$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}_{m \times n}$$

$A_{m \times n}$ $\quad B_{n \times k}$

$C = AB = (c_{ij})$

$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$

$A \quad B$

$a \in \mathbb{Z}$

$b = a^{-1} = \frac{1}{a}$

$a \in \mathbb{R} \ \exists \ b \in \mathbb{R}$ st

$a \cdot b = b \cdot a = 1$

A - B

We say A is invertible if $\exists B$ st

$A \cdot B = B \cdot A = I$

$\Rightarrow$ A and B are square matrices of same size

$A \quad \exists B, C$ st

$A \cdot B = BA = I$

$A \cdot C = C \cdot A = I$

$B = B \cdot I = (BA)C = IC = C$

AB

$A B B^{-1} A^{-1} = A(BB^{-1})A^{-1}$

$= AA^{-1} = I$

$ax + by = c$ line

$ax + by + cz = d$ plane

$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = d$

hyperplane

System of linear equations

$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = d_1$

$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = d_2$

$\vdots$

$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = d_m$

$A = \left(a_{ij} \right)_{m \times n}$ $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$

$d = \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_m \end{pmatrix}$

$AX = d$

(1) — $AX_1 = d_1$ $\qquad$ $AX_2 = d_2$ — (2)

We say (1) and (2) are equivalent.

Claim: Two equivalent systems have same set of solutions.

$y_1, y_2, \dots y_n$ is a solution of (1)

$b_{11}x_1 + b_{12}x_2 + \dots + b_{1n}x_n - e_1$

$= c_1(\ \ ) + c_2(\ \ ) + \dots + c_m(\ \ )$

$b_{11}y_1 + \dots + b_{1n}y_n - e_1 = 0$

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n - d_1 = 0$$
$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n - d_2 = 0$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n - d_m = 0$$
— (1)

$$b_{11}x_1 + b_{12}x_2 + \dots + b_{1n}x_n - e_1 = 0$$
$$b_{21}x_1 + b_{22}x_2 + \dots + b_{2n}x_n - e_2 = 0$$
$$\vdots$$
— (2)

$AX = d$

If $d = 0$ homogeneous

If $d \neq 0$ nonhomogeneous

* Every homogeneous system has a solⁿ.
* If a homogeneous system has a nonzero solution then it has infinitely many sol'ns.

$$\begin{pmatrix} \alpha y_1 \\ \alpha y_2 \\ \vdots \\ \alpha y_n \end{pmatrix}$$

$$A \begin{pmatrix} \alpha y_1 \\ \vdots \\ \alpha y_n \end{pmatrix} = 0$$

* A nonhomogeneous system have a unique solⁿ or no solution or infinitely many sol'ns

$$x + y = 2$$
$$x + y = 4$$

$3\times$ $\quad x + 2y = 1$

$$3x + 4y = 3$$

$$3x + 6y = 3$$
$$3x + 4y = 3$$
$$2y = 0$$

$y \quad z$

$Ay = d \quad Az = d$

$A(y - z) = Ay - Az = 0$

$A(\alpha(y - z)) = 0$

$$A\left(y + \alpha(y-z)\right) = d$$

$$AX = d$$

$$\begin{pmatrix} A & d \end{pmatrix}$$ augmented matrix.

$$\begin{array}{c} \left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & d_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & d_2 \\ \vdots & & & & \\ a_{m1} & a_{m2} & & a_{mn} & d_m \end{array}\right) \begin{array}{l} \longrightarrow R_1 \\ \longrightarrow R_2 \\ \longrightarrow R_m \end{array} \end{array}$$

$m \times (n+1)$

$$3x + 6y = 3$$
$$0 \cdot x + (-2y) = 0$$

2nd equ^n $\longrightarrow$ 2nd equ^n $- 3$ (1st equ^n)

$$\begin{pmatrix} 3 & 6 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \quad R_{21}(-3)$$

R_{ij} = interchange ith and jth row.

$R_i(c)$ = R_i is multiplied by c

$R_{ij}(c)$ = Replace ith row by c times jth row

$R_{ij}(c)$

$$R_i \longmapsto R_i + cR_j$$

$$R_{ij}\begin{pmatrix} 1 & & & & \\ & 1 & & 0 & \\ & & 1 & & \\ & 0 & & & \\ & & & 1 & \end{pmatrix} \qquad E_{ij}$$

$$\begin{pmatrix} 1 & & & & \\ & 1 & & 1 & \\ & & 0 & & \\ 1 & & & 1 & \\ & & & & 1 \end{pmatrix}$$

$$R_i(c)\begin{pmatrix} 1 & & & \\ & 1 & & 0 \\ & & c & \\ 0 & & & \\ & & & 1 \end{pmatrix} - E_i(c)$$

$$E_i\left(\frac{1}{c}\right)$$

$$R_{ij}(c)\begin{pmatrix} 1 & & c & \\ & 1 & & \\ & & 1 & \\ & & & \\ & & & 1 \end{pmatrix} - E_{ij}(c)$$

elementary matrices : $E_{ij}(-c)$