Recall:

$V$

$W \subseteq V$ subspace.

$v_1, v_2 \in V$

$v_1 R v_2$ if $v_1 - v_2 \in W$

$R$ is an equivalence relation.

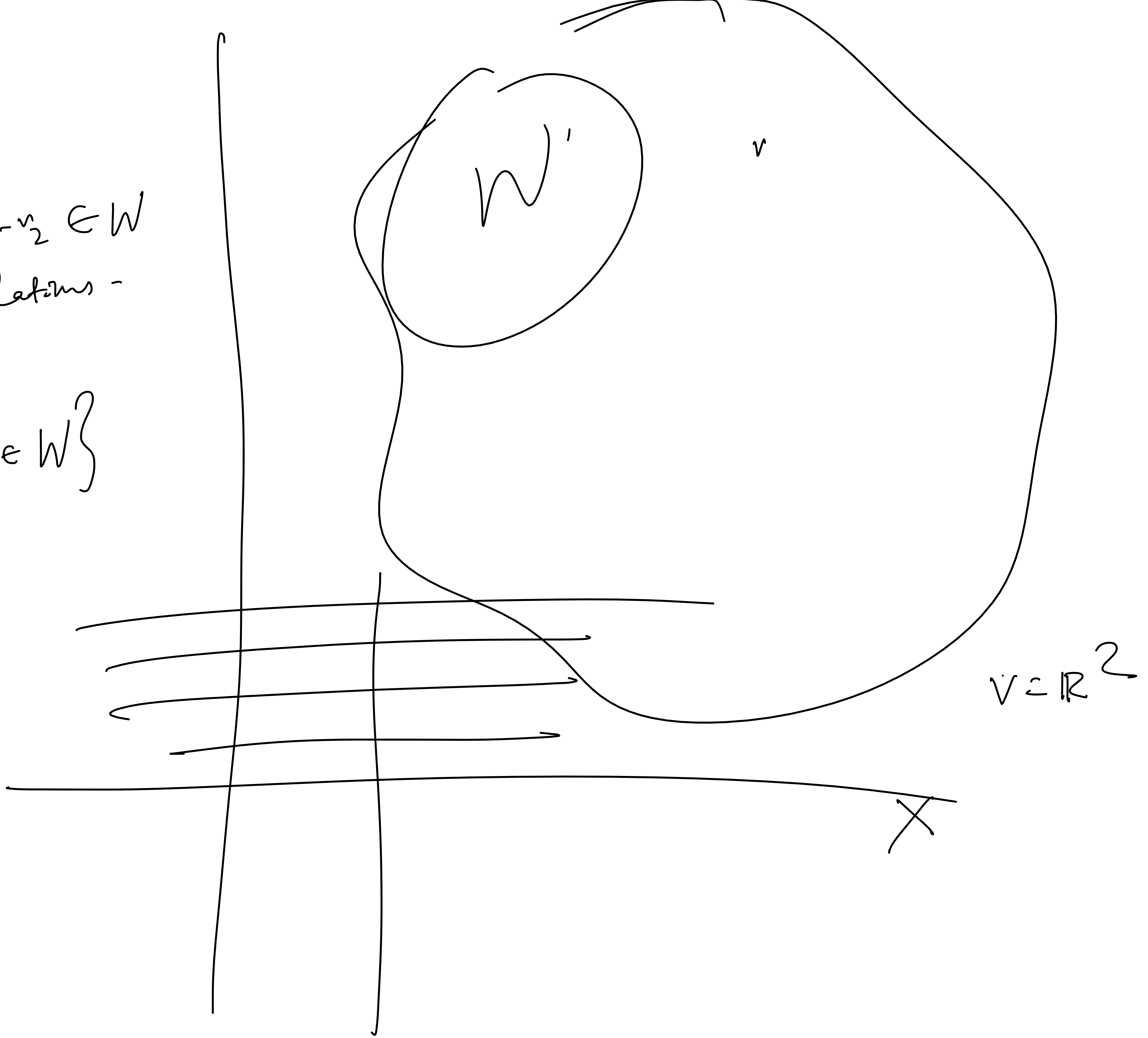
$[v] = \{$

$[v] = v + W = \{ v + w : w \in W \}$

$\frac{V}{W} = \{ v + W : v \in V \}$

$v + W \subseteq V$

$v + W \in \frac{V}{W}$



$$\dim\left(\frac{V}{W}\right) = \dim V - \dim W.$$

$W_1, W_2 \subseteq V$

$$\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$$

Ordered basis: A basis of $V$ with an ordering

$\{v_1, v_2, v_3, \ldots v_n\}$ $\dim V = n.$

$B = \{v_1, v_2, \ldots v_n\}$ - ordered

$v \in V$

$v = a_1 v_1 + a_2 v_2 + \ldots + a_n v_n$

$[v]_B = (a_1, a_2, \ldots a_n)$

co-ordinate vector of $v$ w.r.t $B$

$$W_1 \cap W_2 \subseteq W_1$$
$$\cap \qquad\qquad$$
$$W_2$$

$$V \to \mathbb{F}^n$$
$$v \mapsto (a_1, a_2, \ldots a_n)$$

$(a_1, a_2, \ldots a_n) = (b_1, b_2, \ldots b_n)$

bijection $\Rightarrow a_i = b_i$

$f : S_1 \to S_2$ finite

$f$ - one-one

$|S_1| \leq |S_2|$

$|S_1| \geq |S_2|$

F

$T: V_1 \longrightarrow V_2$

$\oplus \qquad \oplus$

$V_1 \quad + \quad \cdot$

$V_2 \quad \oplus \quad \times$

A map $T: V_1 \to V_2$ is called a linear map

$T(v_1 + v_2) = T(v_1) \oplus T(v_2)$

$T(a \cdot v_1) = a \times T(v_1)$

Examples: (1) $T: V_1 \to V_2$

$v \longmapsto 0$

(2) $T: V_1 \to V_1$

$v \longmapsto v$

(3) $T: V_1 \to V_1$ $\qquad a \in F$

$v \longmapsto a \cdot v$

$T(v+w) = a(v+w) = a\cdot v + a\cdot w$

$= T(v) + T(w)$

$T(c\cdot v) = a\cdot c \cdot v = c \cdot a \cdot v = c \cdot T(v)$

(4) $V_1 = \mathbb{R}^n \qquad V_2 = \mathbb{R}^m \qquad m < n$

$T: V_1 \to V_2$

$(x_1, x_2, \dots x_n) \longmapsto (x_1, x_2, \dots x_m)$.

$(y_1, y_2, \dots y_n)$

$T\big((x_1, x_2, \dots x_n) + (y_1, y_2, \dots y_n)\big)$

$= T(x_1+y_1, x_2+y_2, \dots x_n+y_n)$

$= (x_1+y_1, x_2+y_2, \dots x_m+y_m) = (x_1, x_2, \dots x_m) + (y_1, y_2, \dots y_m)$

$= T(x_1, x_2 \dots x_n) + T(y_1, y_2 \dots y_n)$

$T : \mathbb{R}^n \to \mathbb{R}^n$
$(x_1, x_2, \dots x_n) \mapsto (x_1, x_2, 0, 0 \dots 0)$

$A_{m\times n}$ - fixed.

$T : \mathbb{R}^n \to \mathbb{R}^m$

$X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}_{n\times 1} \mapsto A_{m\times n} X_{n\times 1}$ linear map

(6) $P(x)$ - Set of all polynomials in $x$

$\frac{d}{dx} : P(x) \to P(x)$
$f \mapsto \frac{d}{dx} f$ linear map

(7) $P(x) \to P(x)$
$a, b \in \mathbb{R}$ $f \mapsto \int_a^b f \cdot dx$

linear map

$T : V_1 \to V_2$
$T(0) = 0$ ?

$T(0+0) = T(0) + T(0)$
$T(0)$ $\Rightarrow T(0) = 0$

$T : V_1 \to V_2$
$\text{Image}(T) = \text{Range}(T)$
$= \{ v \in V_2 : v = T(v_1) \text{ for some } v_1 \in V_1 \}$
$\subseteq V_2$
a subspace of $V_2$ ?

$w_1, w_2 \in \text{Range}(T)$ $aw_1 + w_2$
$\exists$ $v_1$ s.t $T(v_1) = w_1$
$\exists$ $v_2$ s.t $T(v_2) = w_2$
$aw_1 + w_2 = aT(v_1) + T(v_2)$
$= T(av_1 + v_2) \in \text{Range}(T)$

$$T : V_1 \to V_2$$

$$\text{Kernel}(T) = \{ v_1 \in V_1 : T(v_1) = 0 \}$$

Null space $(T)$

$$= T^{-1}(\{0\}) \subseteq V_1$$

$$0 \in \text{Ker}(T) = \text{Null}(T)$$

a subspace.

$v_1, v_2 \in \text{Ker}(T)$ — subspace

$$T(a v_1 + v_2) = a T v_1 + T v_2$$
$$= a \cdot 0 + 0 = 0$$

$$T : V_1 \to V_2$$

$$\text{Ker}(T) \subseteq V_1, \quad \text{Range}(T) \subseteq V_2$$

$$\text{Nullity}(T) = \dim(\text{Ker}(T))$$
$$\text{Rank}(T) = \dim(\text{Range}(T))$$

Rank Nullity Theorem ;

let $T : V \to W$ be a linear map. $\dim V < \infty$

Then $\text{Rank}(T) + \text{Nullity}(T) = \dim V$.

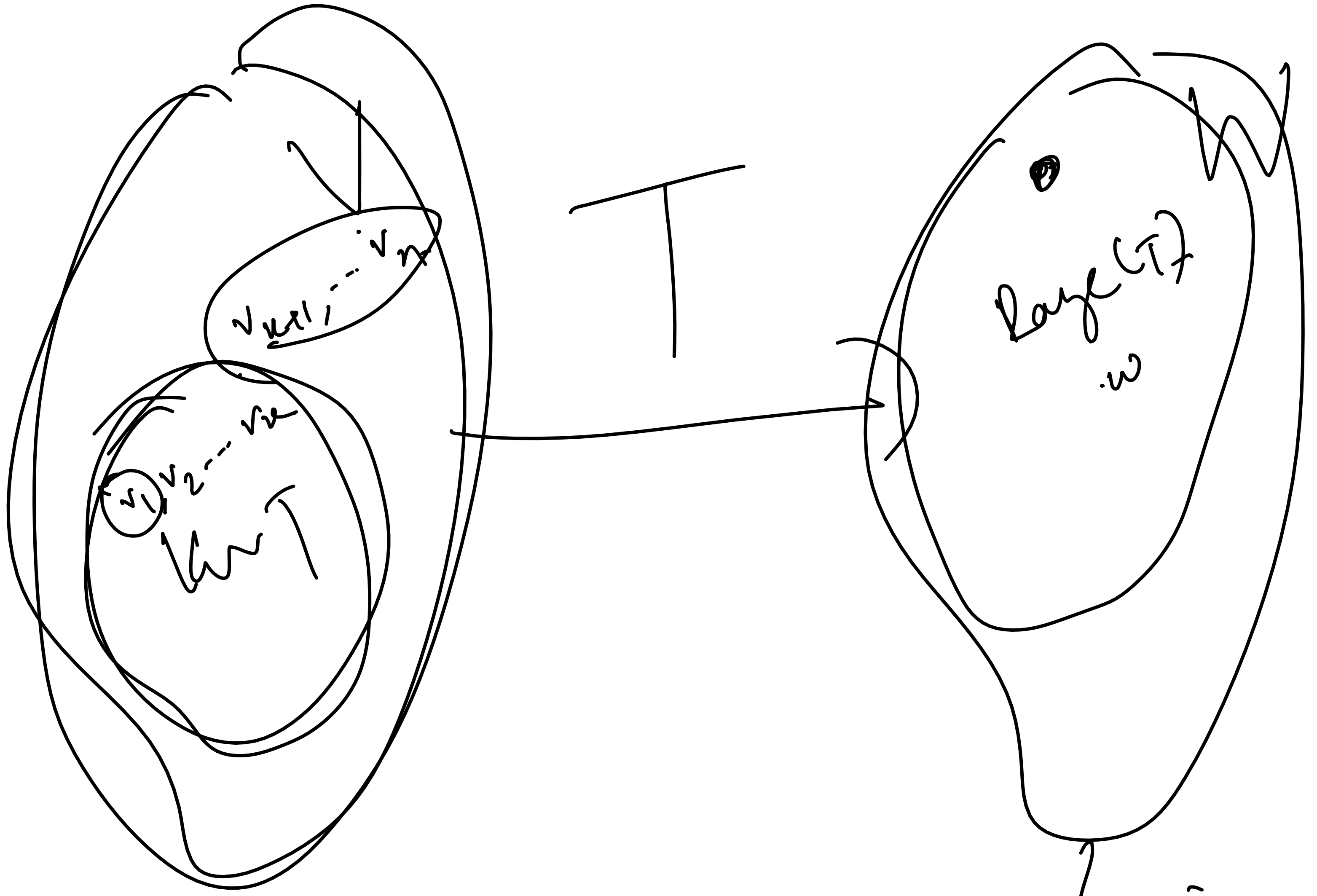
Proof :

$$T : V \to W$$

$$\text{Ker } T \subseteq V, \quad \text{Image}(T) \subseteq W$$

$\{v_1, v_2, \ldots v_k\}$ be a basis of $\text{Ker } T$

let $\{v_1, v_2, \ldots v_k, v_{k+1}, \ldots v_n\}$ be a basis of $V$




Claim: $\{ T(v_{k+1}), T(v_{k+2}), \dots T(v_n) \}$ is a basis of Range(T)

$a_{k+1}T(v_{k+1}) + a_{k+2}T(v_{k+2}) + \dots + a_n T(v_n) = 0$

$\Rightarrow T(a_{k+1}v_{k+1} + \dots + a_n v_n) = 0$

$\Rightarrow a_{k+1}v_{k+1} + \dots + a_n v_n \in Ker(T)$

$T(c_1 v_1 + \dots + c_n v_n) = c_1 T(v_1) + \dots + c_n T(v_n)$

$\Rightarrow a_{k+1}v_{k+1} + \dots + a_n v_n = b_1 v_1 + b_2 v_2 + \dots + b_k v_k$

$\Rightarrow a_{k+1}v_{k+1} + \dots + a_n v_n - b_1 v_1 - b_2 v_2 - \dots - b_k v_k = 0$

$\Rightarrow a_i = b_j = 0 \quad \forall i, j$

let $w \in Range(T)$

$\Rightarrow \exists v \in V$ s.t. $T(v) = w$

$\Rightarrow v = a_1 v_1 + a_2 v_2 + \dots + a_k v_k + a_{k+1}v_{k+1} + \dots + a_n v_n$

$\Rightarrow T(v) = T(\quad)$

$= a_1 T(v_1) + a_2 T(v_2) + \dots + a_k T(v_k) + a_{k+1}T(v_{k+1}) + \dots + a_n T(v_n)$

$= a_{k+1}T(v_{k+1}) + \dots + a_n T(v_n)$

$T : V \to W$ linear map

$\operatorname{Ker} T = \{0\}$

Claim: $T$ is one-one

$T(v_1) = T(v_2)$

$\Rightarrow T(v_1) - T(v_2) = \underline{T(v_1 - v_2) = 0}$

$\Rightarrow v_1 - v_2 \in \operatorname{Ker} T = \{0\}$

$\Rightarrow v_1 - v_2 = 0 \Rightarrow v_1 = v_2$

$T$ is one-one iff $\operatorname{Ker} T = \{0\}$

$T : V \to W \qquad \dim V < \infty$

$$\underbrace{\dim(\operatorname{Ker} T)}_{0} + \underline{\dim(\operatorname{Image}(T))} = \dim V$$

$\{0\}$

If $T$ is onto $\Rightarrow W = \operatorname{Image}(T)$