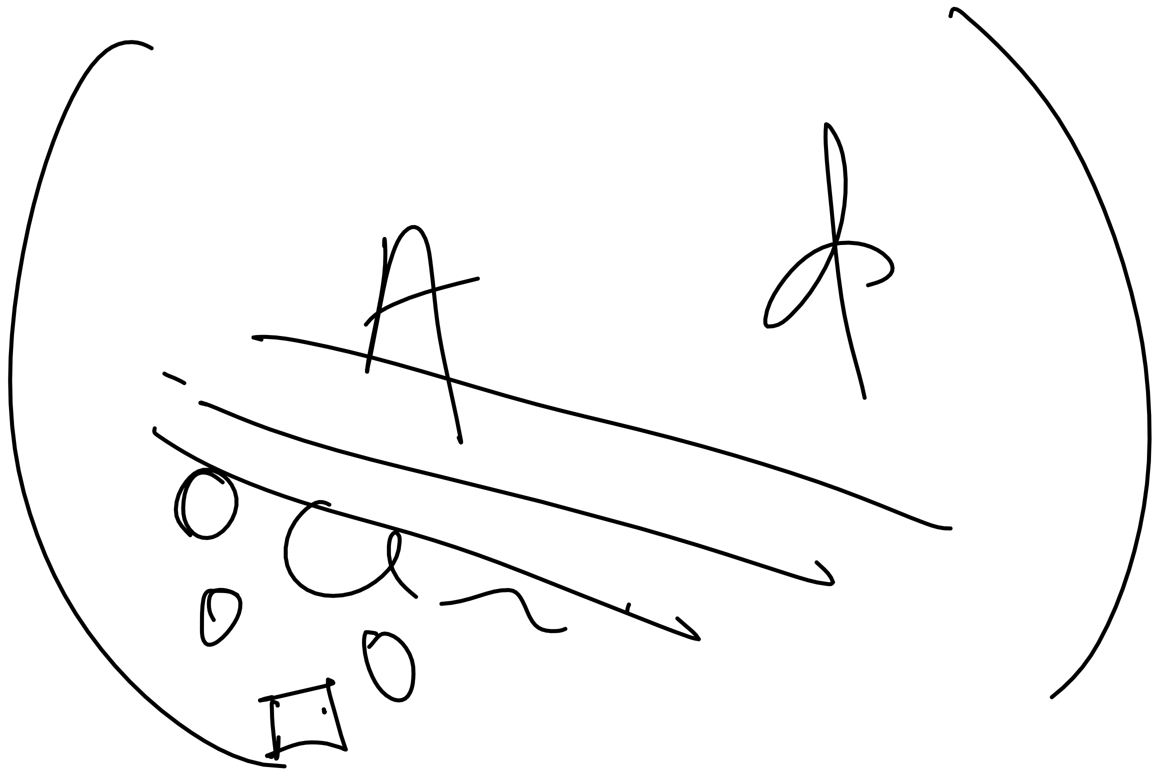
$$A_{m\times n} X = d$$

$$\left(A \;\middle|\; d \right)_{m\times(n+1)}$$

(Augmented Matrix)



Row Operations

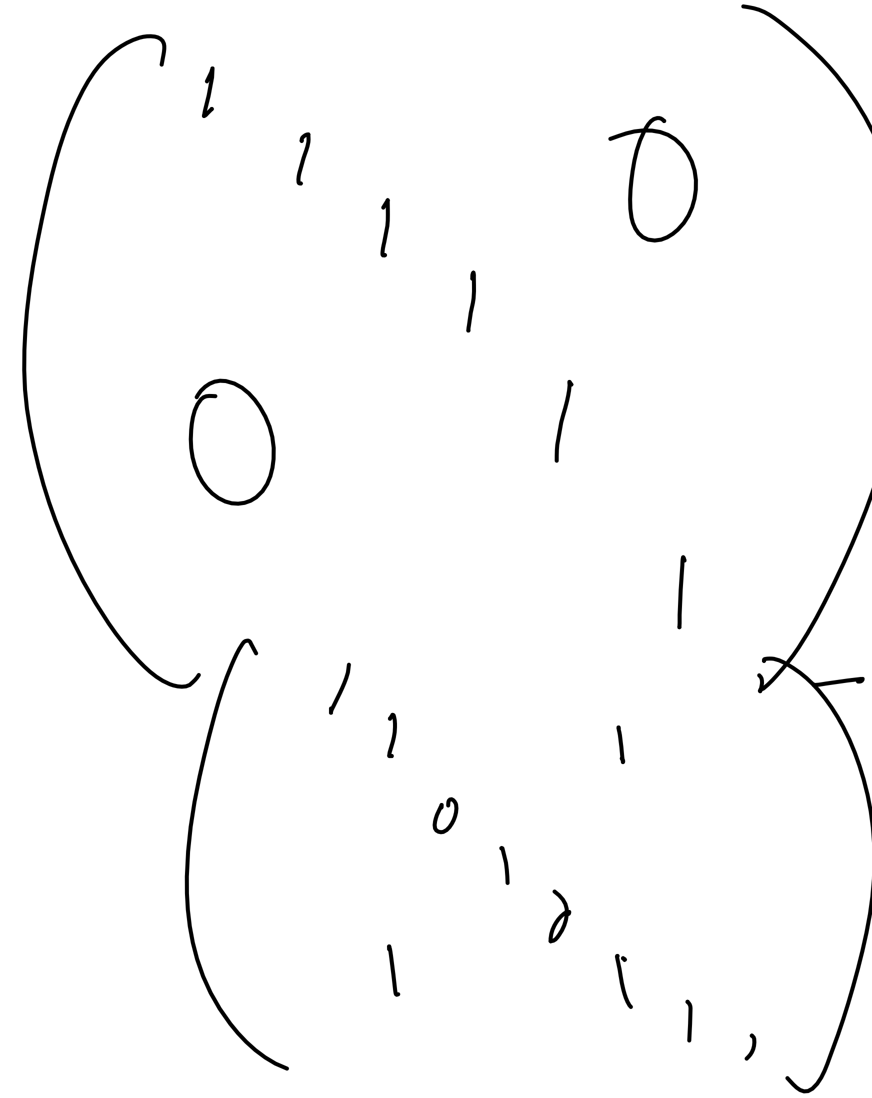
(1) R_{ij} = Interchange ith row and jth row

(2) $R_i(c)$ = Multiplying c to ith row

(3) $R_{ij}(c)$ = ith row is replaced by ith row + c times jth row

$$R_{ij} = R_{ji}$$

R_{ij}



$=$ E_{ij}

$$R_{ij} \begin{pmatrix} & & \\ & & \end{pmatrix}$$

$$= E_{ij} \begin{pmatrix} & & \\ & & \end{pmatrix}$$

$$R_{12} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}_{2\times 3} = \begin{pmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \end{pmatrix}$$

$$E_{12} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

$$R_i(c)\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & & & & & 0 \\ & 1 & & & & \\ & & \ddots & & & \\ & & & c & & \\ & 0 & & & 1 & \\ & & & & & 1 \end{pmatrix}}_{E_i(c)}$$

$$E_i(c)\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

$$R_{ij}(c)\begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix}$$

$$= \underbrace{\begin{array}{c} \begin{pmatrix} 1 & & & & & & \\ & 1 & & c & & & \\ & & \ddots & & & & \\ & & & 1 & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix} \end{array}}_{E_{ij}(c)} \quad \text{ith row}$$

$$R_{ij}(c)\,A = E_{ij}(c)\,A$$

Elementary Matrices.

E_{ij}

$E_{ij} = E_{ji}$

$E_{ij} E_{ij} I = I$

$E_{ij} E_{ij} = I$

$E_i(c) E_i(1/c) = I = E_i(1/c) \cdot E_i(c)$

$E_{ij}(c) \; E_{ij}(-c) = I = E_{ij}(-c) E_{ij}(c)$

$AX = d$ $\quad (A|d)$

Row Echelon form of a Matrix

A matrix is said to be in row echelon form if

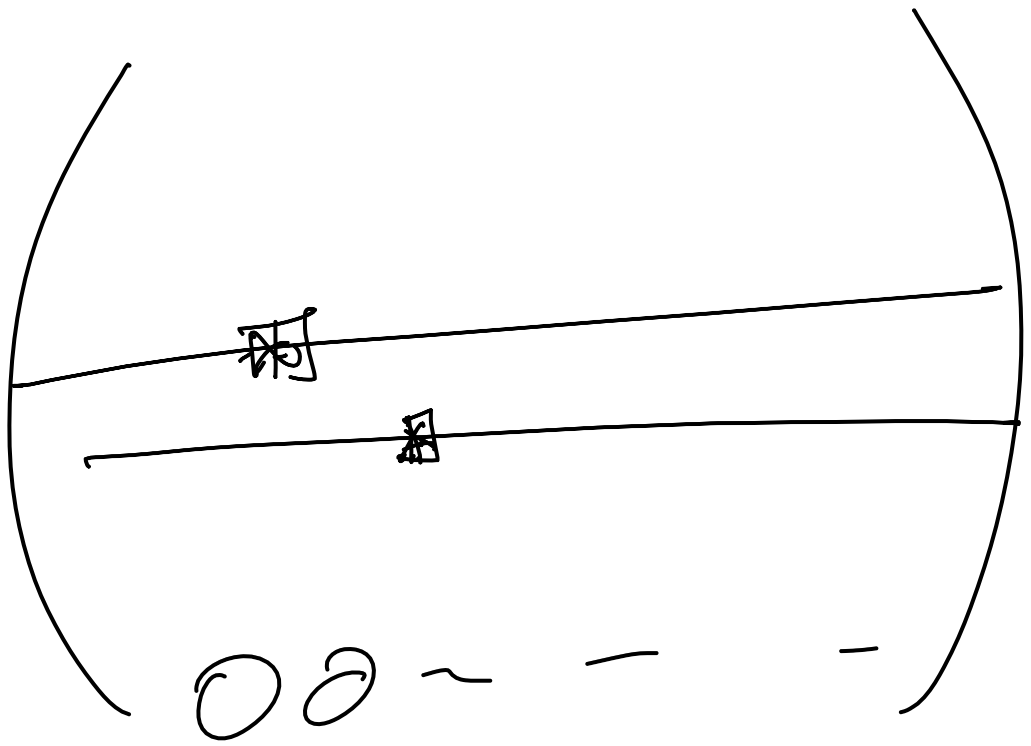
(1) Zero rows are below the nonzero rows

(2) The leading coefficient of a nonzero row is strictly right to the leading coeffnt of the row above it.

Pivot = 1st non-zero entry of the row.

Example

$$\begin{pmatrix} 3 & 0 & 5 & 2 & 3 \\ 0 & 0 & 2 & 1 & 4 \\ 0 & 0 & 0 & 2 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$



Row Reduced Echelon form (RREF)

A Matrix is said to be in RREF

(1) It is in row echelon form

(2) All the leading coefficients are 1

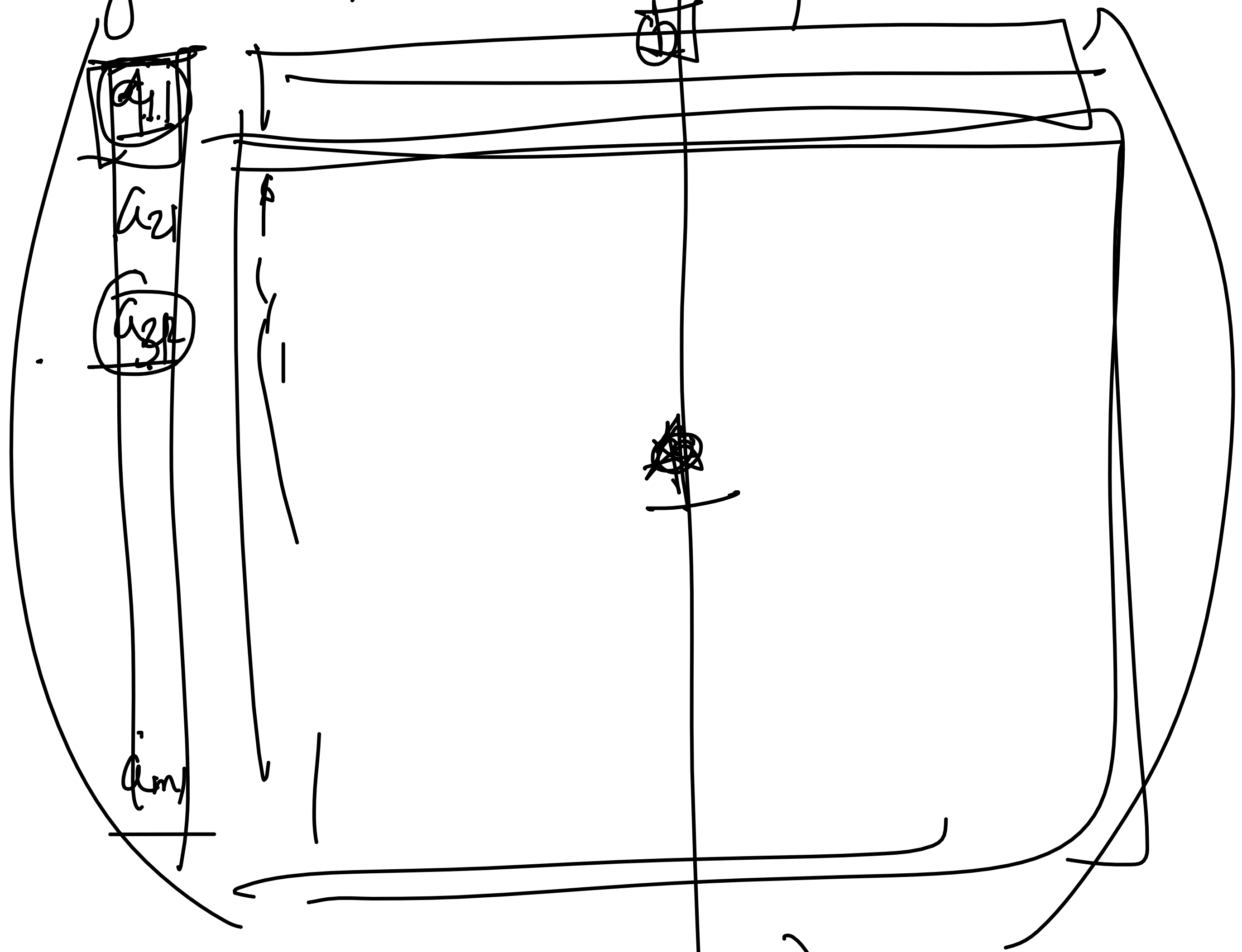
(3) The pivots are the only nonzero entries of the respective columns.

Example :

$$\begin{pmatrix} 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

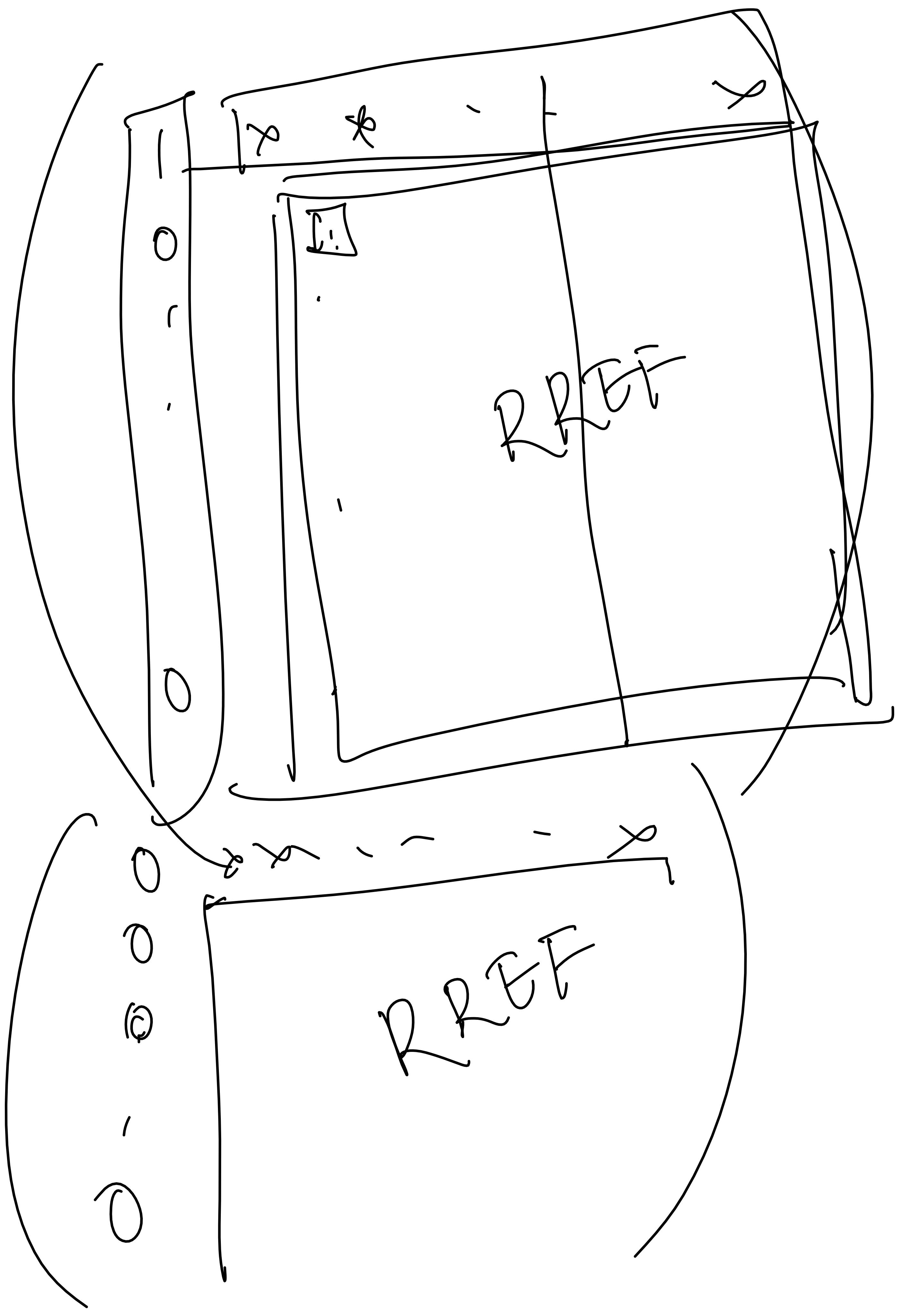
★ Every matrix has a unique RREF.

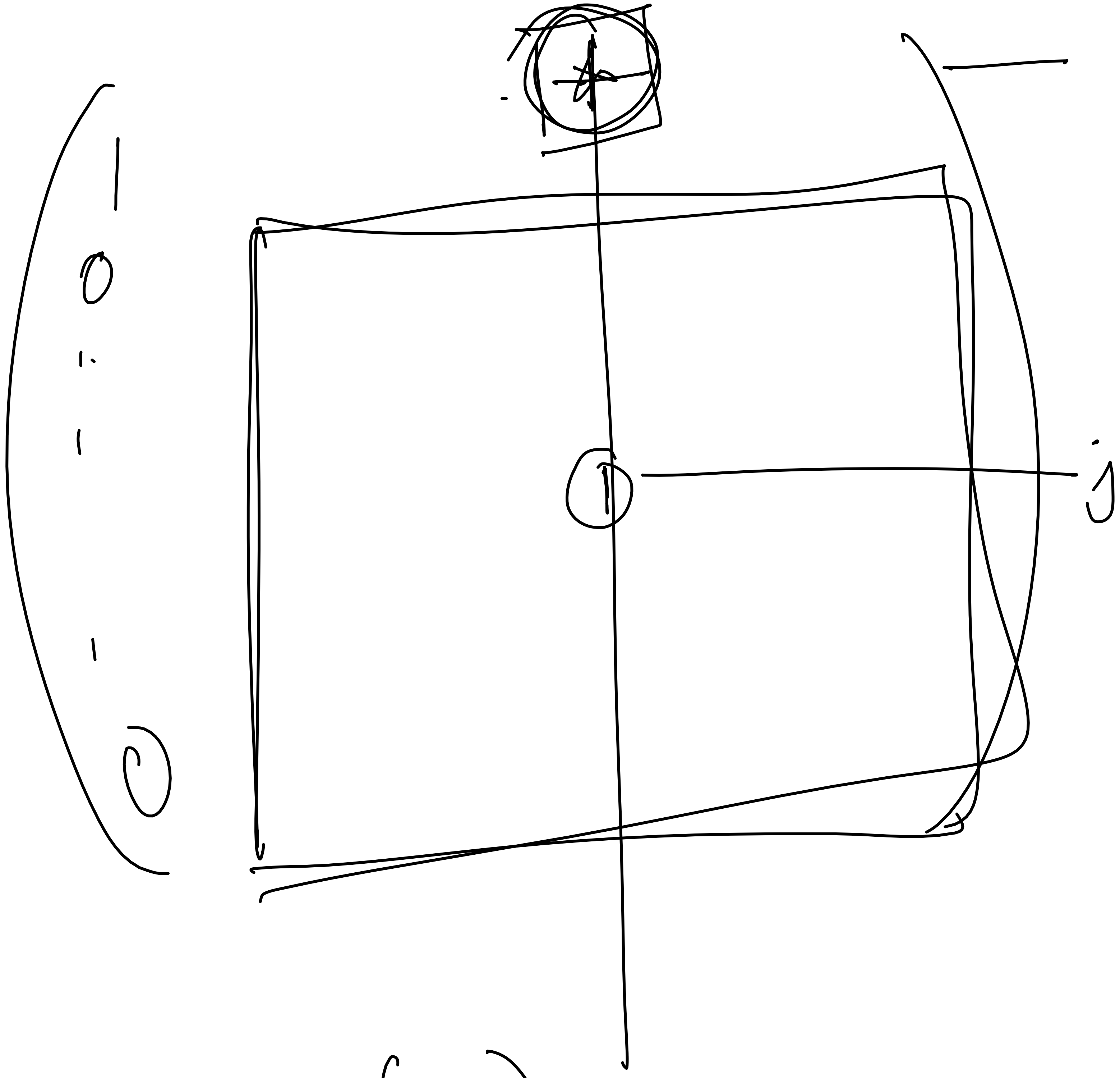
$A =$



$a_{11} \neq 0$ $\quad R_1\left(\frac{1}{a_{11}}\right)$

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad E_1\left(\frac{1}{a_{11}}\right) \quad \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$





$R_{ij}(\alpha)$

$$AX = d$$

$$R_{ij}\left(A \mid d\right) = \left(C \mid d\right)$$

$$R_i(c)\left(A \mid d\right)$$

$$R_{ij}(c)\left(A \mid d\right)$$

$$AX = d$$

$$\left(A \mid d\right)$$

$$\left(C \mid d\right) \text{ RREF}$$

$$CX = d$$

$\begin{pmatrix} & \vdots \\ 0\ 0\ 0\ 0 \cdots 0 & 1 \\ 0 \cdots & \end{pmatrix} \xrightarrow{\text{RREF}}$

$0 \cdot x_1 + 0 \cdot x_2 + \cdots + 0 \cdot x_n = 1$

no solutions

Def^n: A column containing a pivot is called a pivot column.

A variable associated to a pivot column is called a basic variable. Otherwise we call a free variable.

$$
\begin{array}{cccc}
x_1 & x_2 & x_3 & \\
\left(\begin{array}{c} 1 \\ 0 \\ 0 \end{array}\right. & \begin{array}{c} 2 \\ 0 \\ 0 \end{array} & \begin{array}{c} 0 \\ 1 \\ 0 \end{array} & \left|\begin{array}{c} 2 \\ 3 \\ 0 \end{array}\right)
\end{array}
$$

Pivot

$$x_1 + 2x_2 = 2$$

$$x_3 = 3$$