Recall: $T : V \longrightarrow W$ a map is called linear if

$$T(v_1 \pm v_2) = T(v_1) \pm T(v_2)$$

$$T(a \cdot v) = a \cdot T(v)$$

$$\text{Range}(T) = \text{Image}(T) = \{ T(v) : v \in V \}$$

$$\text{Kernel}(T) = \text{Null space}(T) = \{ v \in V : T(v) = 0 \}$$

$$= T^{-1}(\{0\}).$$

$$T(0) = 0$$

$$T : V \longrightarrow W$$

$$V \supseteq \text{Ker}(T) \qquad W \supseteq \text{Range}(T)$$

$$\dim(\text{Ker}(T)) + \dim(\text{Range}(T)) = \dim V.$$

$$\text{Nullity}(T) + \text{Rank}(T) = \dim V.$$

Rank Nullity Theorem.

$T : V \longrightarrow W$ linear.

$$V \supseteq \text{Ker}(T)$$

$$\frac{V}{\mathrm{Ker}(T)} = \{ v + \mathrm{Ker}\,T : v \in V \}$$

$$(v_1 + \mathrm{Ker}\,T) + (v_2 + \mathrm{Ker}\,T) = (v_1 + v_2) + \mathrm{Ker}\,T$$

$$a \cdot (v + \mathrm{Ker}\,T) = a \cdot v + \mathrm{Ker}\,T$$

Claim: $\frac{V}{\mathrm{Ker}\,T} \cong \underline{\mathrm{Range}(T)} \subseteq W$

$$T' : \frac{V}{\mathrm{Ker}\,T} \to \mathrm{Range}(T)$$

$$v + \mathrm{Ker}\,T \longmapsto T(v)$$

<u>Isomorphism</u>: A <u>linear map</u> between two vector spaces $T : V_1 \to V_2$ is an <u>isomorphism</u> if T is a bijection.

(1) $\underline{T \text{ linear}}$

(2) T is one-one

(3) T is onto

We say V_1 and V_2 are <u>isomorphic</u> if $\exists$ an isomorphism $T : V_1 \to V_2$

$$\underline{V_1 \cong V_2}$$

$$v_1 + \mathrm{Ker}\,T = v_2 + \mathrm{Ker}\,T$$

$$\Leftrightarrow v_1 - v_2 \in \mathrm{Ker}\,T \Leftrightarrow T(v_1 - v_2) = 0$$

$$\Leftrightarrow T(v_1) - T(v_2) = 0$$

$$\Leftrightarrow T(v_1) = T(v_2)$$

$$T'\big((v_1 + \mathrm{Ker}\,T) + (v_2 + \mathrm{Ker}\,T)\big) = T'\big((v_1 + v_2) + \mathrm{Ker}\,T\big)$$

$$= T(v_1 + v_2) = T(v_1) + T(v_2)$$

$$= T'(v_1 + \mathrm{Ker}\,T) + T'(v_2 + \mathrm{Ker}\,T)$$

$T'(a\cdot(v+\mathrm{Ker}\,T)) = a\cdot T'(v+\mathrm{Ker}\,T)$.

$\Rightarrow T'$ is linear.

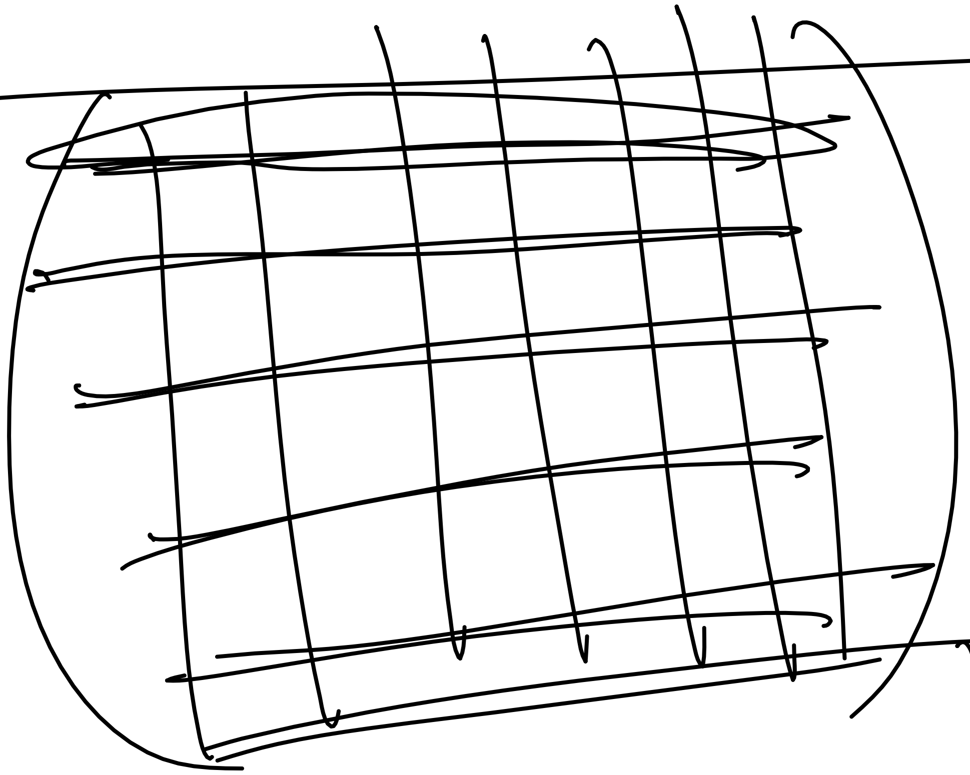
T' is one-one

$T(v) \in \mathrm{Rang}(T)$

$v+\mathrm{Ker}\,T \longmapsto T(v)$

T' is onto

T' is an isomorphism



$m\times n$

$A_{m\times n}$ $\qquad$ $B_{m\times n}$

We say A and B are row-equivalent if every row of A is a L.C of rows of B and vice-versa.

Ex. $E_k \cdots E_2 E_1 A = R$

RREF

Given a matrix A we define the row span of A to be span of all the rows of A

Row span of A = Span of all the rows $\subseteq \mathbb{R}^n$

Column span of A = Span of all the columns $\subseteq \mathbb{R}^m$

Row rank(A) = dim(Row space(A))

Column rank(A) = dim(Column space(A))

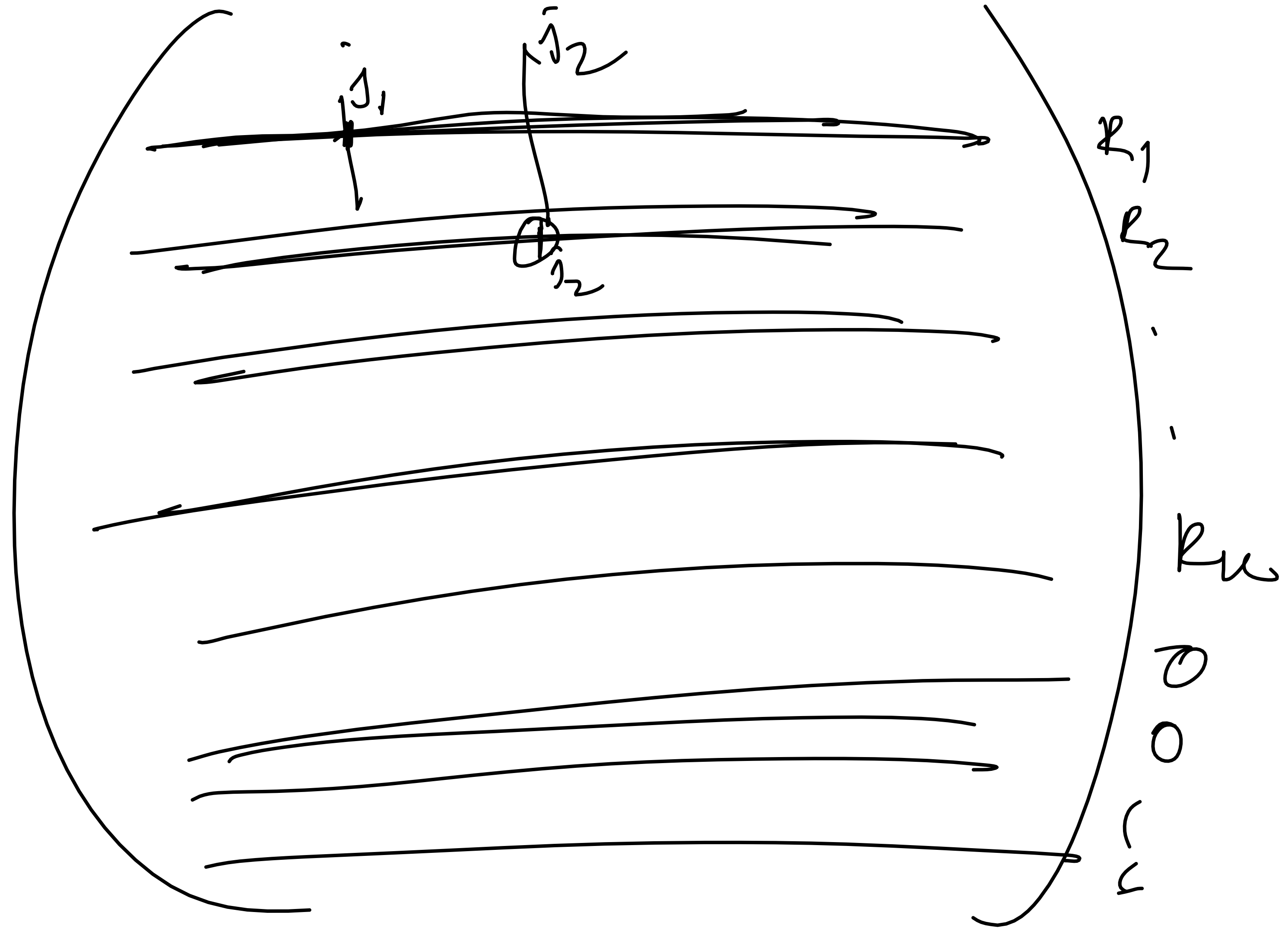
Row space(A) = Row space(R)

$\Rightarrow$ Row rank(A) = row rank(R)

* The non zero rows of R forms a basis of row space of A.

Ex: $E_2 E_1 A = R$ — RREF

R =



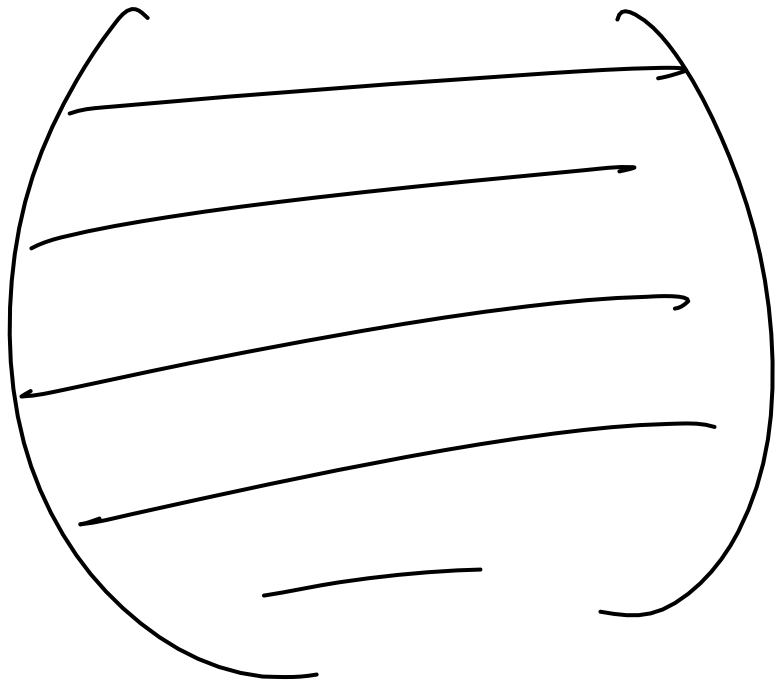
$c_1 R_1 + c_2 R_2 + \dots + c_k R_k = 0$

$c_1 = 0 \qquad c_i = 0 \ \forall i$

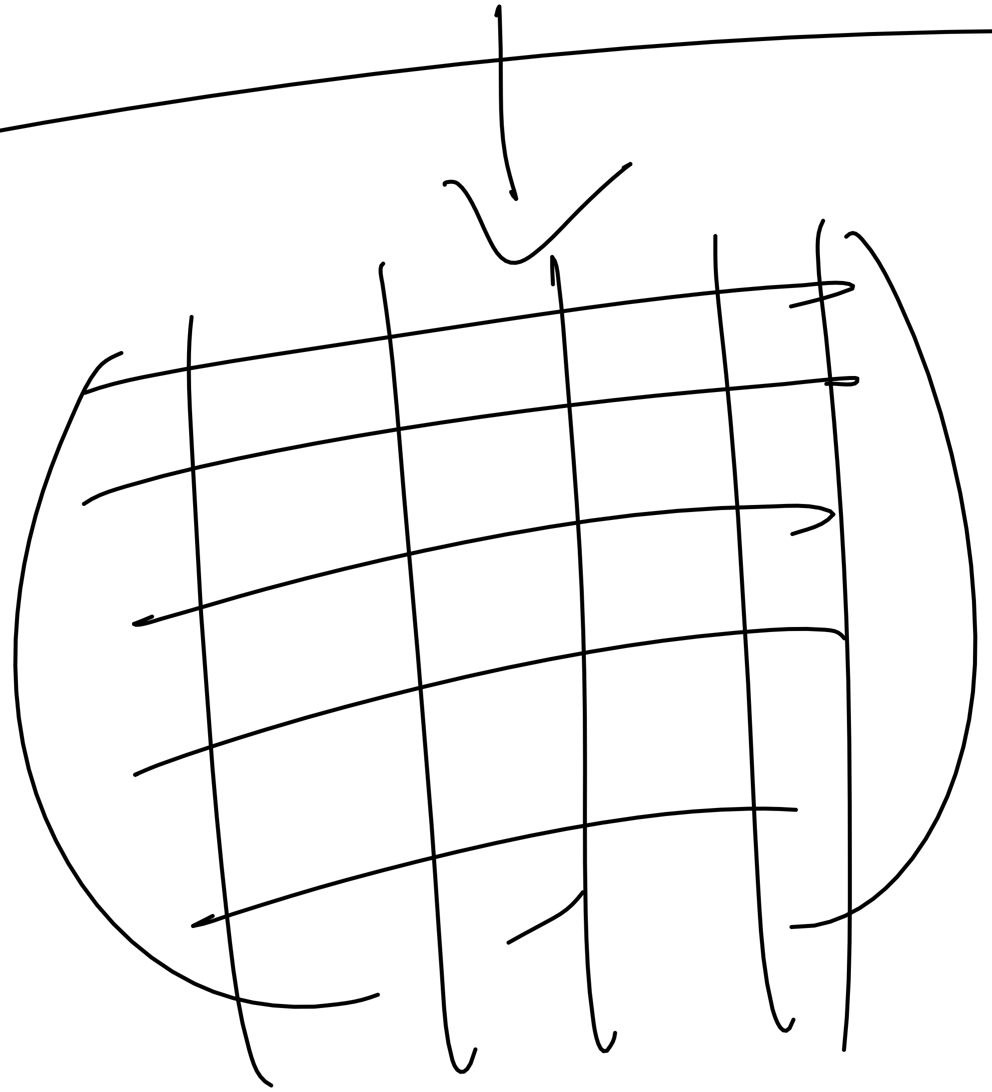
$$A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$

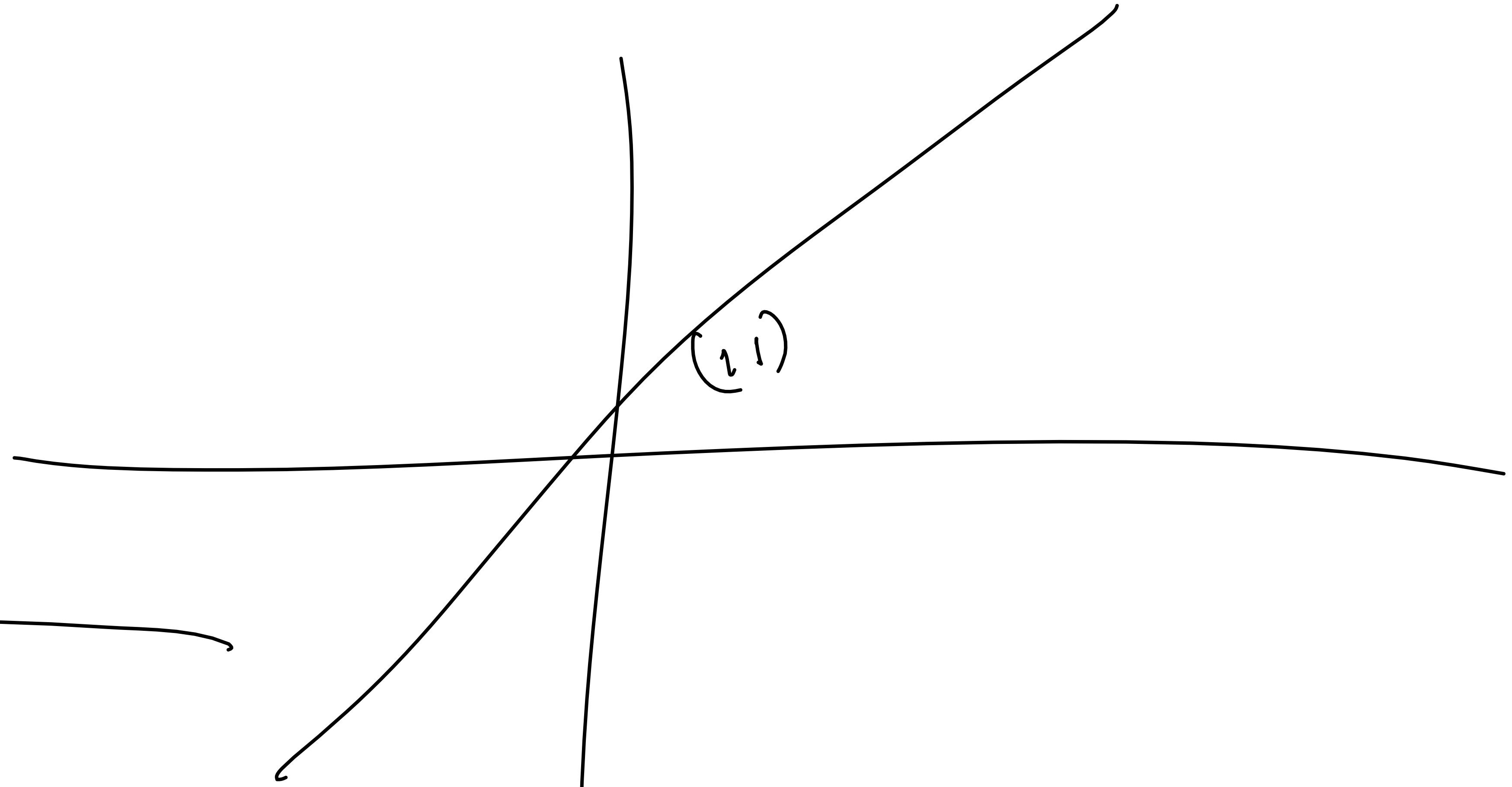
$$\downarrow$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$



R =





Column space (R)

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 9 & 0 \\ & & & 1 \\ & . & 1 & 0 \end{pmatrix} \cdots$$

Claim: The pivot columns form a basis for the column space of R.

no of pivot columns = no of nonzero rows.

column rank (R) = row rank (A)

no of basic variables + no of free variables = no of columns of A

$$AX = 0$$

$$\{ X_{n\times 1} : A_{m\times n} X_{n\times 1} = 0 \} \subseteq \mathbb{R}^n$$

non-pivot column

b_1

b_2

$\vdots$

b_r

0

0

$\vdots$

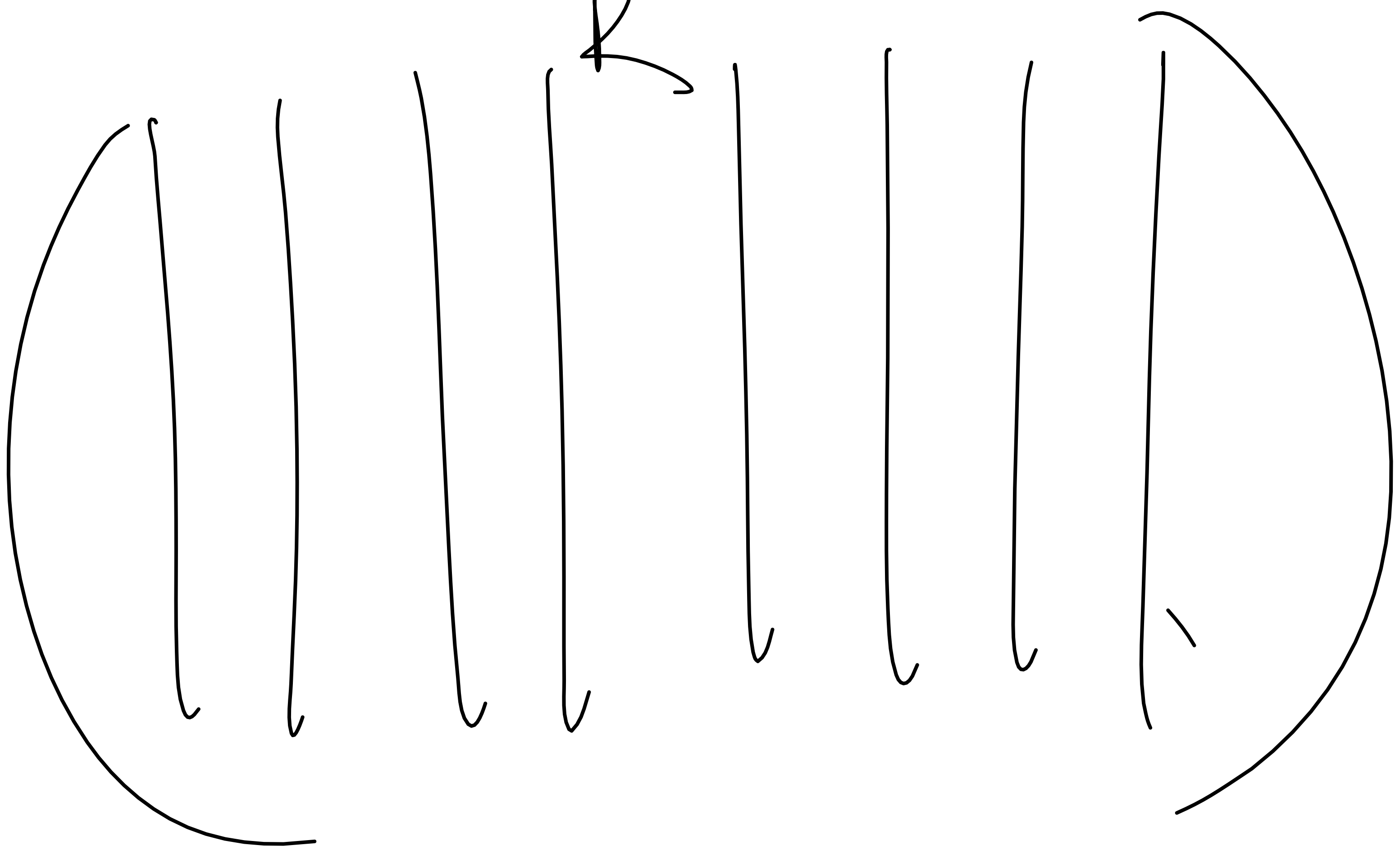
0

RREF

Given a matrix $A_{m\times n}$

$$\text{Null space}(A) = \{ X_{n\times 1} : A X_{n\times 1} = 0 \} \subseteq \mathbb{R}^n$$

$$E_k \cdots E_2 E_1 A \longrightarrow R$$



$RX = 0$

basic variable

$x_{j_1}, x_{j_2}, \ldots, x_{j_k}$ are free variables

$$x_2 = c_{j_1} x_{j_1} + c_{j_2} x_{j_2} + \cdots + c_{j_k} x_{j_k}$$

$$X = \begin{pmatrix} \cancel{x_1} \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \\ x_{j_1} \\ \\ x_{j_2} \\ \\ x_{j_k} \end{pmatrix}$$

$$= c_{j_1} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_{j_2} \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} + \cdots$$

$\dim(\text{Null}(A))$ = no of free variable

$\text{Nullity}(A)$ = n − no of basic variable

= n − row rank

Rank-Nullity Theorem for matrices: Let A be a $m \times n$ matrix. Then

$$\text{row rank}(A) + \text{Nullity}(A) = n$$

Proof ✓

$\text{Nullity}(A) = \dim(\text{Null}(A))$

Thm: Given a $m \times n$ matrix A.

$$\text{Row rank}(A) = \text{Column rank}(A)$$

Proof: Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be the map

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}_{n\times 1} \longmapsto A_{m\times n} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}_{n\times 1}$$

$$\text{Null}(T) = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} : A \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = 0 \right\}$$

$$= \text{Null}(A)$$

$$\text{Range}(T) = \{ AX : X \in \mathbb{R}^n \}$$

$$= x_1 C_1 + x_2 C_2 + \cdots + x_n C_n$$

$$= \text{Column Span of } A$$

$$\text{rank}(T) = \text{column rank}(A)$$

$$\text{rank}(T) + \text{Nullity}(T) = n$$

$$\text{column rank}(A) + \text{Nullity}(T) = n$$

$$\text{column rank}(A) + \underbrace{\text{Nullity}(A)}_{n - \text{row rank}(A)} = n$$

$$= n$$

$$\begin{array}{cccc}
C_1 & C_2 & & C_n \\
a_{11} & a_{12} & - & a_{1n} \\
a_{21} & a_{22} & & a_{2n} \\
| & & & \\
| & & & \\
a_{m1} & a_{m2} & , & a_{mn}
\end{array}$$