ASSIGNMENT 3
MTH102A

(1) In $\mathbb{R}$, consider the addition $x \oplus y = x + y - 1$ and the scalar multiplication $\lambda.x = \lambda(x-1)+1$. Prove that $\mathbb{R}$ is a vector space over $\mathbb{R}$ with respect to these operations. What is the additive identity (the $\mathbf{0}$ vector in the definition) in this case ?

Solution: Easy verification. Here the $\mathbf{0}$ vector is $1 \in \mathbb{R}$.

(2) Show that $W = \{(x_1, x_2, x_3, x_4) : x_4 - x_3 = x_2 - x_1\}$ is a subspace of $\mathbb{R}^4$ spanned by vectors $(1, 0, 0, -1), (0, 1, 0, 1), (0, 0, 1, 1)$.

Solution:

$$\begin{aligned} (x_1, x_2, x_3, x_4) &\in \{(x_1, x_2, x_3, x_4) : x_4 - x_3 = x_2 - x_1\} \\ \Leftrightarrow (x_1, x_2, x_3, x_4) &= (x_1, x_2, x_3, -x_1 + x_2 + x_3) \text{ as } x_4 = -x_1 + x_2 + x_3 \\ &= x_1(1, 0, 0, -1) + x_2(0, 1, 0, 1) + x_3(0, 0, 1, 1) \end{aligned}$$

Moreover, $\{(x_1, x_2, x_3, x_4) : x_4 - x_3 = x_2 - x_1\}$ is a subspace of $\mathbb{R}^4$ because it is a linear span of vectors in $\mathbb{R}^4$.

(3) Describe all the subspaces of $\mathbb{R}^3$.

Solution: $\{0\}$ and $\mathbb{R}^3$ are the trivial subspaces of $\mathbb{R}^3$. Any line passing through origin is an one-dimensional subspace of $\mathbb{R}^3$ and any plane passing through origin is a 2-dimensional subspace in $\mathbb{R}^3$. We claim that these are all the subspaces of $\mathbb{R}^3$.

Let W be a non-trivial subspace of $\mathbb{R}^3$. If $dim(W) = 1$ choose a basis $\{v\}$ of W. Then $W = \{a.v : a \in \mathbb{R}\}$. So W represents a line passing through origin in the direction of v. If $dim(W) = 1$ then choose a basis $\{v_1, v_2\}$ of W. Then $W = Span\{v_1, v_2\} = \{av_1 + bv_2 : a, b \in \mathbb{R}\}$. So W represents a plane passing through origin with normal vector $v_1 \times v_2$.

(4) Find the condition on real numbers a, b, c, d so that the set $\{(x, y, z)|ax + by + cz = d\}$ is a subspace of $\mathbb{R}^3$.

Solution: Let $W = \{(x, y, z)|ax + by + cz = d\}$. If W is a subspace then $(0, 0, 0) \in W$ and so $d = 0$.

(5) Discuss the linear dependence/independence of following set of vectors:
(i) $\{(1, 0, 0), (1, 1, 0), (1, 1, 1)\}$ in $\mathbb{R}^3$ as a vector space over $\mathbb{R}$,

Ans: Linearly independent since the determinant of the matrix formed by taking $\{(1, 0, 0), (1, 1, 0), (1, 1, 1)\}$ as row vectors is non zero. So they are linearly independent.

(ii) $\{(1,0,0,0),(1,1,0,0),(1,1,1,0),(3,2,1,0)\}$ in $\mathbb{R}^4$ as a vector space over $\mathbb{R}$,

Ans: Linearly dependent since $(3,2,1,0) = (1,0,0,0) + (1,1,0,0) + (1,1,1,0)$.

(iii) $\{(1,i,0),(1,0,1),(i+2,-1,2)\}$, in $\mathbb{C}^3$ as a vector space over $\mathbb{C}$,
Ans: $(i+2,-1,2) = i(1,i,0) + (1,0,1)$. So they are linearly dependent.

(iv) $\{(1,i,0),(1,0,1),(i+2,-1,2)\}$, in $\mathbb{C}^3$ as a vector space over $\mathbb{R}$,
Ans: If $a.(1,i,0) + b(1,0,1) + c(i+2,-1,2) = 0$ for $a,b,c \in \mathbb{R}$. Then we have $a = b = c = 0$. So they are linearly independent.

(v) The sets $\{1, sinx, cosx\}$ and $\{2, sin^2x, cos^2x\}$ in the vector space of real valued functions $F = \{f : f : \mathbb{R} \to \mathbb{R}\}$.

Solution: Suppose $a.1 + b.sinx + c.cosx = 0$. Then the identity is true for all $x \in \mathbb{R}$.

For $x = 0$ we have $a + c = 0$. For $x = \pi/2$ we have $a + b = 0$ and for $x = -\pi/2$ we have $a - b = 0$. From these linear equations we have $a = b = c = 0$. So the set $\{1, sinx, cosx\}$ is linearly independent. On the other hand we have $2sin^2x + 2cos^2x - 2 = 0$. So the set $\{2, sin^2x, cos^2x\}$ is linearly dependent.

(v) $\{u+v, v+w, w+u\}$ in a vector space V given that $\{u,v,w\}$ is linearly independent.

Ans: If $a(u+v)+b(v+w)+c(w+u) = 0$ for some scalars a,b,c. Then we have $a+b = b+c = a+c = 0$ and hence $a = b = c = 0$. So $\{u+v, v+w, w+u\}$ is linearly independent.

(6) Let $W_1 = Span\{(1,1,0),(-1,1,0)\}$ and $W_2 = Span\{(1,0,2),(-1,0,4)\}$. Prove that $W_1 + W_2 = \mathbb{R}^3$.

Solution: The three vectors $(1,1,0),(-1,1,0),(1,0,2)$ are in W_1+W_2 and are linearly independent. So $Span\{(1,1,0),(-1,1,0),(1,0,2)\} = W_1+W_2 = \mathbb{R}^3$.

(7) Find 3 vectors u, v and w in $\mathbb{R}^4$ such that $\{u,v,w\}$ is linearly dependent whereas $\{u,v\}, \{u,w\}$, and $\{v,w\}$ are linearly independent. Extend each of the linearly independent sets to a basis of $\mathbb{R}^4$.

Solution: Let $u = (1,0,0,0), v = (0,1,0,0)$ and $w = (1,1,0,0)$. Then since $w = u + v$, the set $\{u,v,w\}$ is linearly dependent whereas the sets $\{u,v\}, \{u,w\}$, and $\{v,w\}$ are linearly independent.

Extending $\{u,v\}$ we have the set $\{(1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1)\}$ a basis of $\mathbb{R}^4$.

Extending $\{v, w\}$ we have the set $\{((0,1,0,0),(1,1,0,0),(0,0,1,0),(0,0,0,1)\}$ a basis of $\mathbb{R}^4$.

Extending $\{u, w\}$ we have the set $\{((1,0,0,0),(1,1,0,0),(0,0,1,0),(0,0,0,1)\}$ a basis of $\mathbb{R}^4$.

(8) Let A be a $n \times n$ matrix over $\mathbb{R}$. Then A is invertible iff the row vectors are linearly independent over $\mathbb{R}$ iff the column vectors are linearly independent over $\mathbb{R}$.

Solutions: We know that A is invertible iff the row reduced echelon form in the identity matrix iff the system $Ax = 0$ has only the trivial solution $x = 0$. Let $C_1, C_2, \cdots, C_n$ be the column vectors of A. So A is invertible iff $b_1C_1 + b_2C_2 + \cdots + b_nC_n = 0$ for some $b_i \in \mathbb{R}$ implies $b_i = 0$ for all i. So A is invertible iff the column vectors of A are linearly independent over $\mathbb{R}$.

A is invertible iff A^T is invertible. So the row vectors of A are linearly independent over $\mathbb{R}$ iff the column vectors of A are linearly independent over $\mathbb{R}$.

(9) Determine if the set $T = \{1, x^2 - x + 5, 4x^3 - x^2 + 5x, 3x + 2\}$ is a basis for the vector space of polynomials in x of degree ≤ 4. Is this set a basis for the vector space of polynomials in x of degree ≤ 3 ?

Solution If $a.1 + b(x^2 - x + 5) + c(4x^3 - x^2 + 5x) + d(3x + 2) = 0$ then equating the coefficients we get $a = b = c = d = 0$. So the set is linearly independent. But $x^4 \notin Span(T)$. So it is not a basis for the vector space of polynomials in x of degree ≤ 4. However since dimension of the vector space of polynomials in x of degree ≤ 3 is 4 the linearly independent set $T = \{1, x^2 - x + 5, 4x^3 - x^2 + 5x, 3x + 2\}$ is a basis.