=1}^{n} d_{kk} = Tr(BA)$.

(ii) $Tr(ABA^{-1}) = Tr((AB)A^{-1}) = Tr(A^{-1}(AB)) = Tr(B)$

(7) Apply Gauss elimination method to solve the system $2x + y + 2z = 3$, $3x - y + 4z = 7$ and $4x + 3y + 6z = 5$.
Solution. The augmented matrix is

$$\left[\begin{array}{ccc|c} 2 & 1 & 2 & 3 \\ 3 & -1 & 4 & 7 \\ 4 & 3 & 6 & 5 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 2 & 1 & 2 & 3 \\ 3 & -1 & 4 & 7 \\ 4 & 3 & 6 & 5 \end{array}\right] \xrightarrow{R_2 \to R_2 - (3/2)R_1} \left[\begin{array}{ccc|c} 2 & 1 & 2 & 3 \\ 0 & -5/2 & 1 & 5/2 \\ 4 & 3 & 6 & 5 \end{array}\right] \xrightarrow{R_3 \to R_3 - 2R_1} \left[\begin{array}{ccc|c} 2 & 1 & 2 & 3 \\ 0 & -5/2 & 1 & 5/2 \\ 0 & 1 & 2 & -1 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 + (2/5)R_2} \left[\begin{array}{ccc|c} 2 & 1 & 2 & 3 \\ 0 & -5/2 & 1 & 5/2 \\ 0 & 0 & 12/5 & 0 \end{array}\right]$$

We can thus obtain the solution to the given linear system by solving the equivalent system

$$\begin{aligned} 2x + y + 2z &= 3 \\ (-5/2)y + z &= 5/2 \\ (12/5)z &= 0 \end{aligned}$$

The solution is $x = 2, y = -1$ and $z = 0$.

(8) Find the row reduced echelon form of the matrix $\begin{bmatrix} 2 & 6 & -2 \\ 3 & -2 & 8 \end{bmatrix}$.

Solution.

$$\begin{bmatrix} 2 & 6 & -2 \\ 3 & -2 & 8 \end{bmatrix} \xrightarrow{R_1 \to \frac{1}{2}R_1} \begin{bmatrix} 1 & 3 & -1 \\ 3 & -2 & 8 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 3R_1} \begin{bmatrix} 1 & 3 & -1 \\ 0 & -11 & 11 \end{bmatrix}$$

$$\xrightarrow{R_2 \to \frac{-1}{11}R_2} \begin{bmatrix} 1 & 3 & -1 \\ 0 & 1 & -1 \end{bmatrix} \xrightarrow{R_1 \to R_1 - 3R_2} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix}$$