ASSIGNMENT 2
MTH102A

(1) Using Gauss Jordan elimination method find the inverse of $\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 2 & 1 & 2 & 0 & 1 & 0 \\ 2 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & -3 & -2 & -2 & 1 & 0 \\ 0 & -2 & -3 & -2 & 0 & 1 \end{array}\right] \xrightarrow{R_3 \to R_3 - (2/3)R_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & -3 & -2 & -2 & 1 & 0 \\ 0 & 0 & -5/3 & -2/3 & -2/3 & 1 \end{array}\right] \xrightarrow{R_3 \to (-3/5)R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & -3 & -2 & -2 & 1 & 0 \\ 0 & 0 & 1 & 2/5 & 2/5 & -3/5 \end{array}\right]$$

$$\xrightarrow[R_1 \to R_1 - 2R_3]{R_2 \to R_2 + 2R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1/5 & -4/5 & 6/5 \\ 0 & -3 & 0 & -6/5 & 9/5 & -6/5 \\ 0 & 0 & 1 & 2/5 & 2/5 & -3/5 \end{array}\right] \xrightarrow{R_2 \to (-1/3)R_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1/5 & -4/5 & 6/5 \\ 0 & 1 & 0 & 2/5 & -3/5 & 2/5 \\ 0 & 0 & 1 & 2/5 & 2/5 & -3/5 \end{array}\right] \xrightarrow{R_1 \to R_1 - 2R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -3/5 & 2/5 & 2/5 \\ 0 & 1 & 0 & 2/5 & -3/5 & 2/5 \\ 0 & 0 & 1 & 2/5 & 2/5 & -3/5 \end{array}\right]$$

Thus, the inverse is $\begin{bmatrix} -3/5 & 2/5 & 2/5 \\ 2/5 & -3/5 & 2/5 \\ 2/5 & 2/5 & -3/5 \end{bmatrix}$.

(2) Let $\sigma \in S_5$ be given by

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 1 & 2 & 3 \end{pmatrix}$$

(a) Find sign of σ and sign of σ^{-1},

(b) Find $\sigma^2 = \sigma \circ \sigma$.

Ans: (a) In the cycle notation the above permutation can be written as $\sigma = (153)(24) = (15)(53)(24)$. There are odd number of transpositions required to express σ. So $sgn(\sigma) = -1$.

$\sigma^{-1} = (24)(53)(15)$. So $sgn(\sigma^{-1}) = -1$.

(b) $\sigma^2(1) = 3, \sigma^2(2) = 2, \sigma^2(3) = 5, \sigma^2(4) = 4, \sigma^2(5) = 1$. So $\sigma^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 5 & 4 & 1 \end{pmatrix}$.

(3) Using the definition compute the determinant of $\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$.

Ans: We have $S_3 = \{identity, (12), (23), (13), (123), (132)\}$

Recall that $Sgn(\sigma) = (-1)^k$ where k is the number of transpositions required to express σ.

$Sgn(identity) = 1$, $Sgn(12) = -1, Sgn(23) = -1, Sgn(13) = -1$

$(123) = (12)(23)$. So $Sgn(123) = 1$

$(132) = (13)(23)$. So $Sgn(132) = 1$

For a 3×3 matrix $A = (a_{ij})$ we have $det(A) = \sum_{\sigma \in S_3} Sgn(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} a_{3\sigma(3)}$

For $\sigma = identity$ we have $a_{1\sigma(1)} a_{2\sigma(2)} a_{3\sigma(3)} = a_{11}a_{22}a_{33} = 1$

For $\sigma = (12)$ we have $a_{1\sigma(1)} a_{2\sigma(2)} a_{3\sigma(3)} = a_{12}a_{21}a_{33} = 4$

For $\sigma = (13)$ we have $a_{1\sigma(1)} a_{2\sigma(2)} a_{3\sigma(3)} = a_{13}a_{22}a_{31} = 4$

For $\sigma = (23)$ we have $a_{1\sigma(1)} a_{2\sigma(2)} a_{3\sigma(3)} = a_{11}a_{23}a_{32} = 4$

For $\sigma = (123)$ we have $a_{1\sigma(1)} a_{2\sigma(2)} a_{3\sigma(3)} = a_{12}a_{23}a_{31} = 8$

For $\sigma = (132)$ we have $a_{1\sigma(1)} a_{2\sigma(2)} a_{3\sigma(3)} = a_{13}a_{21}a_{32} = 8$

So $det(A) = \sum_{\sigma \in S_3} Sgn(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} a_{3\sigma(3)} = 1 - 4 - 4 - 4 + 8 + 8 = 5$

(4) Let A be a square matrix of order n. Show that $det(A) = 0$ if and only if there exists a non-zero vector $X = (x_1, x_2, ..., x_n)$ such that $AX^T = 0$.

Ans: If $det(A) \neq 0$ then A is invertible and so the only solution of the system $AX^T = 0$ is $\underline{0}$.

Conversely if $det(A) = 0$ then A is not invertible and hence the rref of A has a zero row and hence the system has a free variable. We can pick the value of the free variable as we please, specifically not 0, and get a non-trivial solution.

(5) (Vandermonde Matrix) Find the determinant of the following matrix:

$$\begin{pmatrix} 1 & x_1 & x_1^2 & ... & x_1^{n-1} \\ 1 & x_2 & x_2^2 & ... & x_2^{n-1} \\ .. & .. & .. & ... & .. \\ 1 & x_n & x_n^2 & ... & x_n^{n-1} \end{pmatrix}$$

Ans: Let

$$A_n = \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ . & . & . & \cdots & . \\ . & . & . & \cdots & . \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{bmatrix}.$$

If $n = 2$, $\det(A_2) = x_2 - x_1$. We will prove that

$$\det(A_n) = \prod_{i<j} (x_j - x_i).$$

Assume the result for $n - 1$ and define

$$F(x) = \begin{vmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ . & . & . & \cdots & . \\ . & . & . & \cdots & . \\ 1 & x & x^2 & \cdots & x^{n-1} \end{vmatrix}.$$

Then F is a polynomial of degree $n-1$ with roots $x_1, x_2, \ldots, x_{n-1}$. So, $F(x) = c\prod_{i=1}^{n-1}(x - x_i)$ where c is coefficient of x^{n-1} which is clearly $\det(A_{n-1})$. Therefore,

$$F(x) = \det(A_{n-1})\prod_{i=1}^{n-1}(x - x_i).$$

The result follows for n as

$$\det(A_n) = F(x_n) = \det(A_{n-1})\prod_{i=1}^{n-1}(x_n - x_i).$$

(6) Let A be a $n \times n$ real matrix. Show that $det(adj(A)) = (det(A))^{n-1}$ and $adj(adj(A)) = (det(A))^{n-2}.A$.

Ans: Note that for a matrix A we have $adj(A).A = det(A).I_n$. If $det(A) = 0$ then $adj(A)$ is not invertible and hence $det(adj(A)) = 0$.

Assume that $det(A) \neq 0$. Then taking determinant both sides of the above identity we have $det(adj(A)).det(A) = (det(A))^n$. So $det(adj(A)) = (det(A))^{n-1}$.

Again $adj(A).adj(adj(A)) = det(adj(A)).I_n = (det(A))^{n-1}.I_n$.

Multiplying by A we get $A.adj(A).adj(adj(A)) = (det(A))^{n-1}.A$. So $det(A).adj(adj(A)) = (det(A))^{n-1}.A$ and hence $adj(adj(A)) = (det(A))^{n-2}.A$.

(7) Using Cramer's rule solve the following system:

$$x + 2y + 3z = 1$$
$$-x + 2z = 2$$
$$-2y + z = -2$$

Ans: Here $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 2 \\ 0 & -2 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $\underline{\mathbf{d}} = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$

We have $A_1 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 2 \\ -2 & -2 & 1 \end{bmatrix}$, $A_2 = \begin{bmatrix} 1 & 1 & 3 \\ -1 & 2 & 2 \\ 0 & -2 & 1 \end{bmatrix}$ and $A_3 = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 2 \\ 0 & -2 & -2 \end{bmatrix}$

Note that $det(A) = 12, det(A_1) = -20, det(A_2) = 13$ and $det(A_3) = 2$.

So we have $x = \frac{det(A_1)}{det(A)} = \frac{-20}{12}$, $y = \frac{det(A_2)}{det(A)} = \frac{13}{12}$ and $z = \frac{det(A_3)}{det(A)} = \frac{2}{12}$.