A map $T: V \longrightarrow W$ is called a linear map if $T(v_1 + v_2) = T(v_1) + T(v_2)$, $T(a \cdot v) = a \cdot T(v)$

V and W are v.s over F.

$\text{Range}(T) = \text{Image}(T) = \{T(v) : v \in V\}$

$\text{Kernel}(T) = \text{Null space}(T) = \{v \in V : T(v) = 0\} = T^{-1}(\{0\})$

Rank-Nullity Theorem:

$$\text{Rank}(T) + \text{Nullity}(T) = \dim(V)$$

$$T: V \longrightarrow W$$

$$V \supseteq \text{Ker}(T) \qquad W \supseteq \text{Range}(T)$$

$\text{Nullity}(T) = \dim(\text{Ker}(T))$ $\qquad$ $\text{Rank}(T) = \dim(\text{Range}(T))$

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* A linear map $T: V \to W$ is called an isomorphism if T is a bijection

T - linear

T - one-one

T - onto

* Two vector space V and W are said to be isomorphic if $\exists$ an isomorphism between them

$$V \cong W$$

$\exists\ T: V \to W$ $\qquad$ T linear, T - one one, T - onto

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$$T: V \longrightarrow W$$

$$V \supseteq \text{Ker}(T)$$

$$\frac{V}{\mathrm{Ker}(T)} = \{ v + \mathrm{Ker}\,T : v \in V \}$$

$$(v_1 + \mathrm{Ker}\,T) + (v_2 + \mathrm{Ker}\,T) = (v_1 + v_2) + \mathrm{Ker}\,T$$

$$a\cdot(v + \mathrm{Ker}\,T) = a\cdot v + \mathrm{Ker}\,T$$

Theorem: $T : V \to W$ linear. Then

$$\frac{V}{\mathrm{Ker}\,T} \cong \mathrm{Range}(T)$$

Proof: $T' : \frac{V}{\mathrm{Ker}\,T} \to \mathrm{Range}(T)$

$$v + \mathrm{Ker}\,T \longmapsto T(v)$$

T' is well defined

$$T : V \to W$$

$$\frac{V}{\mathrm{Ker}\,T} \to W$$

$\underline{v_1 + \mathrm{Ker}\,T} = \underline{v_2 + \mathrm{Ker}\,T}$

$\Rightarrow v_1 - v_2 \in \mathrm{Ker}\,T \Rightarrow T(v_1 - v_2) = 0$

$\Rightarrow T(v_1) - T(v_2) = 0$

$\Rightarrow \underline{T(v_1)} = \underline{T(v_2)}$

T' is one-one

$$v + \mathrm{Ker}\,T \longmapsto T(v) \in \mathrm{Ran}(T)$$

T' is onto

$$T'\big((v_1 + \mathrm{Ker}\,T) + (v_2 + \mathrm{Ker}\,T)\big)$$

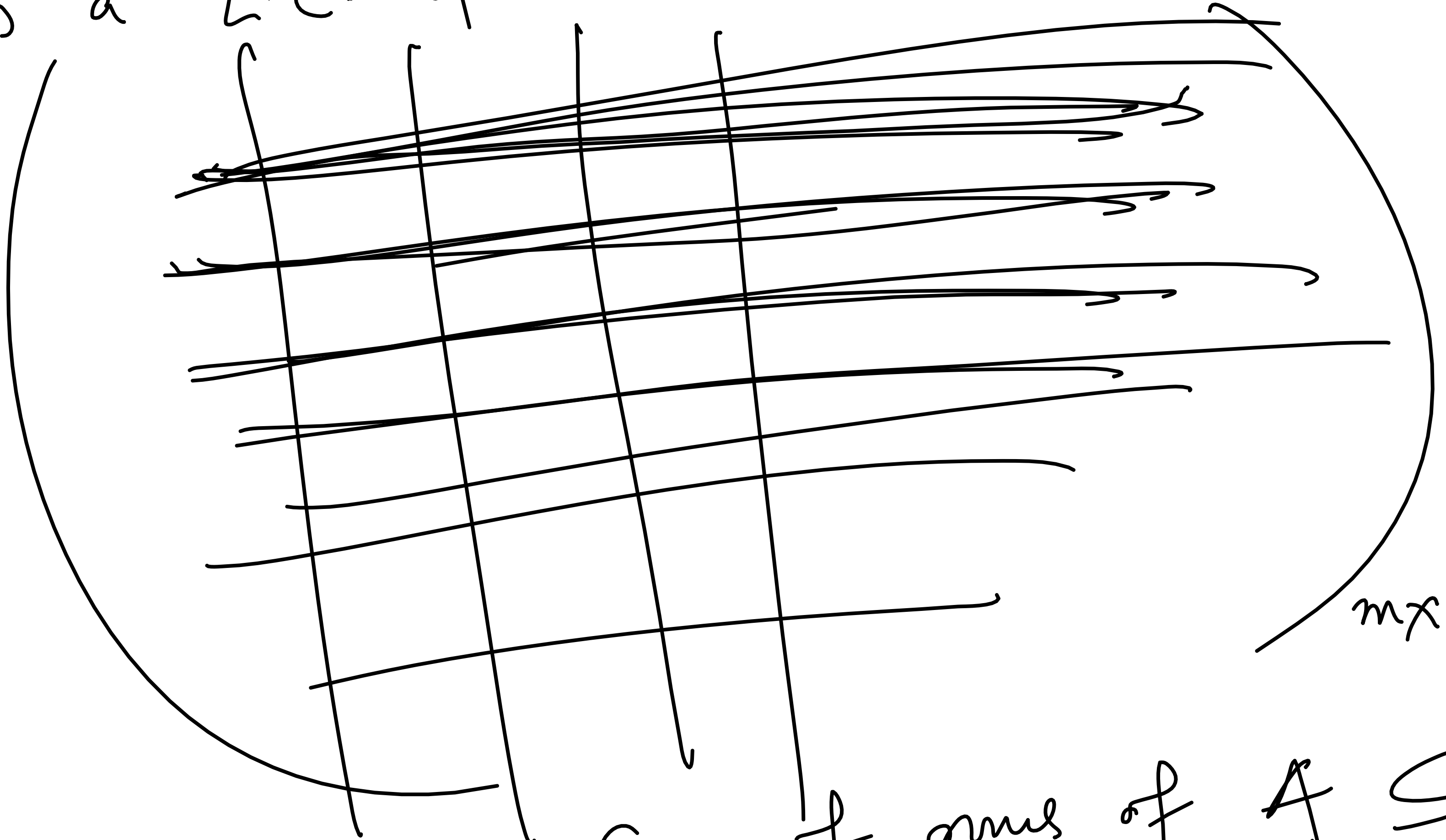
$$= T'\big((v_1 + v_2) + \mathrm{Ker}\,T\big)$$

$$= T(v_1 + v_2) = T(v_1) + T(v_2)$$

$$= T'(v_1 + \mathrm{Ker}\,T) + T'(v_1 + \mathrm{Ker}\,T)$$

$A_{m\times n}$ $\quad B_{m\times n}$

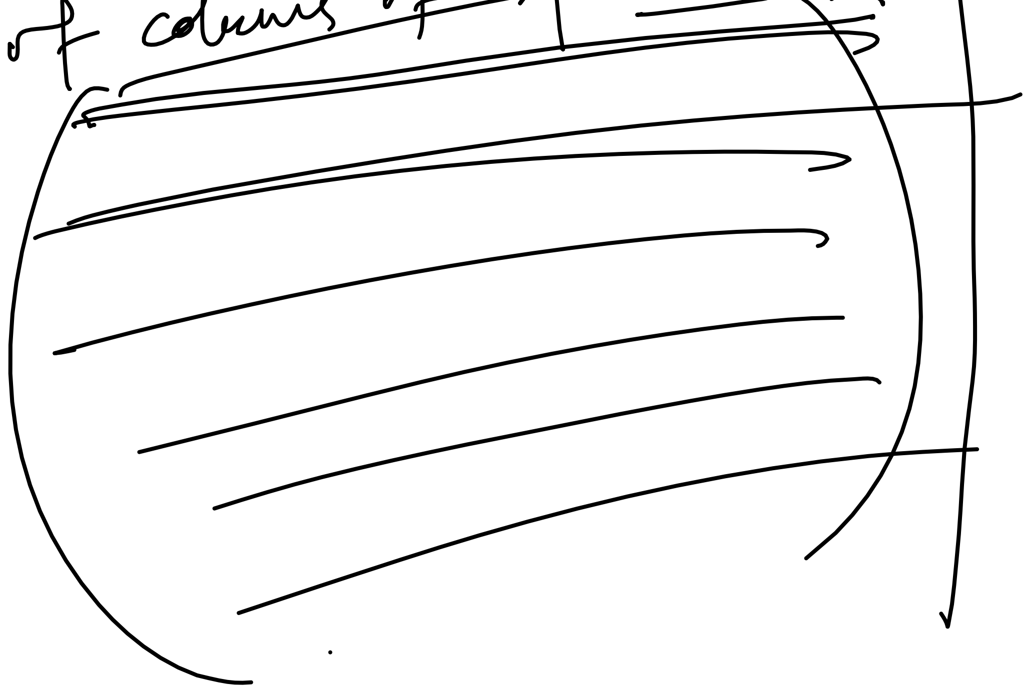
A and B are row-equivalent if every row of A is a L.C. of rows of B and vice-versa

$A =$ 

$m\times n$

Row-Space of A = Span of rows of $A \subseteq \mathbb{R}^n$

Column Space of A = Span of columns of $A \subseteq \mathbb{R}^m$

$R =$ 

Row-rank(A) = dim(Row-space of A)

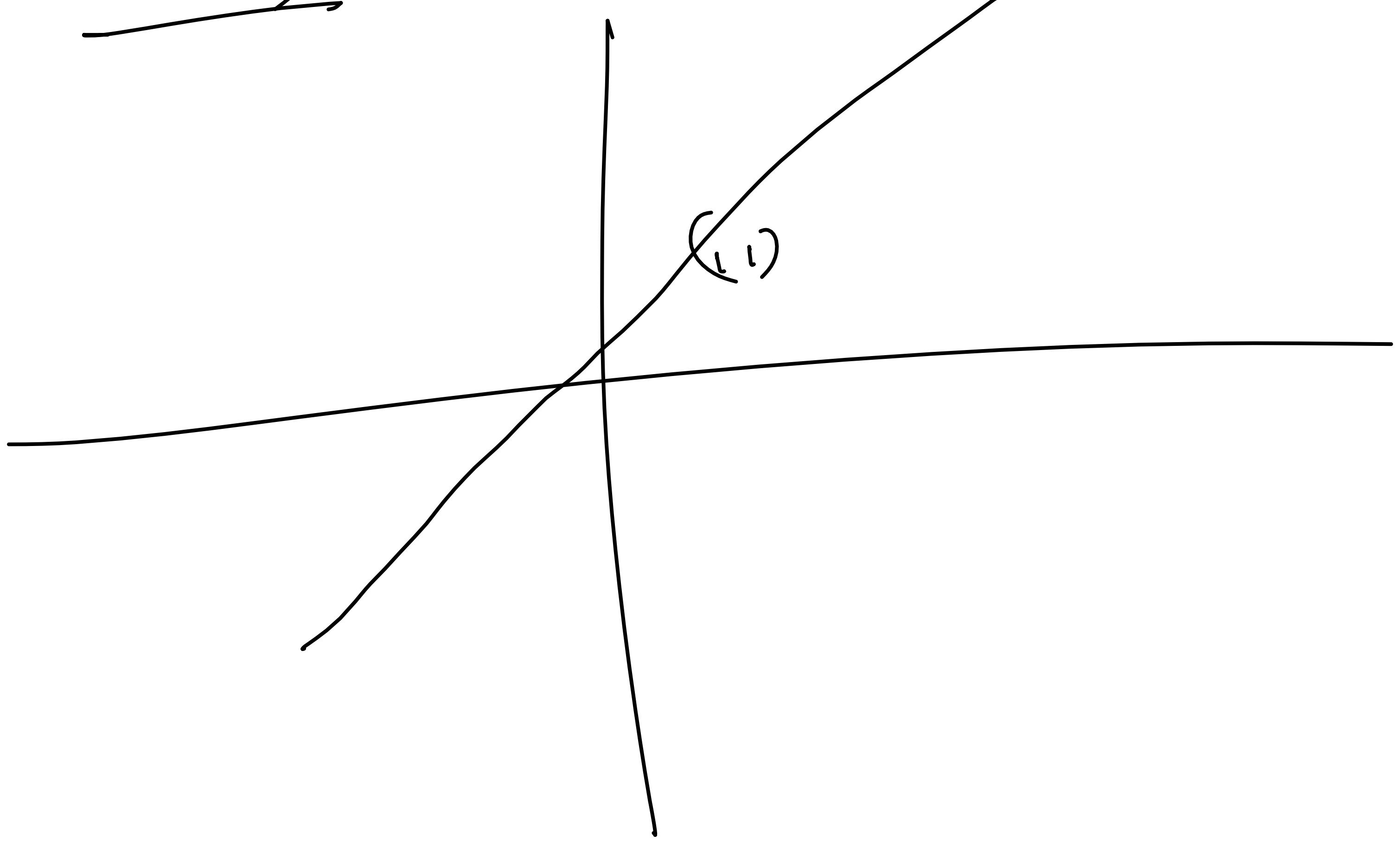
Column rank(A) = dim(Column-space of A)

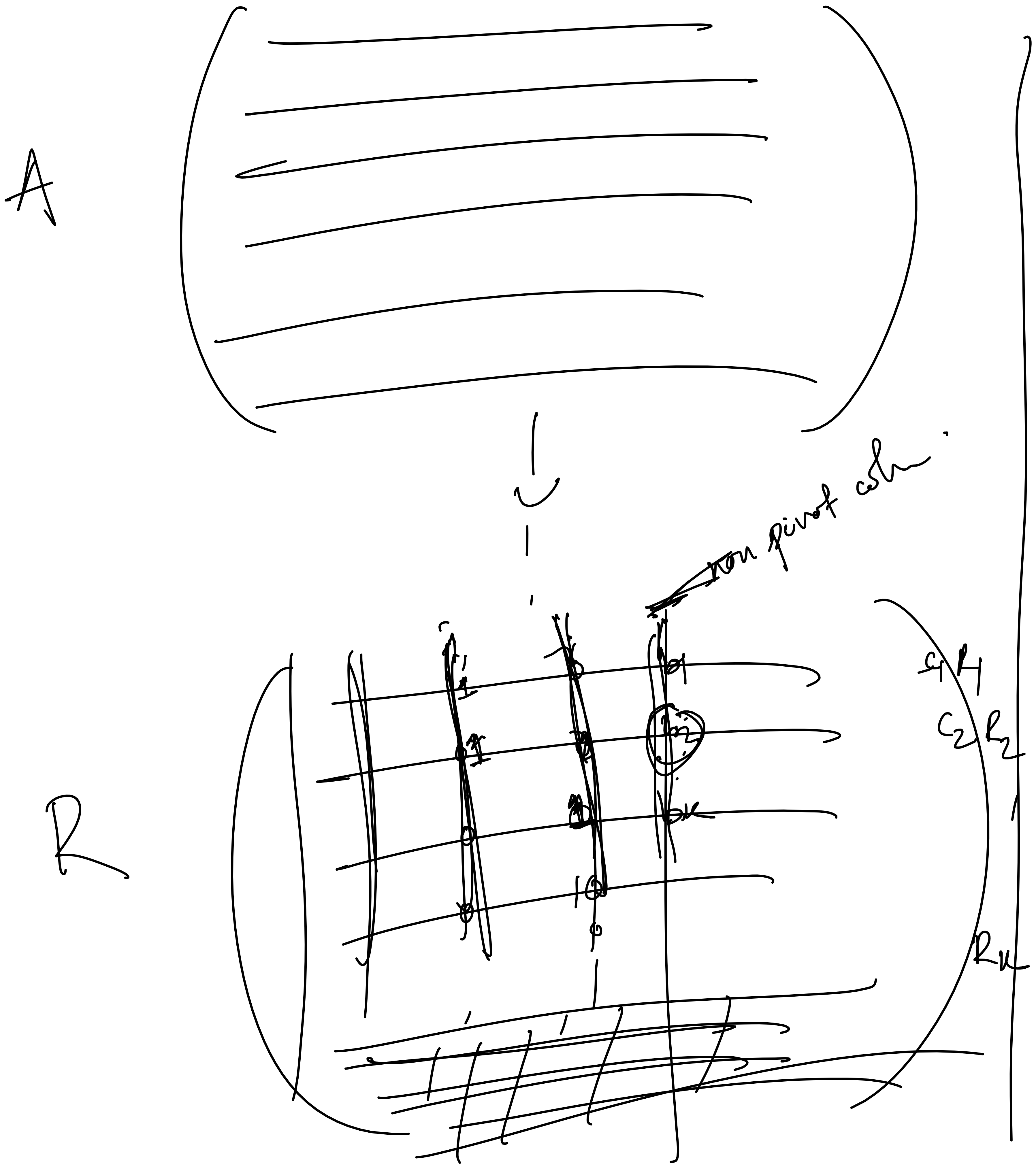
Ex. $\;\; - \; E_2 E_1 A_{m\times n} = R$

RREF

Row-space(A) = Row-space(R)

$A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ $\qquad R = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$





Row Span(A) = Row span(R)

Claim: The non-zero rows of R give a basis for Row. Space(A) = Row. space(R)

$$c_1R_1 + c_2R_2 + \cdots + c_kR_k = 0$$

$c_1 = 0$ $\quad$ $c_i = 0$

Claim: The pivot columns form a basis for the column space of R.

No of pivot column = No of non-zero rows
= Row rank(A)

no of basic variables + no of free variables = n

Row rank(A) + no of free variables = n

Null space $A_{m\times n}$

$$\mathrm{Null}(A) = \{ X \in \mathbb{R}^n : \underset{m\times n}{A} X_{n\times 1} = 0 \} \subseteq \mathbb{R}^m$$

$AX = 0$

x_{j_1} — free $\quad x_{j_2}$ — free $\quad x_{j_k}$

R

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11} x_{j_1} + a_{12} x_{j_2} + \cdots \\ a_{21} x_{j_1} + a_{22} x_{j_2} + \cdots \\ \vdots \\ x_{j_1} \\ \vdots \\ x_{j_2} \\ \vdots \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 0 \\ \vdots \\ \vdots \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ \vdots \end{pmatrix}$$

$\dim(\text{Null}(A)) = \text{Nullity}(A)$

$= \text{no of free variables}$

$= n - \text{row-rank}(A)$

Theorem: Let A be a $m \times n$ matrix.

Then row rank (A) = column rank (A)

Theorem: Let A be a $m \times n$ matrix. Then

$\text{Nullity}(A) + \text{row rank}(A) = n.$

Proof: $T : \mathbb{R}^n \to \mathbb{R}^m$ by

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}_{n\times 1} \longmapsto A_{m\times n} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}_{n\times 1}$$

T is a linear map.

$\text{Range}(T) = \{ AX : X \in \mathbb{R}^n \}$

$$\begin{array}{cccc} C_1 & C_2 & & C_n \\ \end{array}$$

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_n \end{pmatrix}$$

$= x_1 C_1 + x_2 C_2 + \cdots + x_n C_n.$

$=$ Column space of A

$\text{rank}(T) = \text{column-rank}(A)$

$$\operatorname{Ker} T = \{ X : AX = 0 \} = \text{Null space}(A)$$

$$\dim(\operatorname{Ker} T) = \text{Nullity}(A)$$

$$\text{Nullity}(T) = \text{Nullity}(A)$$

$$\text{rank}(T) + \text{Nullity}(T) = n$$

$$\text{Column rank}(A) + \text{Nullity}(A) = n$$

$$\text{row rank}(A) + \text{Nullity}(A) = n.$$