ASSIGNMENT 4
MTH102A

(1) Let $\{w_1, w_2, ..., w_n\}$ be a basis of a finite dimensional vector space V. Let v be a non zero vector in V. Show that there exists w_i such that if we replace w_i by v in the basis it still remains a basis of V.

Solution. Let $v = \sum_1^n a_i w_i$ for some $a_1, ..., a_n \in \mathbb{F}$. Since v is non-zero, $a_i \neq 0$ for some $1 \leq i \leq n$. Assume $a_1 \neq 0$. Write $w_1 = \frac{1}{a_1} v - \sum_{i=2}^n \frac{a_i}{a_1} w_i$. Replace w_1 by v. Clearly $\{v, w_2, ..., w_n\}$ spans V. Now we show that this set is L.I.. Let $b_1, ..., b_n$ be such that $b_1 v + \sum_{j=2}^n b_j w_j = 0$. Then $b_1 \sum_{i=1}^n a_i w_i + \sum_{j=2}^n b_j w_j = 0$ which implies $b_1 a_1 w_1 + \sum_{j=2}^n (b_1 a_j + b_j) w_j = 0$. Since $w_1, w_2, \cdots w_n$ are LI we have $b_1 a_1 = 0, b_1 a_j + b_j = 0$ for $2 \leq j \leq n$. Since $a_1 \neq 0$ we have $b_j = 0$ for all $1 \leq j \leq n$.

(2) Find the dimension of the following vector spaces :
(i) X is the set of all real upper triangular matrices,
(ii) Y is the set of all real symmetric matrices,
(iii) Z is the set of all real skew symmetric matrices,
(iv) W is the set of all real matrices with $Tr(A) = 0$

Solution. Let E_{ij} be the matrix with ij^{th} entry one and all other entries are zero , F_{ij} be the matrix with ij^{th} and ji^{th} entries are 1 and all other entries are zero and for $i \neq j$ define D_{ij} to be the matrix with ij^{th} entry 1, ji^{th} entry -1 and all other entries are zero.
(i) The set $\{E_{ij}; i \leq j\}$ forms a basis for X. Hence $dim(X) = n + \frac{n^2-n}{2} = \frac{n(n+1)}{2}$.
(ii) The set $\{Fij; i \leq j\}$ is a basis for Y. Hence $dim(Y) = \frac{n(n+1)}{2}$.
(iii) For a real skew-symmetric matrix all diagonal entries are zero. Then the set $\{D_{ij}; i < j\}$ is a basis. Hence $dim(Z) = \frac{n^2-n}{2} = \frac{n(n-1)}{2}$.
(iv) Let A be a matrix with trace zero. Then $\sum_{i=1}^n a_{ii} = 0$. So $a_{11} = -(a_{22} + ... + a_{nn})$. The set $\{E_{ij}, E_{ji} : i \neq j\} \cup \{E_{ii} - E_{i+1,i+1} : 1 \leq i \leq n-1\}$ is a basis of W. Hence $dim(W) = n^2 - 1$.

(3) Let $\mathcal{P}_2(X, \mathbb{R})$ be the vector space of all polynomials in X of degree less or equal to 2. Show that $B = \{X+1, X^2 - X + 1, X^2 + X - 1\}$ is a basis of $\mathcal{P}_2(X, \mathbb{R})$. Determine the coordinates of the vectors $2X - 1, 1 + X^2, X^2 + 5X - 1$ with respect to B.

Solution. First we show that the set B is L.I.. Let $a_0, a_1, a_2 \in \mathbb{R}$ such that $a_0(X+1)+a_1(X^2-X+1)+a_2(X^2+X-1)=0$. Then $a_0+a_1-a_2+(a_0-a_1+a_2)X+(a_1+a_2)X^2=0$. From here we will get $a_0=a_1=a_2=0$.

Let $p(X)=a_0+a_1X+a_2X^2$ be any element in $\mathcal{P}_2(X,\mathbb{R})$. Let $b_0, b_1, b_2 \in \mathbb{R}$ such that $p(X)=a_0+a_1X+a_2X^2=b_0(X+1)+b_1(X^2+X-1)+b_3(X^2-X+1)$ then we get $b_0=\frac{a_0+a_1}{2}, b_1=\frac{a_0-a_1+2a_2}{4}, b_2=\frac{a_1-a_0+2a_2}{4}$. Hence B spans $\mathcal{P}_2(X,\mathbb{R})$.

We have $2X-1=\frac{1}{2}(X+1)-\frac{3}{4}(X^2-X+1)+\frac{3}{4}(X^2+X-1)$. So $[2X-1]_B=(\frac{1}{2},-\frac{3}{4},\frac{3}{4})$.
$1+X^2=\frac{1}{2}(X+1)+\frac{3}{4}(X^2-X+1)+\frac{1}{4}(X^2+X-1)$. So $[1+X^2]_B=(\frac{1}{2},\frac{3}{4},\frac{1}{4})$.
$X^2+5X-1=2(X+1)-1(X^2-X+1)+(X^2+X-1)$. So $[X^2+5X-1]_B=(2,-1,1)$.

(4) Let W be a subspace of a finite dimensional vector space V
(i) Show that there is a subspace U of V such that $V=W+U$ and $W\cap U=\{0\}$,
(ii) Show that there is no subspace U of V such that $W\cap U=\{0\}$ and $dim(W)+dim(U)>dim(V)$.

Solution.
(i) Let $\dim(V)=n$, since V is finite dimensional, W is also finite dimensional. Let $\dim(W)=k$ and let $B=\{w_1,...,w_k\}$ be a basis for W. In case $k=n$ we take $U=\{0\}$. If $k<n$ we extend B to a basis $B_1=\{w_1,...,w_k,v_{k+1},...,v_n\}$ of V. Let U be the subspace of V generated by $\{v_{k+1},...,v_n\}$.

Let $v\in V$. Then there exist scalars $a_1,...,a_n\in\mathbb{F}$ such that $v=a_1w_1+....+a_kw_k+a_{k+1}v_{k+1}+...+a_nv_n=(a_1w_1+....+a_kw_k)+(a_{k+1}v_{k+1}+...+a_nv_n)\in W+U$.

Now we show that $W\cap U=\{0\}$. Let $v\in W\cap U$. Then there exist scalars $a_1,...,a_k$ and $b_{k+1},...,b_n$ such that $a_1w_1+...+a_kw_k=v=b_{k+1}v_{k+1}+....+b_nv_n$. We have $a_1w_1+...+a_kw_k-b_{k+1}v_{k+1}-....-b_nv_n=0$ Since B_1 is a basis, $a_1=...=a_k=b_{k+1}=...=b_n=0$. Hence $W\cap U=\{0\}$.
(ii) Let $W\cap U=\{0\}$. Then $dim(W+U)=dim(W)+dim(U)-dim(W\cap U)=dim(W)+dim(U)$. So if $dim(W)+dim(U)>dim(V)$ we have $dim(W+U)>dim(V)$, which is a contradiction as $W+U$ is a subspace of V.

(5) Decide which of the following maps are linear transformations:
(i) $T:\mathbb{R}^3\to\mathbb{R}^3$ defined by $T(x,y,z)=(x+2y,z,|x|)$,
(ii) Let $M_n(\mathbb{R})$ be the set of all $n\times n$ real matrices and $T:M_n(\mathbb{R})\to M_n(\mathbb{R})$ defined by
(a) $T(A)=A^T$,
(b) $T(A)=I+A$, where I is the identity matrix of order n,
(c) $T(A)=BAB^{-1}$, where $B\in M_n(\mathbb{R})$ is an invertible matrix.

Solution.
(i) Not a linear transformation.

Take $a = -1$. Then $T(a(1,0,1)) = T(-1,0,-1) = (-1,-1,1) \neq aT((1,0,1)) = -1(1,1,1) = (-1,-1,-1)$.
(ii) (a) Linear Transformation.
Let $A, B \in M_n(\mathbb{R})$ and $a \in \mathbb{R}$. Then $T(A + aB) = A + aB^T = A^T + aB^T$.
(b) Not a linear transformation.
Let O be the zero matrix. Then $T(O) = I \neq O$.
(c) Linear transformation.
Let $P, Q \in M_n(\mathbb{R})$ and $a \in \mathbb{R}$. Then $T(P + aQ) = B(P + aQ)B^{-1} = BPB^{-1} + aBQB^{-1} = T(P) + aT(Q)$.

(6) Let $T : \mathbb{C} \to \mathbb{C}$ be defined by $T(z) = \overline{z}$. Show that T is $\mathbb{R}$-linear but not $\mathbb{C}$-linear.

Solution. Let $a + ib, c + id \in \mathbb{C}$ and $\alpha \in \mathbb{R}$. Then $T((a + ib) + \alpha(c + id)) = T(a+\alpha c+i(b+\alpha d)) = a+\alpha c-i(b+\alpha d) = (a-ib)+\alpha(c-id) = T(a+ib)+\alpha T(c+id)$. Hence T is $\mathbb{R}$-linear whereas $T(i.i) = T(-1) = -1 \neq iT(i) = i(-i) = 1$, so it is not $\mathbb{C}$-linear.

(7) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation such that $T(1,0,0) = (1,0,0), T(1,1,0) = (1,1,1), T(1,1,1) = (1,1,0)$. Find $T(x,y,z)$, $Ker(T)$, $R(T)$ (Range of T). Prove that $T^3 = T$.

Solution. Let $(x,y,z) \in \mathbb{R}^3$. Then $(x,y,z) = (x-y)(1,0,0) + (y-z)(1,1,0) + z(1,1,1)$. We have $T((x,y,z)) = (x-y)T(1,0,0) + (y-z)T(1,1,0) + zT(1,1,1) = (x-y)(1,0,0) + (y-z)(1,1,1) + z(1,1,0) = (x,y,y-z)$.

Let $x = (a,b,c) \in Ker(T)$, then $T(x) = (a,b,b-c) = (0,0,0)$ implies $(a,b,c) = (0,0,0)$. Hence $Ker(T) = \{(0,0,0)\}$.

$R(T) = \{(x,y,y-z); x,y,z \in \mathbb{R}\} = Span\{(1,0,0),(0,1,1),(0,0,-1)\}$.

$T^3((x,y,z)) = T^2((x,y,y-z)) = T((x,y,y-(y-z))) = T((x,y,z))$.

(8) Find all linear transformations from $\mathbb{R}^n$ to $\mathbb{R}$.

Solution. For $a_1, ..., a_n \in \mathbb{R}$, the map defined by $T((x_1, ..., x_n)) = a_1x_1 + ... + a_nx_n$ is a linear transformation. We claim that all the linear transformations are of this form for some $a_1, ..., a_n \in \mathbb{R}$. Let $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ be the n-tuple with all components equal to 0, except the i-th, which is 1. Let $T : \mathbb{R}^n \to \mathbb{R}$ be a linear map. Suppose $T(e_i) = a_i$ for some $a_i \in \mathbb{R}$, $1 \leq i \leq n$. Then $T(x_1, x_2, \cdots, x_n) = T(x_1e_1 + x_2e_2 + \cdots + x_ne_n) = a_1x_1 + ... + a_nx_n$.

(9) Let $\mathcal{P}_n(X, \mathbb{R})$ be the vector space of all polynomials in X of degree less or equal to n. Let T be the differentiation transformation from $\mathcal{P}_n(X, \mathbb{R})$ to $\mathcal{P}_n(X, \mathbb{R})$. Find $Range(T)$ and $Ker(T)$.

Solution. If $f(X) = a_0 + a_1X + a_2X^2 + \cdots + a_nX^n \in \mathcal{P}_n(X, \mathbb{R})$ then $T(f) = a_1 + 2a_2X + \cdots + na_nX^{n-1}$. So $Range(T) = Span\{1, X, \cdots X^{n-1}\}$.

$Ker(T) = \{f : T(f) = f' = 0\}$. So $Ker(T) = \{f(X) : f(X) = c \; for \; c \in \mathbb{R}\} = Span\{1\} = \mathbb{R}$.