ASSIGNMENT 5
MTH102A

(1) Show that there does not exist a linear map from $\mathbb{R}^5$ to $\mathbb{R}^2$ whose kernel is $\{(x_1, x_2, x_3, x_4, x_5) : x_1 = 3x_2 \ and \ x_3 = x_4 = x_5\}$.

Solutions: If $\phi : \mathbb{R}^5 \to \mathbb{R}^2$ is any linear map, then the rank-nullity theorem tells us that

$$5 = \dim(\mathrm{Ker}(\phi)) + \dim(\mathrm{Im}(\phi)).$$

Since $\mathrm{Im}(\phi) \subset \mathbb{R}^2$, its dimension is at most 2, so that $\dim(\mathrm{Ker}(\phi)) \geq 3$. The subspace in the question is

$$\mathrm{Span}\{(3, 1, 0, 0, 0), (0, 0, 1, 1, 1)\},$$

which is 2-dimensional. So it cannot possibly be the kernel of a linear map $\phi : \mathbb{R}^5 \to \mathbb{R}^2$.

(2) Find a basis for the kernel and the basis for the image of the linear transformation $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ given by $T(p) = p' + p''$ where $P_2(\mathbb{R})$ is the vector space of polynomials in x of degree less than or equal to n.

Solution: Note that $T(ax^2 + bx + c) = 2ax + 2a + b$. Now

$$\ker T = \{ax^2 + bx + c : 2ax + b + 2a = 0\}$$

that is $2a = 0$ and $2a + b = 0$. So $a = b = 0$. So $Ker(T)$ is the set of all constant (degree 0) polynomials which can be identified with $\mathbb{R}$. For the image note that $T(1) = 0$, $T(x) = 1$, $T(x^2) = 2x + 2$. So $Range(T) = Span\{1, x\}$.

(3) Find the matrix of the differentiation map on the vector space of polynomials in x of degree less than or equal to n with respect to the standard basis and verify the Rank-Nullity theorem.

Solution: The standard basis in this case is $B = \{1, x, x^2, \cdots, x^n\}$. Let D denotes the differentiation map. Then $D(1) = 0$, $D(x) = 1, \cdots, D(x^n) = nx^{n-1}$. So the matrix with respect to B is

$$[D]_B = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & n \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$

So $Range(D) = Span\{1, x, \cdots x^{n-1}\}$ and $Ker(D) = Span\{1\} = \mathbb{R}$. Since $\{1, x, \cdots x^{n-1}\}$ is LI, $rank(D) = n$. $Nullity(D) = 1$. So $rank(D) + Nullity(D) = n + 1$.

(4) Determine the quotient vector space $M_3(\mathbb{R})/W$, where $M_3(\mathbb{R})$ is the vector space of all 3×3 real matrices and W is the subspace of symmetric matrices, that is $W = \{A \in M_3(\mathbb{R}) : A = A^t\}$.

Solution: Let $U = \{A \in M_3(\mathbb{R}) : A^T = -A\}$ and define $T : M_3(\mathbb{R}) \to U$ by $A \mapsto A - A^T$. Then T is a linear map since

$A + B \mapsto A + B - (A+B)^T = (A - A^T) + (B - B^T)$ and $a.A \mapsto a.A - a.A^T = a.(A - A^T)$. Again T is onto because if $B \in U$ then $B = -B^T$ and $\frac{1}{4}(B - B^T) \mapsto B$.

We also have $Ker(T) = W$. So $M_3(\mathbb{R})/W \cong U$.

(5) Find the matrix of the linear transformation T: $\mathbb{R}^4 \to \mathbb{R}^4$, with respect to the standard basis of $\mathbb{R}^4$ such that $Ker(T) = Span\{(2,1,1,2),(1,2,1,1)\}$ and $Range(T) = Span\{(1,0,1,0),(0,1,1,1)\}$.

Solution: Note that the standard basis of $\mathbb{R}^4$ is

$$B = \{(1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1)\}.$$

Since $\{(2,1,1,2),(1,2,1,1)\}$ is LI we extend it to a basis of $\mathbb{R}^4$. So we take $\{(1,0,0,0),(2,1,1,2),(1,2,1,1),(0,0,0,1)\}$ to be a basis of $\mathbb{R}^4$. Now we define:

$$T(1,0,0,0) = (1,0,1,0),$$
$$T(2,1,1,2) = (0,0,0,0),$$
$$T(1,2,1,1) = (0,0,0,0),$$
$$T(0,0,0,1) = (0,1,1,1).$$

Now, we have to compute the value of T on the vectors $(0,1,0,0),(0,0,1,0)$. In fact, we have that: $(0,1,0,0) = (1,2,1,1) - (2,1,1,2) + (1,0,0,0) + (0,0,0,1)$ and since T is linear, we have:

$$T(0,1,0,0) = T(1,2,1,1) - T(2,1,1,2) + T(1,0,0,0) + T(0,0,0,1) \tag{1}$$
$$= T(1,0,0,0) + T(0,0,0,1) \tag{2}$$
$$= (1,0,1,0) + (0,1,1,1) \tag{3}$$
$$= (1,1,2,1). \tag{4}$$

Again $(0,0,1,0) = (1,2,1,1) - (1,0,0,0) - (0,0,0,1) - (0,1,0,0)$, so that:

$$T(0,0,1,0) = T(1,2,1,1) - T(1,0,0,0) - T(0,0,0,1) - T(0,1,0,0) \tag{5}$$
$$= -T(1,0,0,0) - T(0,0,0,1) - T(0,1,0,0) \tag{6}$$
$$= -(1,0,1,0) - (0,1,1,1) - (1,1,2,1) \tag{7}$$
$$= (-2,-2,-4,-2). \tag{8}$$

Summarizing the above, we have obtained:

$$T(1,0,0,0) = (1,0,1,0) \tag{9}$$
$$T(0,1,0,0) = (1,1,2,1) \tag{10}$$
$$T(0,0,1,0) = (-2,-2,-4,-2) \tag{11}$$
$$T(0,0,0,1) = (0,1,1,1). \tag{12}$$

Therefore, we have that $Range(T) = \text{Span}\{T(1,0,0,0), T(0,0,0,1)\}$
$= \text{Span}\{(1,0,1,0),(0,1,1,1)\}$ and the kernel is $\text{Span}\{(2,1,1,2),(1,2,1,1)\}$ as required. The matrix is

$$[T]_B = \begin{pmatrix} 1 & 1 & -2 & 0 \\ 0 & 1 & -2 & 1 \\ 1 & 2 & -4 & 1 \\ 0 & 1 & 2 & 1 \end{pmatrix}.$$

(6) Show that for any two matrices A and B $rank(AB) \leq min\{rank(A), rank(B)\}$ and $rank(A+B) \leq rank(A) + rank(B)$.

Solution: Let the columns of A and B be $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ respectively. By definition, the rank of A and B are the dimensions of $Span\{a_1, \ldots, a_n\}$ and $Span\{b_1, \ldots, b_n\}$. Now the rank of $A+B$ is the dimension of the linear span of the columns of $A+B$, i.e. the dimension of $Span\{a_1+b_1, \ldots, a_n+b_n\}$. Since $Span\{a_1+b_1, \ldots, a_n+b_n\} \subseteq Span\{a_1, \ldots, a_n, b_1, \ldots, b_n\}$ the result follows we have $rank(A+B) \leq rank(A) + rank(B)$.

Note that rows of AB are linear combinations of rows of B. So $rank(AB) \leq rank(B)$. Since row rank of a matrix is same as the column rank we have $rank(AB) = rank((AB)^T) = rank(B^T A^T) \leq rank(A^T) = rank(A)$. So $rank(AB) \leq min\{rank(A), rank(B)\}$.

(7) Show that for a matrix A, $rank(AA^T) = rank(A)$.

Solution: Let A be a $m \times n$ matrix and $x \in \mathbb{R}^n$ such that $x \in Null(A)$. Then $Ax = 0$. Multiplying both sides with A^T from the left, we have $A^T Ax = 0$, which means $x \in Null(A^T A)$. Therefore $Null(A) \subseteq Null(A^T A)$.

Now, assume $x \in Null(A^T A)$, which implies $A^T Ax = 0$. Multiplying both sides with x^T from the left, we get

$$x^T A^T Ax = (Ax)^T (Ax) = 0$$

Now, defining $y = Ax$, we see that $y^T y = 0$, or

$$\sum_{i=1}^{m} y_i^2 = 0$$

Since y_i's are real, this means $y_i = 0$ for $i = 1, 2, ..., n$, which means

$$Ax = y = 0$$

which means $x \in Null(A)$. Therefore $Null(A^T A) \subseteq Null(A)$. So we showed that $Null(A^T A) = Null(A)$. By Rank nullity theorem we have $Nullity(A^T A) + rank(A^T A) =$ number of columns of $A^T A$. But number of columns of $A^T A=$ number of columns of A. So $rank(AA^T) = rank(A)$.

(8) Let V be a n dimensional vector space and W be a m dimensional vector space. Let $L(V, W)$ be the vector space of all linear maps from V to W. Find a basis for $L(V, W)$.

Solution: Let $\{v_1, v_2, \cdots, v_n\}$ be a basis of V and $\{w_1, w_2, \cdots, w_m\}$ be a basis for W. Define

$$f_{ij} : V \to W,\ v_k \mapsto \begin{cases} 0 & k \neq i, \\ w_j & k = i, \end{cases}.$$

We claim that $\{f_{ij} : i = 1, 2 \cdots n, j = 1, 2, \cdots, m\}$ is a basis of $L(V, W)$.

Linear Independence: If $f = \sum_{i,j} a_{ij} f_{ij} = 0$ for some scalars a_{ij} then $f(v_i) = \sum_j a_{ij} w_j = 0$. Since $\{w_1, w_2, \cdots, w_m\}$ is LI we have $a_{ij} = 0$ for all j. So the functions f_{ij} are LI.

Spanning: Let $f \in L(V, W)$ then $f(v_i) = \sum_j b_{ij} w_j$ for $i = 1, 2, \cdots, n$ and for some scalars b_{ij}. Then $f = \sum_{i,j} b_{ij} f_{ij}$ since $f(v_k) = \sum_{i,j} b_{ij} f_{ij}(v_k)$ for every $k = 1, 2, \cdots, n$. So $\{f_{ij} : i = 1, 2 \cdots n, j = 1, 2, \cdots, m\}$ is a basis of $L(V, W)$.

(9) Let V be a n dimensional vector space. Let B_1 and B_2 be two bases of V and let T be a linear operator on V. Show that there exists an invertible matrix P such that $[T]_{B_1} = P^{-1}[T]_{B_2} P$.

Solution: Let $B_1 = \{v_1, v_2, \cdots, v_n\}$ and $B_2 = \{w_1, w_2, \cdots, w_n\}$. Let P be the change of basis matrix from B_1 to B_2. Recall that $P = ([w_1]_{B_1}, [w_2]_{B_1}, \cdots, [w_n]_{B_1})$. Then for any $v \in V$ we have $[v]_{B_1} = P[v]_{B_2}$. In particular we have $[T(v)]_{B_1} = P[T(v)]_{B_2}$. Note that $[T(v)]_{B_1} = [T]_{B_1}[v]_{B_1}$. Combining all these we get $[T]_{B_1} P[v]_{B_2} = P[T(v)]_{B_2}$. So $P^{-1}[T]_{B_1} P[v]_{B_2} = [T(v)]_{B_2} = [T]_{B_2}[v]_{B_2}$ and hence $[T]_{B_1} = P^{-1}[T]_{B_2} P$.

(10) Let T be a linear map from V to W. Show that T is non-singular ($Ker(T) = \{0\}$) if and only if T carries each linearly independent subset of V to a linearly independent subset of W.

Solution: Suppose T is non-singular and let $\{v_1, v_2, \cdots, v_k\}$ be a LI subset of V. We claim that $\{T(v_1), T(v_2), \cdots, T(v_k)\}$ is a LI subset of W.

If $a_1 T(v_1) + a_2 T(v_2) + \cdots + a_k T(v_k) = 0$ then $T(a_1 v_1 + \cdots + a_k v_k) = 0$ and so $a_1 v_1 + \cdots + a_k v_k = 0$ since T is non-singular. Then $a_i = 0$ for all i.

Conversely, suppose T carries each linearly independent subset of V to a linearly independent subset of W. Let v be a non-zero vector in V. Since $\{v\}$ is LI, the set $\{T(v)\}$ is linearly independent. So $T(v) \neq 0$ and hence $v \notin Ker(T)$. So $Ker(T) = \{0\}$.