Recall: $(V\ F, +, \cdot)$

W, a subspace.

$a \cdot w_1 + w_2 \in W \quad \forall w_1, w_2 \in W, \ a \in F$

$S \subseteq V$ $\quad \operatorname{Span} S = \{ c_1 v_1 + c_2 v_2 + \cdots + c_n v_n : c_i \in F, \ v_i \in S \}$

↳ subspace

$S \subset V$ $\quad S$ is LD if $\exists\ v_1, v_2, \ldots v_n \in S$ and $c_1, c_2, \ldots c_n \in F$ not all $c_i = 0$

s.t. $\sum_{i=1}^{n} c_i v_i = 0$

If S is not LD we call it LI

Basis: $B \subset V$

(1) B span V $\quad V = \operatorname{Span} B$

(2) B is also LI

* Every VS has a basis.

$\{v_1\}$

* If V is f.d. Then any two bases have cardinality.

* Every LI set can be extended to a basis.

* $W \subseteq V$ a subspace then $\dim W \leq \dim V$

↳ f.d.

* Any spanning set contains a basis.

$S = \{\textcircled{$v_1$}, v_2, \ldots v_n\}$ $\quad S$ is a spanning set

$\mathcal{C} = \{ T \subseteq S \text{ and } T \text{ is LI} \}$

$T_1 \subseteq T_2 \subseteq - - \_\_$

$\bigcup T_i \subset S$

and LI $\quad M$ is a basis

V - V.S

$W \subseteq V$ a subspace.

$v_1, v_2 \in V$

$v_1 R v_2$ if $v_1 - v_2 \in W$

R is an equivalence relation

$[v_1] = \{ v \in V : v R v_1 \}$

$\Rightarrow v - v_1 \in W$

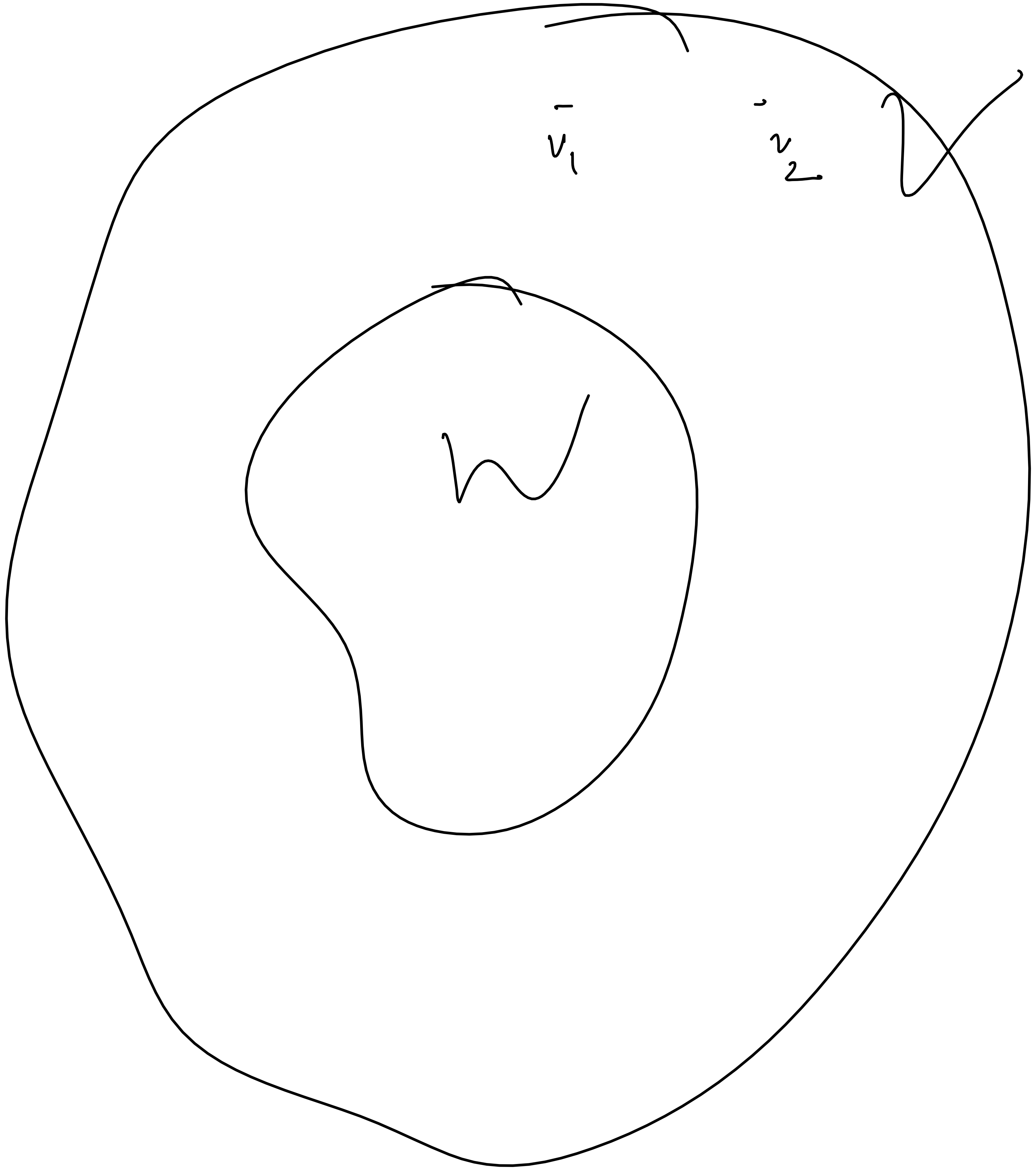
$\Rightarrow v - v_1 = w$ for some $w \in W$

$\Rightarrow v = v_1 + w$

$v_1 + W = \{ v_1 + w : w \in W \}$

$\Rightarrow [v_1] = v_1 + W$

$\Rightarrow V = \bigcup_i [v_i]$



$$\boxed{\frac{V}{W}} = \left\{ \boxed{v+\underline{W}} : v \in V \right\}$$

$$\boxed{v+\underline{W}} = \left\{ v+w : v \in W \right\}$$

Quotient space.

$$\underline{v_1+W} \ \boxed{+} \ \underline{v_2+W} = \boxed{(v_1+v_2)}+W$$

$$\underline{a \in F} \qquad \underline{a}\cdot(v_1+W) = \underline{\boxed{av_1}+W}$$

$\frac{V}{W}$ is a VS.

$0+W$

$v_1 +$

$[v_1] = [v_2]$?

$\{v_1 + w : w \in W\} = \{v_2 + w : w \in W\}$

iff $v_1 - v_2 \in W$

Suppose V is f-d.

What is the $\dim\left(\frac{V}{W}\right)$?

Let $\{w_1, w_2, \dots w_m\}$ be a basis of W

Extend $\{w_1, w_2, \dots w_m\}$ to be basis of V

$\{w_1, w_2, \dots w_m, w_{m+1}, \dots w_n\}$ is a basis of V

$m \leq n$

$m = \dim W$

$n = \dim V$

$S = \{w_{m+1} + W, w_{m+2} + W, \dots, w_n + W\}$

Claim: S is a basis of $\frac{V}{W}$.

$$a_{m+1}(w_{m+1} + W) + a_{m+2}(w_{m+2} + W) + \dots + a_n(w_n + W) = 0 + W$$

$$\Rightarrow (a_{m+1} w_{m+1} + a_{m+2} w_{m+2} + \dots + a_n w_n) + W = 0 + W$$

$$\Rightarrow a_{m+1} w_{m+1} + \dots + a_n w_n \in W$$

$$\Rightarrow a_{m+1}w_{m+1} + a_{m+2}w_{m+2} + \cdots + a_n w_n = b_1 w_1 + \cdots + b_m w_m$$

$$\Rightarrow a_{m+1}w_{m+1} + \cdots + a_n w_n - b_1 w_1 - b_2 w_2 - \cdots - b_m w_m = 0$$

$$\Rightarrow a_{m+1} = a_{m+2} = \cdots = a_n = 0$$

Claim: S spans $\frac{V}{W}$

$$\boxed{v} + W \in \frac{V}{W}$$

$v \in V$

$$\Rightarrow v = \underbrace{a_1 w_1 + a_2 w_2 + \cdots + a_m w_m}_{\overset{\uparrow}{W} \quad w'} + a_{m+1}w_{m+1} + \cdots + a_n w_n$$

$$\Rightarrow v = a_{m+1}w_{m+1} + \cdots + a_n w_n + \boxed{w'}$$

$$\Rightarrow v = a_{m+1}(w_{m+1} + W) + \cdots + a_n(w_n + W).$$

$\Rightarrow S$ is a basis of $\frac{V}{W}$.

V - V.S. V - f.d.

W_1 and W_2 are two subspaces.

$W_1 + W_2$ is also a subspace

$$\dim(W_1 + W_2) = \,?$$

$$W_1 + W_2 = \{w_1 + w_2 : w_1 \in W_1 \text{ and } w_2 \in W_2\}$$

$$\dim(W_1 + W_2) \leq \dim V$$

$$\dim(W_1 + W_2) = \dim W_1 + \dim W_2 \,?$$

$V = \mathbb{R}^3$ W_1 W_2

$\dim V = \boxed{3}$

XY plane 2

YZ plane 2

$\dim(W_1 + W_2) = ?$

$\underline{W_1 \cap W_2 = ?}$

$\underline{\dim(W_1 \cap W_2) = 1}$

$\dim(W_1 + W_2) = \overset{k}{\dim} W_1 + \overset{l}{\dim} W_2 - \overset{\boxed{r}}{\dim(W_1 \cap W_2)}$

~~$W_1 \subseteq V$ a subsp.~~ $W_2 \subseteq V$ a subsp.

~~Let $\{w_1, w_2, \ldots w_k\}$ be a basis of W_1~~

~~Let $\{v_1, v_2, \ldots v_l\}$ be a basis of W_2~~

Let $\{w_1, w_2, \ldots w_r\}$ be a basis of $W_1 \cap W_2$.

$W_1 \cap W_2 \subseteq W_1$

$W_1 \cap W_2 \subseteq W_2$

Let $\{w_1, w_2, \dots w_r, w_{r+1}, \dots w_k\}$ be a basis of W_1

Let $\{w_1, w_2, \dots w_r, v_{r+1}, \dots v_l\}$ be a basis of W_2

$$S = \{w_1, w_2, \dots w_r, w_{r+1}, \dots w_k, v_{r+1}, \dots v_l\}$$

claim: S is a basis of $W_1 + W_2$

✓ $w + w' \in W_1 + W_2$ $\quad w \in W_1 \quad w' \in W_2$

$$a_1 w_1 + a_2 w_2 + \dots + a_r w_r + b_{r+1} w_{r+1} + \dots + b_k w_k + c_{r+1} v_{r+1} + \dots + c_l v_l = 0.$$

$$\Rightarrow a_1 w_1 + a_2 w_2 + \dots + a_r w_r + b_{r+1} w_{r+1} + \dots + b_k w_k = -(c_{r+1} v_{r+1} + \dots + c_l v_l) \in W_1 \cap W_2$$

$$\Rightarrow -(c_{r+1} v_{r+1} + \dots + c_l v_l) = d_1 w_1 + d_2 w_2 + \dots + d_r w_r$$

$$\Rightarrow -c_{r+1} v_{r+1} - \dots - c_l v_l - d_1 w_1 - \dots - d_r w_r = 0$$

$\Rightarrow c_i = 0 \ \forall i$ and $d_i = 0$

$$\Rightarrow a_1 w_1 + \dots + a_r w_r + b_{r+1} w_{r+1} + \dots + b_k w_k = 0$$

$\Rightarrow a_i = 0 \quad b_j = 0 \quad \forall i, j.$

$\mathbb{R}^3$

$$\{(1,0,0), (0,1,0), (0,0,1)\}$$

$$\{(0,1,0), (1,0,0), (0,0,1)\}$$

B with an ordering is called an ordered basis.