$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}_{m \times n}$$

* A and B can be added / subtracted iff they are of same size.

* $A_{m \times n}$ $\quad B_{n \times k}$

$$C = (c_{ij})$$

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

$a \in \mathbb{Z}$

Can we find $b \in \mathbb{Z}$ s.t $a \cdot b = 1$

$$\frac{a}{b} \in \mathbb{Z} \ ?$$

$r \in \mathbb{Q}$. if $\exists r'$ s.t $r \cdot r' = 1$

$$\frac{r}{r'} \in \mathbb{Q} \ ?$$

$r \in \mathbb{Q}$ we can always find $r' \in \mathbb{Q}$ s.t $r \cdot r' = r' \cdot r = 1$

A

We say A is invertible if $\exists$ a matrix B s.t $A \cdot B = B \cdot A = I$

A and B are square matrices

$$\begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{pmatrix}_{n \times n}$$

A

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} & \\ & 0 \end{pmatrix}$$

A Suppose $\exists$ B and C s.t

$$AB = BA = I$$
$$AC = CA = I$$

claim: $B = C$

$$B = B \cdot I = (BA)C = IC = C$$

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$$

$ax + by = c$ straight line

$ax + by + cz = d$ plane

$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = d$ hyperplane

System of linear equations

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = d_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = d_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = d_m$$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \quad X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \quad d = \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_m \end{pmatrix}$$

$$AX = d.$$

$$AX = d \quad \text{——} \quad (1)$$

We say (1) has a solution if $\exists$

$$y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix} \quad \text{s.t} \quad Ay = d$$

$$A_1 X = d_1 \qquad A_2 X = d_2$$

If $\exists\, c_1, c_2, \ldots, c_r$ s.t

$a_{11} y_1 + a_{12} y_2 + \ldots + a_{1m} y_m - d_1$

$= c_1 \begin{pmatrix} 0 \end{pmatrix} + c_2 \begin{pmatrix} \ \end{pmatrix} + \ldots$

$+ c_r \begin{pmatrix} \ \end{pmatrix} = 0$

$$\left\{ \begin{array}{l} a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n - d_1 = 0 \\ \vdots \\ a_{m1} x_1 + a_{m2} x_2 + \ldots + a_{mn} x_n - d_m = 0 \end{array} \right. \quad \text{——} \quad (1)$$

$$\left\{ \begin{array}{l} b_{11} x_1 + b_{12} x_2 + \ldots + b_{1n} x_n - e_1 = 0 \\ \vdots \\ b_{s1} x_1 + b_{s2} x_2 + \ldots + b_{sn} x_n - e_n = 0 \end{array} \right. \quad \text{——} \quad (2)$$

$$x + y = 1 \qquad 2x + 3y = 2$$

$$2x + 3y = 4 \qquad 4x + 6y = 8$$

$A_1 X = d_1 \qquad A_2 X = d_2$

Claim: Equivalent systems have same set of solutions.

If y is a sol^n then αy is also a sol^n.

$$AX = d$$

$d = 0$ homogeneous

$d \neq 0$ non homogeneous

* A homogeneous system has always a sol^n.

$$A \cdot \begin{pmatrix} \alpha y_1 \\ \alpha y_2 \\ \vdots \\ \alpha y_n \end{pmatrix} = 0$$

$AX = d \qquad d \neq 0$

$x + y = 5$
$x + y = 3$

$x + y = 5$

$3 \quad x + 2y = 3 \qquad 3x + 6y = 9$
$\cancel{3}x + 4y = 5 \qquad 3x + 4y = 5$

$2y = 4$
$y = 2$

$$\begin{pmatrix} 3 & 6 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 9 \\ 5 \end{pmatrix}$$

$$Ax = d$$

$$y \neq y'$$

*. A nonhomogeneous system can have no solⁿ or can have a unique solution or can have infinitely many solutions.

$$Ay = d \qquad Ay' = d$$

$$A(y - y') = 0$$

$$A(\alpha(y-y')) = 0$$

$$A(y + \alpha(y-y')) = d$$

$$x + 2y = 3$$
$$3x + 4y = 5$$

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$$

$$x + 2y = 3$$

Replace the 2nd eqⁿ by 2nd eqⁿ -3 (1st eqⁿ)

$$\begin{pmatrix} 1 & 2 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$$

$$Ax = d$$

$$(A \mid d) = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & d_1 \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & d_m \end{array}\right)_{m \times (n+1)}$$

Augmented matrix of the above system.

Aim :

Elementary Row operation

R_{ij} = Interchanging ith row and jth row

$R_i(c)$ = Multiplying c to the ith row

$R_{ij}(c)$ = Replacing ith row by ith row $+ c \cdot j$th row.

$R_i \longleftrightarrow R_j$

$R_i \longmapsto c \cdot R_i$

$R_i \longmapsto R_i + cR_j$

$$R_{ij}\begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix} = \begin{pmatrix} 1 & & & & & & \\ & 1 & & & 1 & & \\ & & 0 & & & & \\ & & & 0 & & & \\ & 1 & & & 1 & & \\ & & & & & 1 & \\ & & & & & & 1 \end{pmatrix}$$

E_{ij}

$$R_i(c)\begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & c & \\ & & & 1 \\ & & & & 1 \end{pmatrix}$$

$E_i(c)$

$$R_{ij}(c)\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} = \begin{pmatrix} 1 & & & c \\ & 1 & & \\ & & 1 & \\ & & & 1 \\ & & & & 1 \end{pmatrix}$$

$E_{ij}(c)$

$E_{ij} \cdot E_{ij} = I$

$E_i(c) \cdot E_i(1/c) = I$

$E_{ij}(c) \cdot E_{ij}(-c) = I$