ASSIGNMENT 6
MTH102A

(1) Let A and B be square matrices of same order. Prove that characteristic polynomials of AB and BA are same. Do AB and BA have same minimal polynomial ?

Solution: If one of them invertible, say A is invertible then $A^{-1}(AB)A = BA$. So AB and BA being similar have same characteristic polynomial. Suppose none of them is invertible.
Define two matrices C and D of order $n \times n$ as follows,
$C = \begin{pmatrix} xI_n & A \\ B & I_n \end{pmatrix}$ and $D = \begin{pmatrix} I_n & 0 \\ -B & xI_n \end{pmatrix}$, where I_n is the identity matrix of order n and x is an indeterminate.
Now check that ,

$$det(CD) = x^n det(xI_n - AB)$$

$$det(DC) = x^n det(xI_n - BA)$$

as det(CD)=det(DC) we get

$$det(xI_n - AB) = det(xI_n - BA).$$

So the characteristic polynomials of AB and BA are same.

Let $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Then $AB = A$ whereas BA is the zero matrix. Since $A^2 = 0$ and $A \neq 0$, the minimal polynomial of AB is x^2 whereas the minimal polynomial of BA is x.

(2) Let A be an $n \times n$ matrix. Show that A and A^T have same eigen values. Do they have the same eigen vectors ?

Solution: $det(A^T - \lambda I) = det(A^T - (\lambda I)^T) = det((A - \lambda I)^T) = det(A - \lambda I)$. So they have same characteristic polynomial and therefore same eigenvalues.
Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Then 1 is an eigenvalue of A and A^T but the eigenvectors with respect to the eigen value 1 are $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ respectively.

(3) Find the characteristic and minimal polynomial of the following matrix and decide if this matrix is diagonalizable.

$$A = \begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}$$

Solution: The characteristic polynomial is $f_A(x) = det(xI - A)$.

Here $xI - A = \begin{bmatrix} x-5 & 6 & 6 \\ 1 & x-4 & -2 \\ -3 & 6 & x+4 \end{bmatrix}$

So $f_A(x) = (x-1)(x-2)^2$. The minimal polynomial is by definition is the smallest degree monic polynomial $m(x)$ such that $m(A) = 0$.

We know that $m(x)$ divides $f_A(x)$ and they have the same roots. So the possibilities for $m(x)$ are $(x-1)(x-2)$ and $(x-1)(x-2)^2$.

Since $(A - I)(A - 2I) = 0$ the minimal polynomial is $(x-1)(x-2)$. Since the minimal polynomial is a product of distinct linear factors, the matrix A is diagonalizable.

(4) Find the inverse of the matrix $\begin{pmatrix} -1 & 2 & 0 \\ 1 & 1 & 0 \\ 2 & -1 & 2 \end{pmatrix}$ using the Cayley-Hamilton theorem.

Solution: The matrix A is: $A = \begin{bmatrix} -1 & 2 & 0 \\ 1 & 1 & 0 \\ 2 & -1 & 2 \end{bmatrix}$,

So the characteristic polynomial $p_A(\lambda)$ is

$p_A(\lambda) = \det(A - \lambda I) = \det \begin{bmatrix} -1-\lambda & 2 & 0 \\ 1 & 1-\lambda & 0 \\ 2 & -1 & 2-\lambda \end{bmatrix} = (-1-\lambda)(1-\lambda)(2-\lambda) - 2(2-\lambda) = (\lambda^2 - 1)(2-\lambda) - 4 + 2\lambda = -\lambda^3 + 2\lambda^2 + 3\lambda - 6$,

and by Cayley-Hamilton theorem we have

$0 = p_A(A) = -A^3 + 2A^2 + 3A - 6I \Rightarrow A(-A^2 + 2A + 3I) = 6I$ or $A(\frac{1}{6}(-A^2 + 2A + 3I)) = I$ which shows that $A^{-1} = \frac{1}{6}(-A^2 + 2A + 3I)$.

(5) Diagonalize $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix}$ and compute A^{2019}.

Solution: The characteristic polynomial is

$$\det(A - \lambda I) = \det \left(\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \right)$$

$$= \det \left(\begin{bmatrix} 1-\lambda & 2 & 0 \\ 2 & 1-\lambda & 0 \\ 0 & 0 & -3-\lambda \end{bmatrix} \right)$$

.

So $f_A(\lambda) = (1-\lambda)((1-\lambda)(-3-\lambda)-0) - 2(2(-3-\lambda)-0)$. So the eigen values are $\lambda = -3$, $\lambda = -1$ and $\lambda = 3$.

$$(A - \lambda I)\mathbf{x} = \begin{bmatrix} 4 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Here we can clearly see that all solutions $\mathbf{x}$ to this system are of the form $t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, for some scalar t. Thus $v_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is the eigenvector associated with the eigenvalue $\lambda = -3$ of the matrix A.

Following the exact same procedure: we see that the other two eigenvectors are $v_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $v_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ with respect to the eigen values -1 and 3 respectively.

Let $P = \begin{bmatrix} 0 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$. Then $P^{-1}AP = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

We have $P^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$.

Then $A = P \begin{bmatrix} -3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix} P^{-1}$. Hence $A^{2019} = P \begin{bmatrix} -3^{2019} & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3^{2019} \end{bmatrix} P^{-1}$.
Multiplying we get the answer.

(6) Let W be the subspace of $\mathbb{R}^4$ spanned by $\{u_1 = (1,1,1,1), u_2 = (2,4,1,5), u_3 = (2,0,4,0)\}$. Using the standard Euclidean inner product on $\mathbb{R}^4$ find an orthogonal basis for W.

Solution: It is an inductive process, so first let's define:

$$v_1 := u_1 = (1,1,1,1).$$

Then, by Gram-Schmidt orthogonalization process:

$$\begin{aligned} v_2 :&= u_2 - \frac{\langle u_2, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 \\ &= u_2 - \frac{2+4+1+5}{4} v_1 = (2,4,1,5) - 3(1,1,1,1) \\ &= (-1, 1, -2, 2). \end{aligned}$$

and finally

$$\begin{aligned} v_3 &= u_3 - \frac{\langle u_3, v_2\rangle}{\langle v_2, v_2\rangle} v_2 - \frac{\langle u_3, v_1\rangle}{\langle v_1, v_1\rangle} v_1 \\ &= u_3 - \frac{-10}{10} v_2 - \frac{6}{4} v_1 \\ &= (2,0,4,0) + (-1,1,-2,2) - \frac{3}{2}(1,1,1,1) \\ &= (-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \end{aligned}$$

(7) Consider $P_2(\mathbb{R})$ together with inner product:

$$\langle p(x), q(x)\rangle = \int_0^1 p(x)q(x)\,dx.$$

Find an orthogonal basis for $P_2(\mathbb{R})$.

Solution: The standard basis for $P_2(\mathbb{R})$ is $B = \{1, x, x^2\}$.

Using the Gram-Schmidt process:

Let $v_1 = 1$

Let

$$v_2 = x - P_{v_1}(x) = x - \frac{\langle x, v_1\rangle}{\langle v_1, v_1\rangle} \cdot v_1 = x - \frac{\langle x, 1\rangle}{\langle 1, 1\rangle} \cdot 1$$

Since $\langle p(x), q(x)\rangle = \int_0^1 p(x)q(x)\,dx$,

$$\langle x, 1\rangle = \int_0^1 x \cdot 1\,dx = \int_0^1 x\,dx = \frac{1}{2}$$

$$\langle 1, 1\rangle = \int_0^1 1 \cdot 1\,dx = \int_0^1 1\,dx = 1$$

$$\Rightarrow v_2 = x - \frac{1}{2}$$

$$v_3 = x^2 - P_{v_1}(x^2) - P_{v_2}(x^2) = x^2 - \frac{\langle x^2, v_1\rangle}{\langle v_1, v_1\rangle} \cdot v_1 - \frac{\langle x^2, v_2\rangle}{\langle v_2, v_2\rangle} \cdot v_2 = x^2 - x + \frac{1}{6}.$$

So the set $\{1, x - \frac{1}{2}, x^2 - x + \frac{1}{6}\}$ is an orthogonal basis for $P_2(\mathbb{R})$.

(8) Is the following matrix orthogonally diagonalizable ? If yes, then find P such that PAP^T is diagonal.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Solutions: Since the matrix A is symmetric, we know that it can be orthogonally diagonalized. We first find its eigenvalues by solving the characteristic equation:

$$0 = \det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = -(\lambda - 3)\lambda^2 \implies \begin{cases} \lambda_1 = 0 \\ \lambda_2 = 0 \\ \lambda_3 = 3 \end{cases}$$

We now find the eigenvectors corresponding to $\lambda = 0$:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right) \implies \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \implies \mathbf{x} = \begin{pmatrix} s \\ t \\ -s-t \end{pmatrix} = s\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

So $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ are the eigen vectors with respect to the eigen value 0.

By orthonormalizing them, we obtain

$$\left\{ \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \frac{1}{\sqrt{6}}\begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} \right\}$$

We finally find the eigenvector corresponding to $\lambda = 3$:

$$\left(\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right) \implies \left(\begin{array}{ccc|c} 0 & -3 & 3 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 3 & -3 & 0 \end{array}\right) \implies \left(\begin{array}{ccc|c} 0 & -1 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \implies \mathbf{x} = \begin{pmatrix} s \\ s \\ s \end{pmatrix} = s\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

By normalizing it, we obtain

$$\left\{ \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

Hence A is orthogonally diagonalized by the orthogonal matrix

$$P = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \\ -1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \end{pmatrix}$$

Furthermore,

$$P^T A P = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

(9) Find the singular value decomposition of the matrix $A = \begin{pmatrix} -2 & 2 \\ -1 & 1 \\ 2 & -2 \end{pmatrix}$.

Solution: Recall that the singular values of A are the square roots of the nonzero eigenvalues of A^TA (or AA^T). In this case

$$A^TA = \begin{pmatrix} 9 & -9 \\ -9 & 9 \end{pmatrix}$$

and the eigenvalues of A^TA are 0 and 18, so the only singular value is $\sqrt{18}$.

To find the matrix V, we need to find an eigenvectors for A^TA and normalize them. For the eigenvalue $\lambda = 18$ an normalized eigenvector is $\frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$. For $\lambda = 0$ an eigenvector of $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and so

$$V = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Σ is the 3×2 matrix whose diagonal is composed of the singular values

$$\Sigma = \begin{pmatrix} \sqrt{18} & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Finally $AV = U\Sigma$ and the colums of U are the eigenvectors of AA^T, solving this system of equations you get that

$$U = \frac{1}{3}\begin{pmatrix} 2 & \frac{3}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 1 & 0 & \sqrt{8} \\ -2 & \frac{3}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

We have $A = U\Sigma V^T$.

(10) Let $M_{n\times n}$ be the vector space of all real $n \times n$ matrices. Show that $\langle A, B\rangle = Tr(A^TB)$ is an inner product on $M_{n\times n}$. Show that the orthogonal complement of the subspace of symmetric matrices is the subspace of skew-symmetric matrices, i.e., $\{A \in M_{n\times n} \mid A \text{ is symmetric}\}^\perp = \{A \in M_{n\times n} \mid A \text{ is skew-symmetric}\}$.

Solution: Showing $\langle A, B\rangle = Tr(A^TB)$ is an inner product is easy.

Let A be symmetric and B be skew-symmetric. First we need to prove that $\langle A, B\rangle = 0$.

$\langle A, B\rangle = Tr(A^TB) = Tr(AB) = Tr(BA) = Tr(-B^TA) =\langle -B, A\rangle = -\langle A, B\rangle . So \langle A, B\rangle = 0$.

Note that $\langle A, B\rangle = Tr(A^TB) = \sum_{i,j} a_{ij}b_{ij}$. Let E_{ij} be the zero matrix except for a one in the (i, j) position.

Suppose $\langle A, S\rangle = 0$ for all symmetric matrices, then it is true for $S = E_{ij} + E_{ji}$. This gives $\langle A, E_{ij}\rangle + \langle A, E_{ji}\rangle = 0$, which gives $a_{ij} + a_{ji} = 0$, from which it follows that $A = -A^T$.