$(V, F, +, \cdot)$

$W \subseteq V$ a subspace

$a \cdot w_1 + w_2 \in W \quad \forall w_1, w_2 \in W, \; a \in F$

$S \subseteq V$ subset

* $\text{Span } S = \left\{ \sum_{i=1}^{n} c_i v_i : c_i \in F, \; v_i \in S \right\}$

subspace

* $S \subseteq V$ LD if $\exists \, v_1, v_2, \ldots v_n \in S$ and $c_1, c_2, \ldots c_n \in F$ s.t not $c_i = 0$ s.t $\sum_i c_i v_i = 0$

* A set which is not LD we call it LI

* $B \subseteq V$. We say B is a basis of V if
  (1) B spans V
  (2) B is LI

* Every VS has a basis

* Any two bases of a f.d. V.S have same cardinality.

* Any LI set can be extended to a basis.

* Every spanning set contains a basis.

$\text{Span } S = V$

S

$\{v_1\}$ $\quad v_2 \in S$

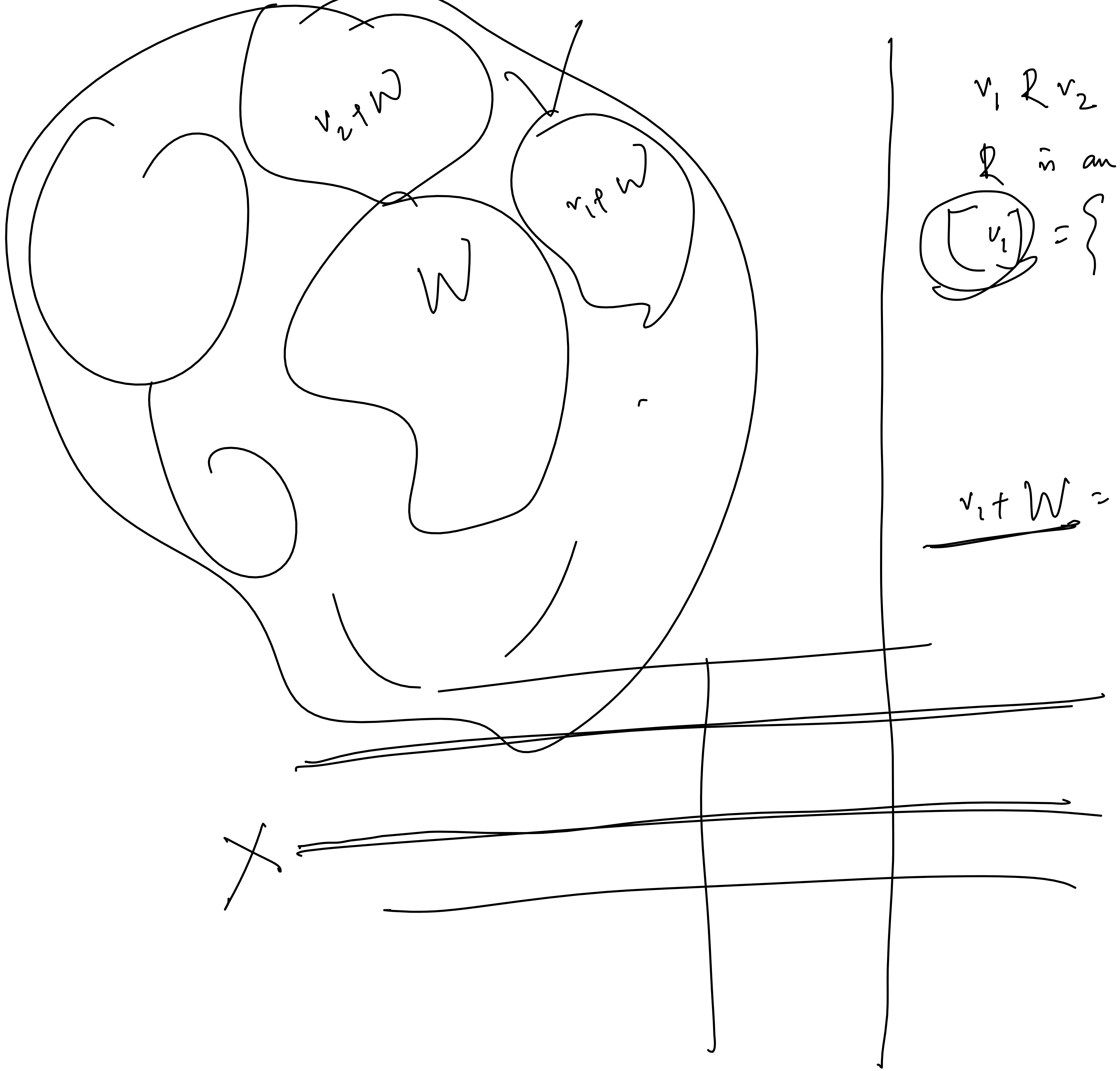
$\{v_1, v_2\}$

$\mathcal{C} = \{ T \subseteq S \text{ and } T \text{ is LI} \}$

$T_1 \subseteq T_2 \subseteq \cdots \subseteq \cdots$

$\bigcup T_i \subseteq S$

M



$v_1 R v_2$ if $v_1 - v_2 \in W$

R is an equivalence relation.

$[v_1] = \{ v \in V : v R v_1 \}$

$v - v_1 \in W$

$\Rightarrow v - v_1 = w \quad w \in W$

$\Rightarrow v = v_1 + w$

$v_1 + W = \{ v_1 + w : w \in W \} = [v_1]$

$V_2 \quad \bigcup [v_i]$.

$[v_1] = [v_2]$

$v_1 + W \qquad v_2 + W$

$v_1 + w \qquad v_2 + w'$

$v_1 + w = v_2 + w'$

$\Rightarrow v_1 - v_2 = w' - w \in W$

$[v_1] = [v_2]$ iff $v_1 - v_2 \in W$

$\frac{V}{W} = \{ v + W \ : v \in V \}$

$(v_1 + W) + (v_2 + W) := (v_1 + v_2) + W \in \frac{V}{W}$

$a \cdot (v + W) = a \cdot v + W \in \frac{V}{W}$

$0 + W$

$\frac{V}{W}$ is a V-S Quotient space

Supp V is f.d. $W \subseteq V$

$\boxed{\dim W \leq \dim V}$

$\{w_1, w_2, \dots w_n\}$ be a basis of W

$\{w_1, w_2, \dots w_n, w_{n+1}, \dots w_r\}$ be a basis of V

$\boxed{\dim \left(\frac{V}{W}\right)} = ?$

$S = \{ w_{n+1} + W, w_{n+2} + W, \dots, w_r + W \}$ $r - n$

Claim: S is a basis of $\frac{V}{W}$

$v + W$ $v \in V$

$v = \underbrace{a_1 w_1 + a_2 w_2 + \dots + a_n w_n}_{\in W} + a_{n+1} w_{n+1} + \dots + a_r w_r$

$v + W = a_{n+1} (w_{n+1} + W) + \dots + a_r \cdot (w_r + W)$

$b_{n+1}(w_{n+1}+W) + \cdots + b_r(w_r+W) = 0+W$

$\Rightarrow (b_{n+1}w_{n+1} + \cdots + b_r w_r) + W = 0+W$

$\Rightarrow (b_{n+1}w_{n+1} + \cdots + b_r w_r) \in W$

$\Rightarrow b_{n+1}w_{n+1} + \cdots + b_r w_r = a_1 w_1 + a_2 w_2 + \cdots + a_n w_n$

$\Rightarrow b_{n+1}w_{n+1} + \cdots + b_r w_r - a_1 w_1 - a_2 w_2 - \cdots - a_n w_n = 0$

$\Rightarrow b_i = a_j = 0 \;\forall i, j$

$\dim\left(\frac{V}{W}\right) = \dim V - \dim W$

W_1, W_2 are subspaces

$W_1 + W_2$ is also a subspace

V is f d $\qquad W_1, W_2$ are subspaces

$\dim(W_1+W_2) = ?$ $\qquad \dim(W_1+W_2) \le \dim V$

$\dim(W_1+W_2) = \dim(W_1) + \dim(W_2)$?

$(X\,Y) \cap (Y\,Z) = 1$

$\underbrace{\dim(W_1+W_2)}_{} = \underbrace{\dim(W_1)}_{n} + \underbrace{\dim(W_2)}_{k} - \underbrace{\dim(W_1 \cap W_2)}_{r}$

$W_1 + W_2 = \{ w_1 + w_2 : w_1 \in W_1 \text{ and } w_2 \in W_2 \}$

let $\{w_1, w_2, \ldots w_r\}$ be a basis of $W_1 \cap W_2$

let $\{w_1, w_2, \ldots w_r, v_{r+1}, \ldots v_n\}$ be a basis of W_1

let $\{w_1, w_2, \ldots w_r, u_{r+1}, \ldots u_k\}$ be a basis of W_2

$S = \{w_1, w_2, \ldots w_r, v_{r+1}, \ldots v_n, u_{r+1}, \ldots u_k\}$ $n+k-r$

claim: S is a basis for $W_1 + W_2$

$w \in W_1,\ w' \in W_2:\ w + w'$

S is LI ?

$W_1 \cap W_2 \subseteq W_1$

$W_1 \cap W_2 \subseteq W_2$

$a_1 w_1 + a_2 w_2 + \cdots + a_r w_r + b_{r+1} v_{r+1} + \cdots + b_n v_n + c_{r+1} u_{r+1} + \cdots + c_k u_k = 0$

$\Rightarrow \underbrace{a_1 w_1 + \cdots + a_r w_r + b_{r+1} v_{r+1} + \cdots + b_n v_n}_{\in W_1} = \underbrace{-c_{r+1} u_{r+1} - \cdots - c_k u_k}_{\in W_2}$

$\Rightarrow -c_{r+1} u_{r+1} - \cdots - c_k u_k \in W_1 \cap W_2$

$\Rightarrow -c_{r+1} u_{r+1} - \cdots - c_k u_k = d_1 w_1 + d_2 w_2 + \cdots + d_r w_r$

$\Rightarrow -c_{r+1}u_{r+1} - \cdots - c_k u_k - d_1 w_1 - d_2 w_2 - \cdots - d_r w_r = 0$

$\Rightarrow c_j = d_i = 0 \quad \forall\, i, j$

$\Rightarrow a_1 w_1 + \cdots + a_r w_r + b_{r+1} v_{r+1} + \cdots + b_n v_n = 0$

$\Rightarrow a_j = b_k = 0 \quad \forall\, j, k$.

$\mathbb{R}^3$

$\{(1,0,0), (0,1,0), (0,0,1)\}$ ✓

$\{(0,1,0), (1,0,0), (0,0,1)\}$ ✓

V. $B \subseteq V$.

!

$v = c_1 v_1 + c_2 v_2 + \cdots + c_k v_k \qquad v_i \in B$

$= d_1 v_1 + d_2 v_2 + \cdots + d_n v_n$

$\Rightarrow c_1 v_1 + c_2 v_2 + \cdots + c_k v_k = d_1 v_1 + d_2 v_2 + \cdots + d_n v_n$

$\Rightarrow (c_1 - d_1) v_1 + \cdots + (c_k - d_k) v_k + \cdots d_n v_n = 0$

$c_i = d_i \quad \forall\, i$

$v \qquad (c_1, c_2 \cdots c_k)$

$\mathbb{R}^3 \qquad (x_1, x_2, x_3)$

$= x_1(1,0,0) + x_2(0,1,0) + x_3(0,0,1)$

V = Set of all polynomials in x of degree ≤ 2

$\{1, x, x^2\}$ $\quad$ $\{1, x^2, x\}$

$\{1+x,\ 1-x,\ x^2\}$

$\{1-x,\ 1+x,\ x^2\}$

$3+x+x^2 = a_0(1+x) + a_1(1-x) + a_2 x^2$

$b_0(1-x) + b_1(1+x) + b_2 x^2$

(a_0, a_1, a_2) $\quad$ (b_0, b_1, b_2)

Ordered basis : A basis with an ordering.

$B \subseteq V$ ordered basis

$v \in V$ $\quad$ $v = \sum_{i=1}^{n} a_i v_i$

$[v]_B = (a_1, a_2, \ldots a_n)$

Coordinate vectors.