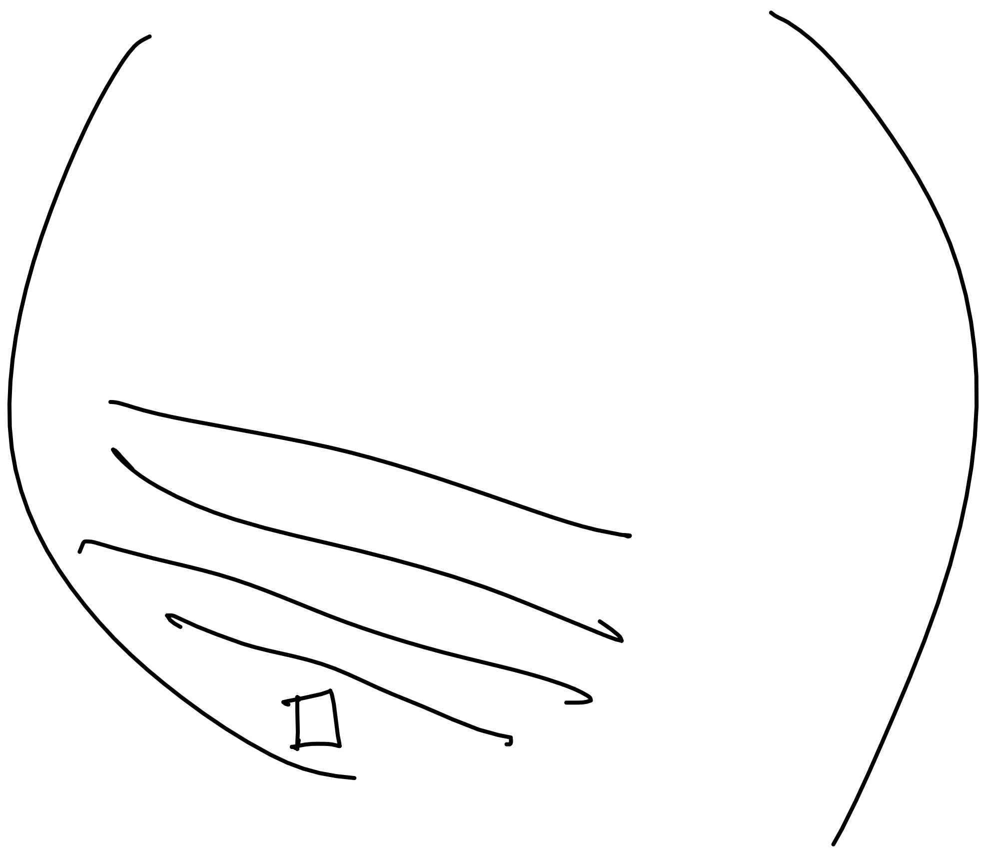
Recall:

$$AX = d.$$

$$\left(A \mid d\right)_{m \times (n+1)} \text{ — Augmented Matrix}$$



$m \times (n+1)$

## Row Operations

(1) $R_{ij}$ = Interchange $i$th row and $j$th row

(2) $R_i(c)$ = Multiply $c$ to row $i$

(3) $R_{ij}(c)$ = Replace $i$th row by $i$th row + $c$ times $j$th row.

$$R_{ij} = R_{ji}$$

$$R_{ij}\begin{pmatrix} 1 & & & & 0 \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ 0 & & & & 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & & & & & 1 \\ & 1 & & & & \\ & & 1 & & & \\ & & & 0 & & \\ 1 & & & & 0 & \\ & & & & & \ddots \\ & & & & & & 1 \end{pmatrix}}_{E_{ij}}$$

$$E_{ij} \cdot E_{ij} = I$$

$R_i(c)$

$$R_i(c)\begin{pmatrix}1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1\end{pmatrix} = \begin{pmatrix}1 & & & & \\ & 1 & & 0 & \\ & & c & & \\ 0 & & & 1 & \\ & & & & 1\end{pmatrix}$$

$E_i(c)$

$$E_i(c)\cdot E_i(1/c) = E_i(1/c)\cdot E_i(c) = I$$

$$E_{ij}(c)\cdot E_{ij}(-c)$$
$$= E_{ij}(-c)\cdot E_{ij}(c) = I$$

Elemetary matrices.

$$R_{ij}(c)\begin{pmatrix}1 & & \\ & 1 & \\ & & 1 \\ 0 & & & 1\end{pmatrix} = \begin{pmatrix}1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & e & \\ & & & & 1\end{pmatrix}$$

# Row Echelon form of a Matrix

A matrix is said to be in row echelon form if

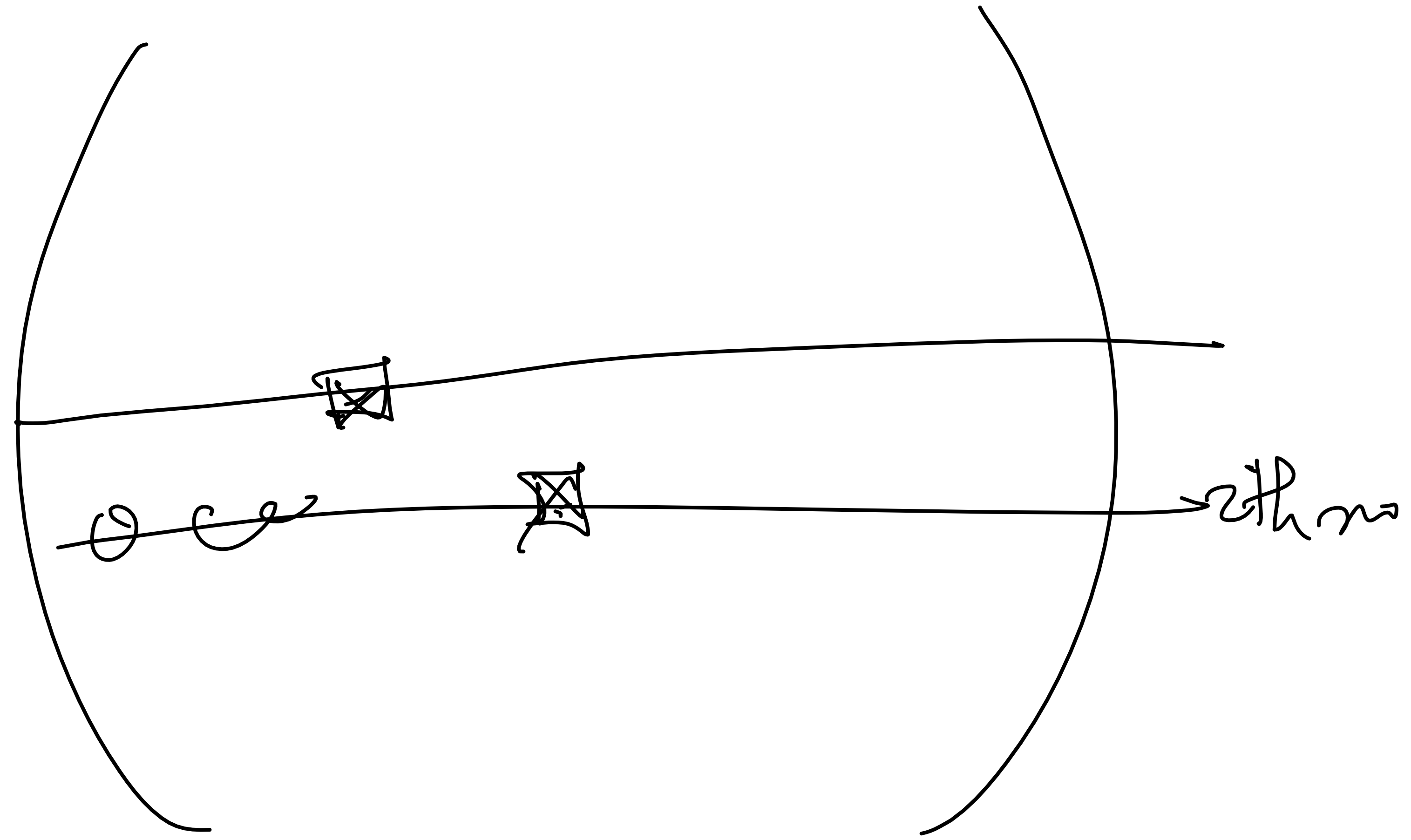
(1) Zero rows are below to all nonzero rows.

(2) The leading coefficient of a nonzero row is right to the leading coefficient of the row above it.

$$\begin{pmatrix} 1 & 2 & 0 & 2 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 5 \end{pmatrix}$$ — row echelon form

The 1st non-zero entry from left

Pivot



## Row reduced Echelon Form (rref)

A matrix is said to be in rref if

(1) It is in row echelon form

(2) All the leading coefficients are 1

(3) Pivot is the only nonzero element in that column.

$$\begin{pmatrix} 1 & 0 & \\ 0 & 1 & \\ & & 1 \end{pmatrix}$$

Example:

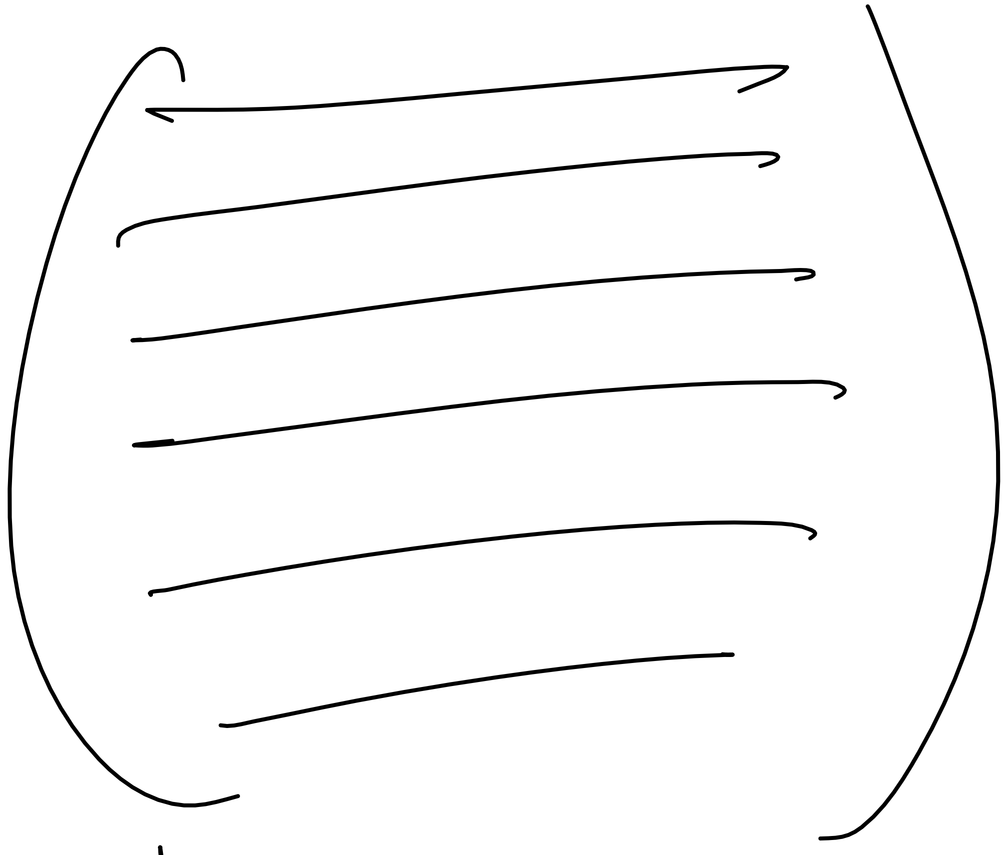
$$\begin{pmatrix} 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$Ax = d$

$(A|d)$ – Augmented matrix

$R_{ij}\, A$

$R_{ij} A$

$R_{ij}$ 

$= E_{ij} A$

$R_i(c) A = E_i(c) A$

$R_{ij}(c) A = E_{ij}(c) A$

$$R_{12}\begin{pmatrix}2 & 3 & 4\\ 1 & 2 & 3\end{pmatrix} = \begin{pmatrix}1 & 2 & 3\\ 2 & 3 & 4\end{pmatrix}$$

$$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}2 & 3 & 4\\ 1 & 2 & 3\end{pmatrix} = \begin{pmatrix}1 & 2 & 3\\ 2 & 3 & 4\end{pmatrix}$$

$$Ax = d$$

$$\left(A \mid d\right) \longrightarrow$$

$R_i(c)$ $\cancel{R_{ij}}$

$$\left(C \mid b\right) \longrightarrow$$

$$Ax = d$$

$$(A \mid d) \quad \left(\begin{array}{c|c} A & d \end{array}\right)$$

RREF $\longrightarrow$ $\left(\begin{array}{c|c} C & b \end{array}\right)$

$$\left(\begin{array}{cccc|c} & & & & \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & & 0 \end{array}\right)$$

$$0\cdot x_1 + 0\cdot x_2 + \cdots + 0\cdot x_n = 1$$

$$\Rightarrow 0 = 1$$

$$Ax = d$$

$$(C \mid d) \quad \text{RREF}$$

A column containing a pivot is called a pivot column.

A variable associated to a pivot column is called a basic variable.

Otherwise we call ~~a~~ variable free.

$$\begin{array}{cccc} x_1 & \text{free} & x_3 & \\ \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$x_3 = 0$$

$$x_1 + 2x_2 = 3$$

$$Ax = d$$

$$(A \mid d)$$

$$\vdots$$

$$(C \mid d) \text{ — RREF}$$

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$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}_{m\times n}$$

If $a_{11} \neq 0$ $\quad R_1\left(\frac{1}{a_{11}}\right)$

$$\begin{pmatrix} 1 & * & * & * & \cdots \\ 0 & & & & \\ 0 & 0 & & & \\ 0 & & 1 & & \\ \vdots & & & & \\ 0 & & & & \end{pmatrix}$$

If $a_{21} \neq 0$

$R_{21}(-a_{21})$

$$\begin{pmatrix} 1 & x & x & \cdots \\ 0 & 1 & & \\ 0 & 0 & 1 & \\ \vdots & \vdots & & \\ 0 & 0 & & 1 \end{pmatrix}$$

$R_{ij}(-a_{ij})$

$$A$$

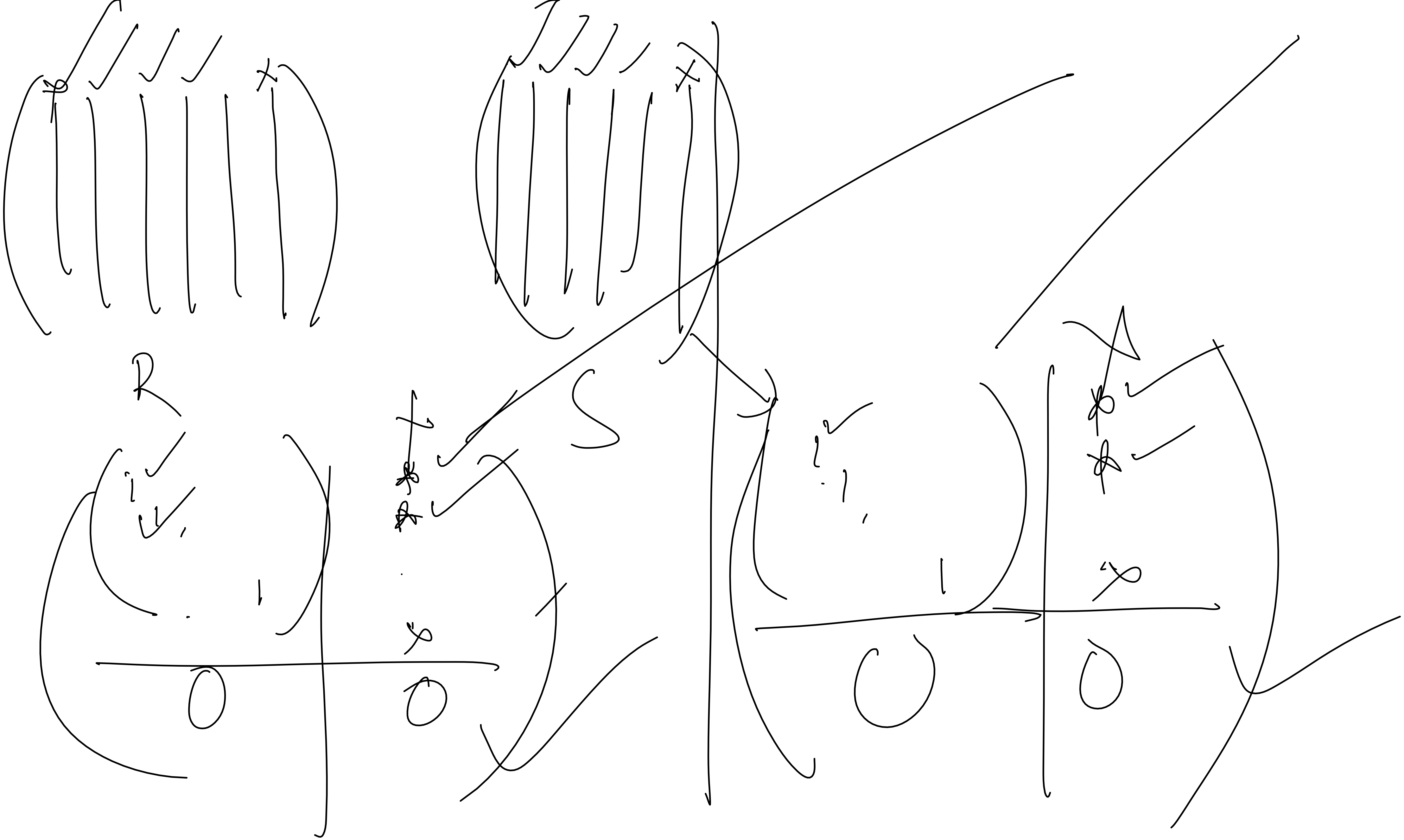
$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & \cancel{5} \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

**Theorem**: The RREF of a matrix is unique

$$\begin{array}{ccc} & A & \\ \swarrow & & \searrow \\ B - \text{RREF} & & C - \text{RREF} \end{array}$$

**claim**: $B = C$

B C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