Recall: Matrices

Operation

Systems of linear equations

$$AX = d.$$

$(A|d)$ — Augmented Matrix

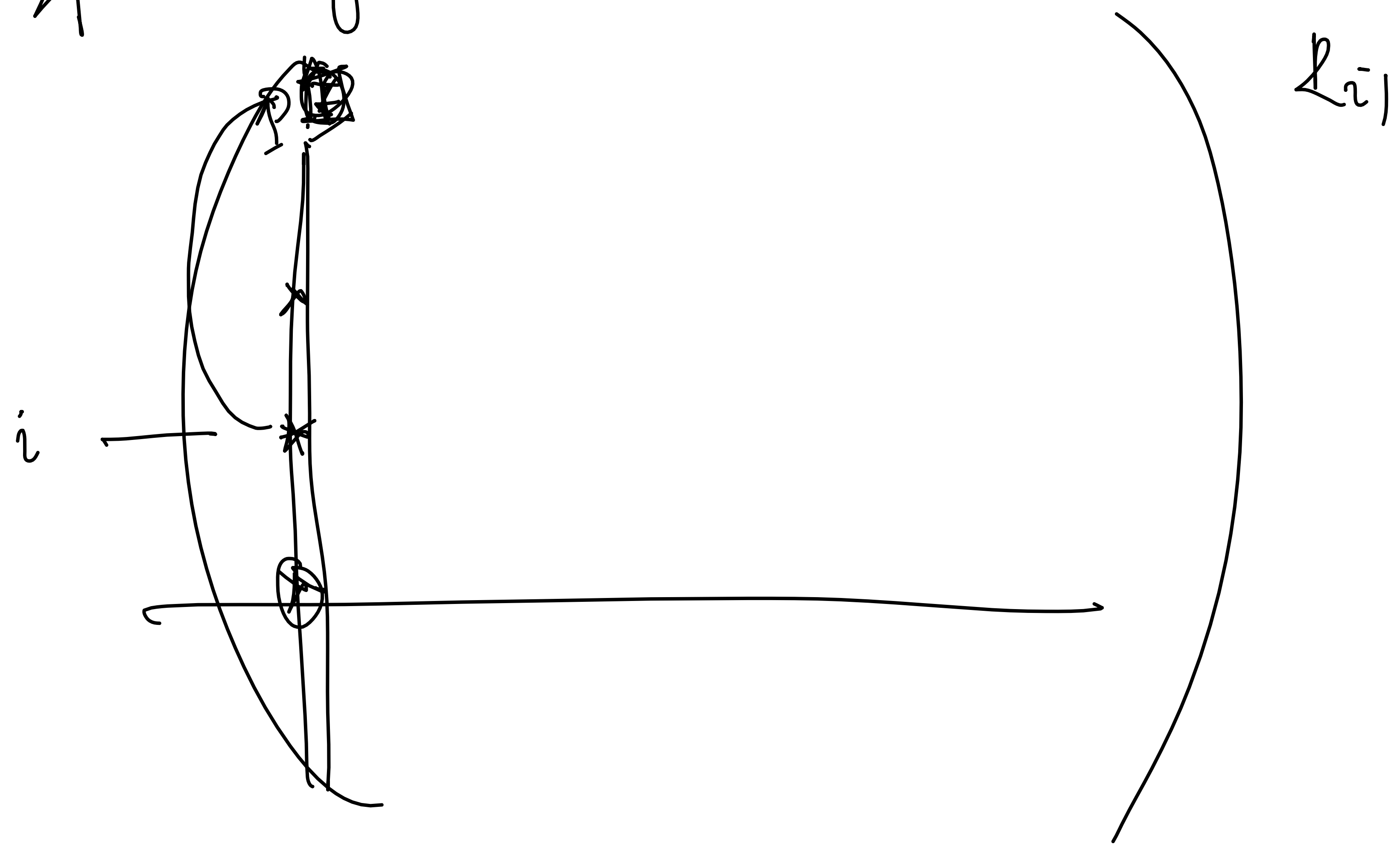
REF   RREF

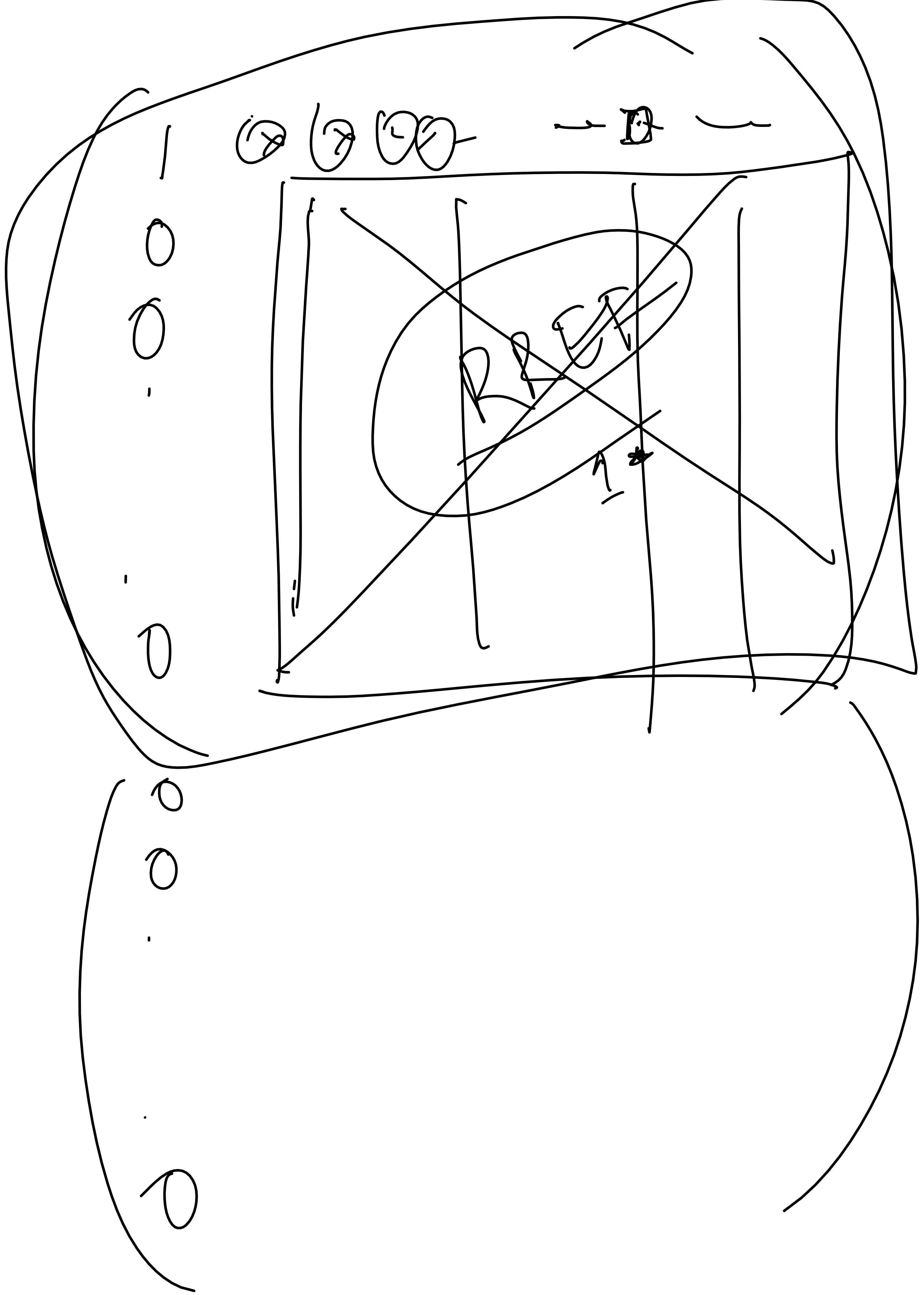
R_{ij} —— E_{ij}

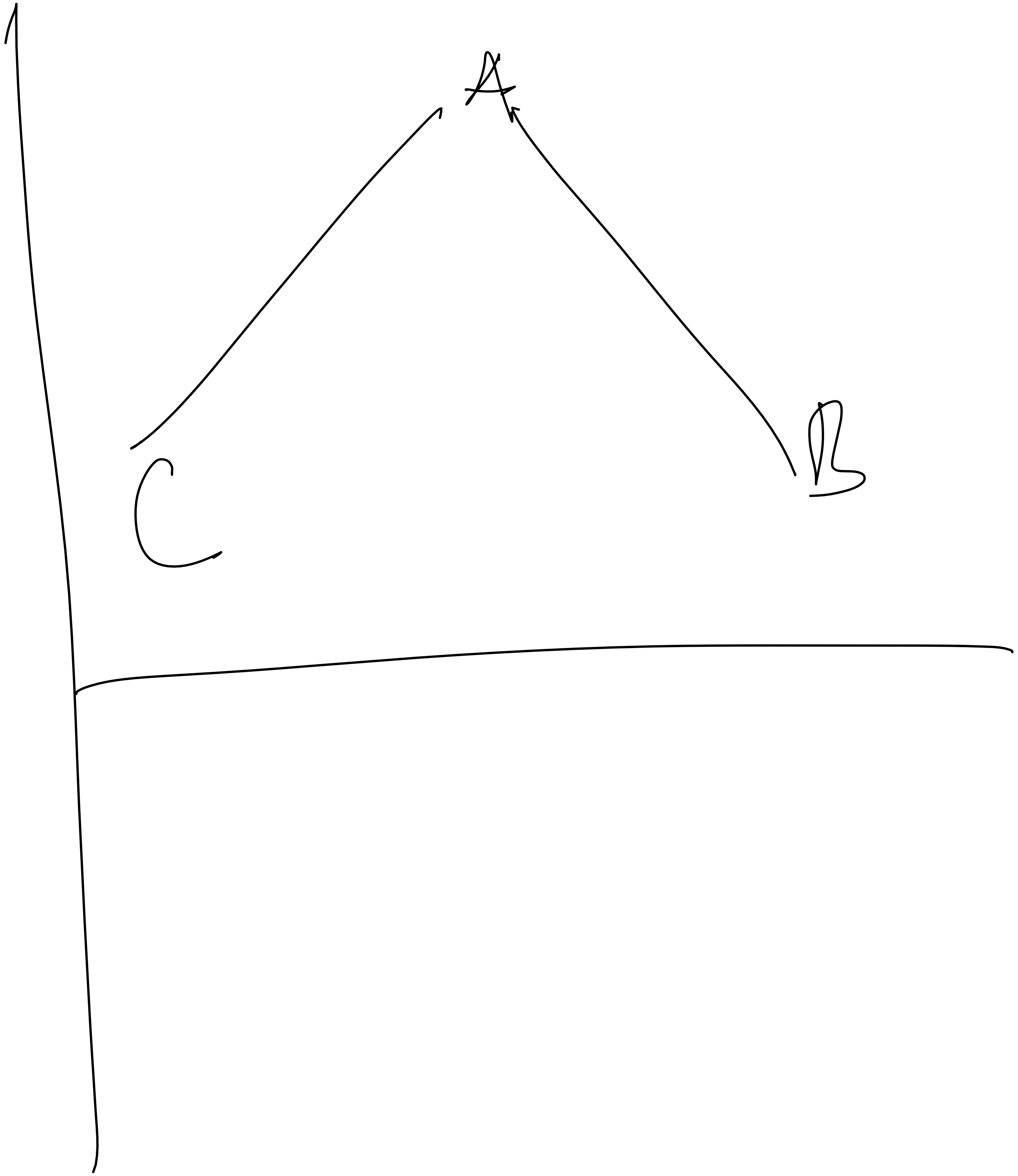
$R_i(c)$ —— $E_i(c)$

$R_{ij}(c)$ —— $E_{ij}(c)$

★ Every matrix has a RREF and it is unique.



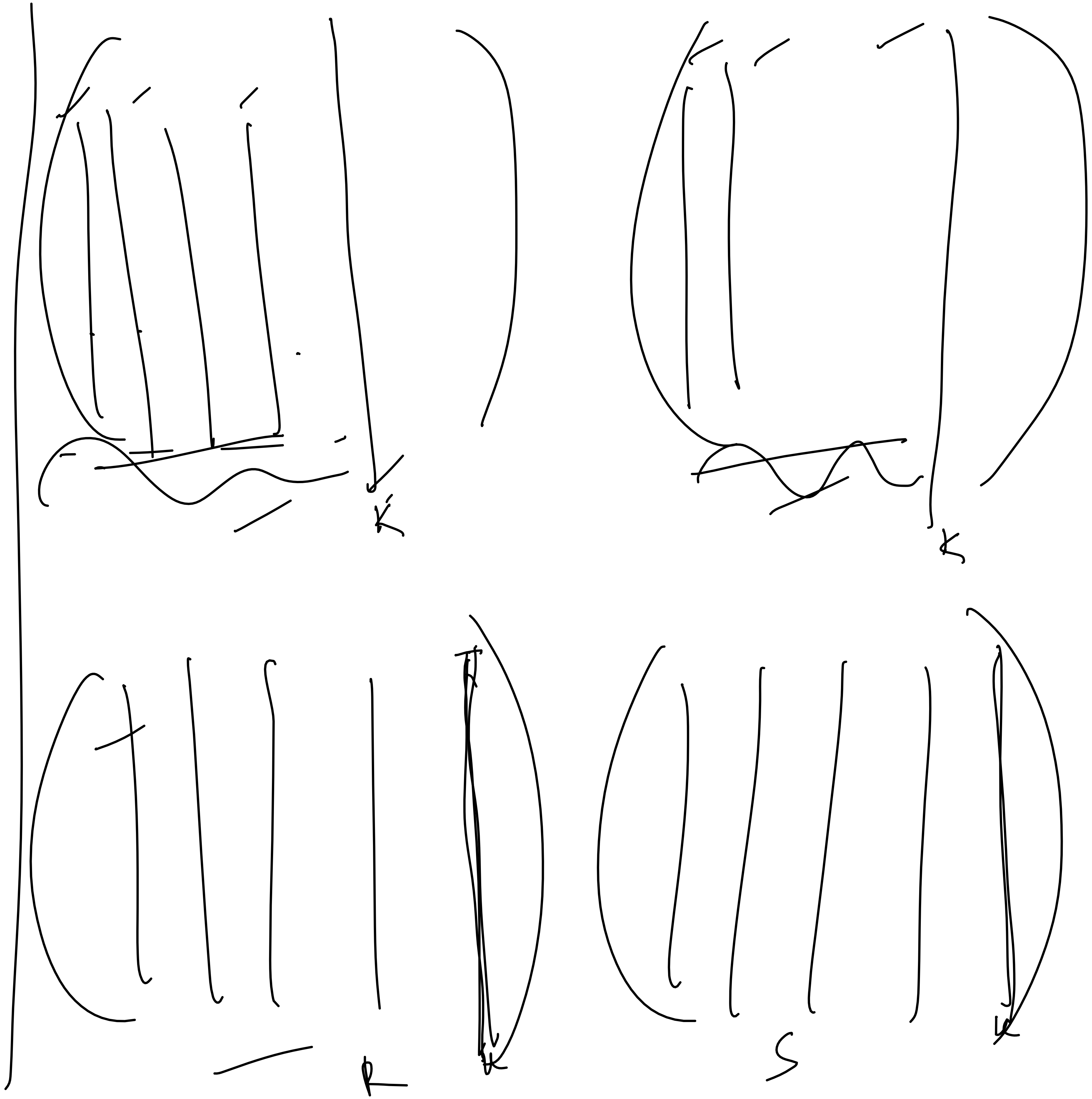



A
C
B

A

B RREF C

$B \neq C$



$$\left(\begin{array}{c|c} I_{r\times r} & \begin{matrix} * \\ * \\ * \end{matrix} \\ \hline 0 & 0 \end{array}\right) \qquad \left(\begin{array}{c|c} I_{r\times r} & \begin{matrix} * \\ * \\ * \end{matrix} \\ \hline 0 & 0 \end{array}\right) \qquad \begin{pmatrix} 1 & 0 & \\ 0 & 0 & \\ 0 & 1 & \\ 1 & & \end{pmatrix}$$

$R' \quad R \qquad\qquad S' \quad S$

$R'X = d_1 \qquad S'X = d_2$

C_{ij} = Interchanging ith column and jth column

$C_i(c)$ = Multiply c to ith column

$C_{ij}(c)$ = Replacing ith column by ith column + c times jth column

F_{ij} $\quad F_i(c)$, $\quad F_{ij}(c)$

$$E_{ij} = F_{ij}$$
$$E_i(c) = F_i(c)$$
$$E_{ij}(c) = F_{ji}(c)$$

Theorem: Any matrix A can be reduced to the following matrix

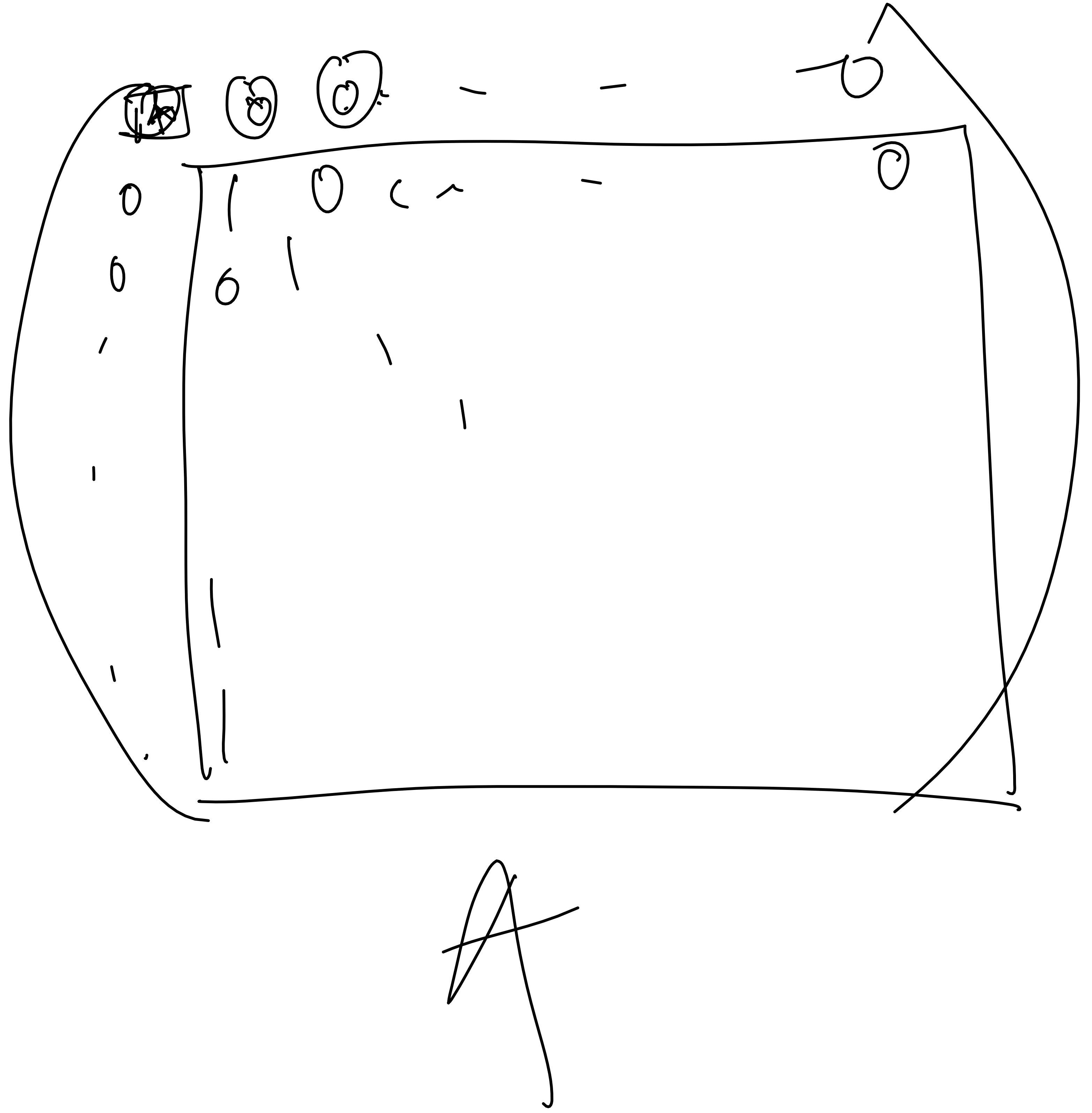
$$\left(\begin{array}{cccc|c} 1 & & & & \\ & 1 & & & \\ & & \ddots & & 0 \\ & & & 1 & \\ \hline & & 0 & & 0 \end{array}\right)$$

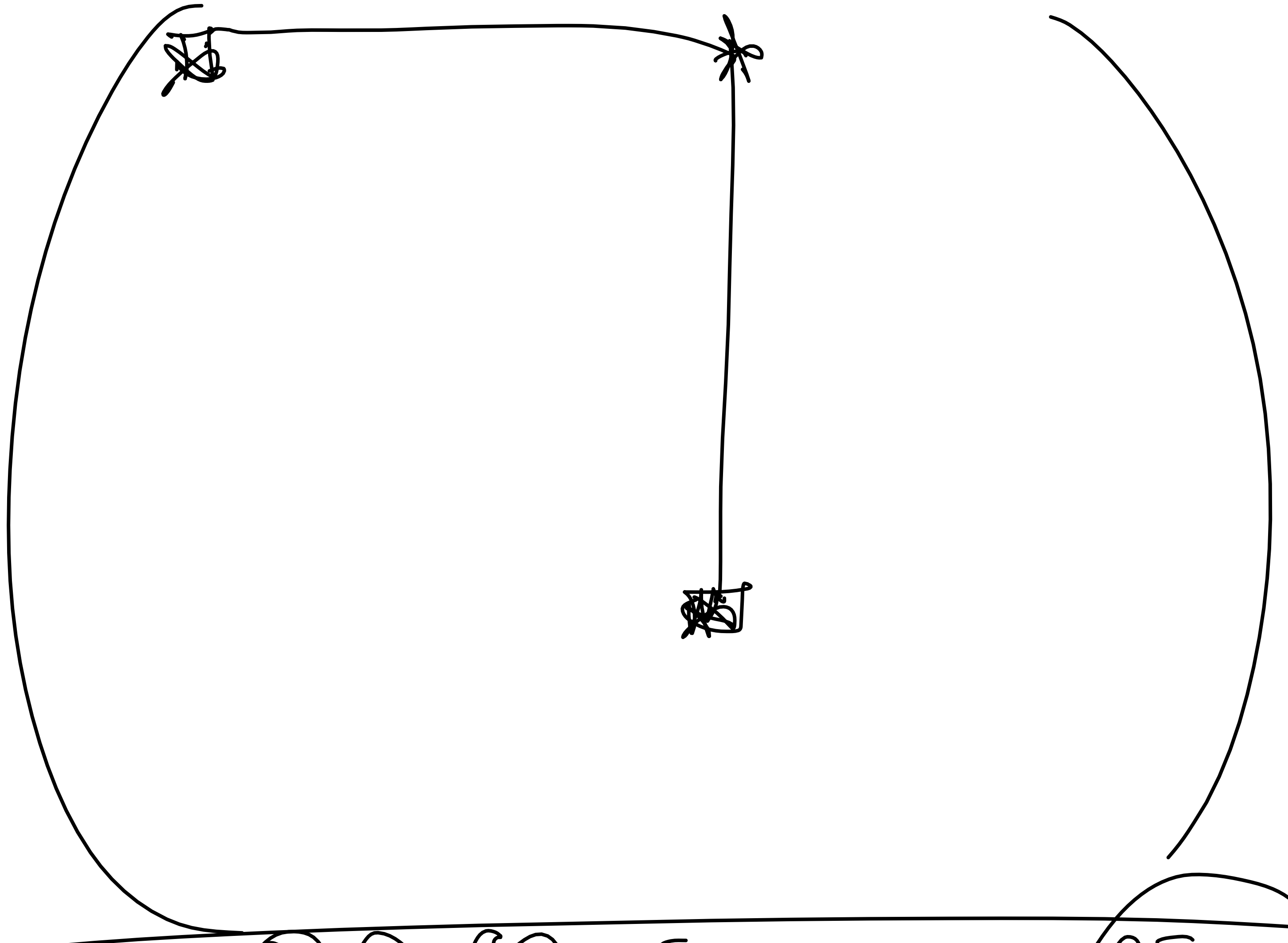
by applying a sequence of row and column operations.

$E_{ij}(c)$

$$\begin{array}{c} \\ \\ \\ \end{array}\left(\begin{array}{cccc} 1 & & c & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{array}\right) \begin{array}{l} \leftarrow i\text{th row} \\ \\ \\ \\ \end{array}$$

$$\left(\begin{array}{c|c} I_{r \times r} & 0 \\ \hline 0 & 0 \end{array}\right)$$





$$E_k \cdots E_2 E_1 A F_1 F_2 \cdots F_l = \begin{pmatrix} I_{r\times r} & 0 \\ 0 & 0 \end{pmatrix}$$

Suppose $A_{n\times n}$ is invertible

$$\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \\ 0 & 0 & c & c & 0 & 0 \end{pmatrix}_C \begin{pmatrix} 1 \end{pmatrix} \neq \quad = \begin{pmatrix} 1 & 1 & 0 \\ & 1 & \\ 0 & 1 & 0 \end{pmatrix}$$

$E_k \cdots E_2 E_1 A F_1 F_2 \cdots F_l = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$

iff $r = n$

$F_j = E_k$

✓ $E_k E_{k-1} \cdots E_2 E_1 A F_1 F_2 \cdots F_l = I_{n\times n}$

$A = E_1^{-1} \cdots E_{k-1}^{-1} E_k^{-1} I F_l^{-1} F_{l-1}^{-1} \cdots F_1^{-1}$

$= E_1 E_2 \cdots \cdots E_p$

$A = E_1 E_2 \cdots \cdots E_p I_{n\times n}$

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* A is invertible iff A is a product of elementary matrices.

* A is invertible iff the RREF of A is $I_{n\times n}$.

$E_p^{-1} \cdots E_2^{-1} E_1^{-1} A = I_{n\times n}$

RREF

$$\underbrace{E_1 E_2 \cdots E_p}\, A = I$$

$$A^{-1} = \underline{E_1 \cdots E_{p-1} E_p}\, I$$

$$E_1 \cdots E_{p-1} E_p \left(\begin{array}{c|c} A & I \end{array} \right)$$

$$\left(\begin{array}{c|c} I & A^{-1} \end{array} \right)$$

$$E_1 \cdots E_{p-1} E_p \left(\begin{array}{c|c} A & I \end{array} \right)$$

$$\left(\begin{array}{c|c} I & A^{-1} \end{array} \right)$$

S.

$f: S \longrightarrow S$

$S = \{a_1, a_2 - - \quad a_n\}$

$S = \{1, 2, \ldots \quad n\}$

S_n = Set of all bijection from S to S
= Set of all permutation on S

$$\begin{Bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ \vdots \\ n \end{Bmatrix} \xrightarrow{\text{bijection}} \begin{Bmatrix} 2 \\ 3 \\ 1 \\ 4 \\ 5 \\ 1 \\ n \end{Bmatrix}$$

$|S_n| = n!$

If $\sigma \in S_n$ $\quad \tau \in S_n$

$\sigma \circ \tau \in S_n$ and $\tau \circ \sigma \in S_n$

identity $\in S_n$.

$S_n =$

$$S \xrightarrow{g} S \xrightarrow{f} S$$

$f \circ g \in S_n \quad g \circ f \in S_n$

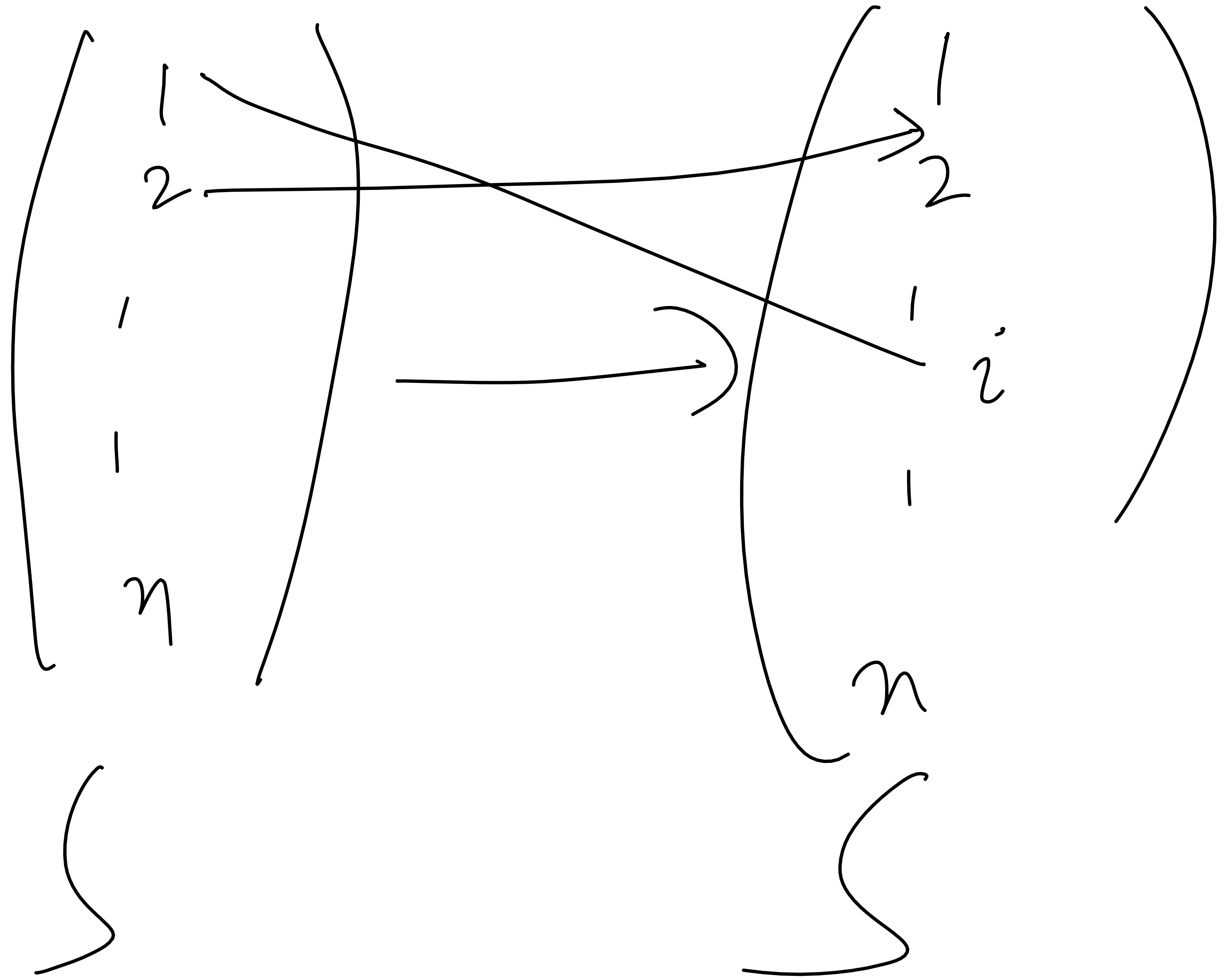
$f^{-1} \in S_n$

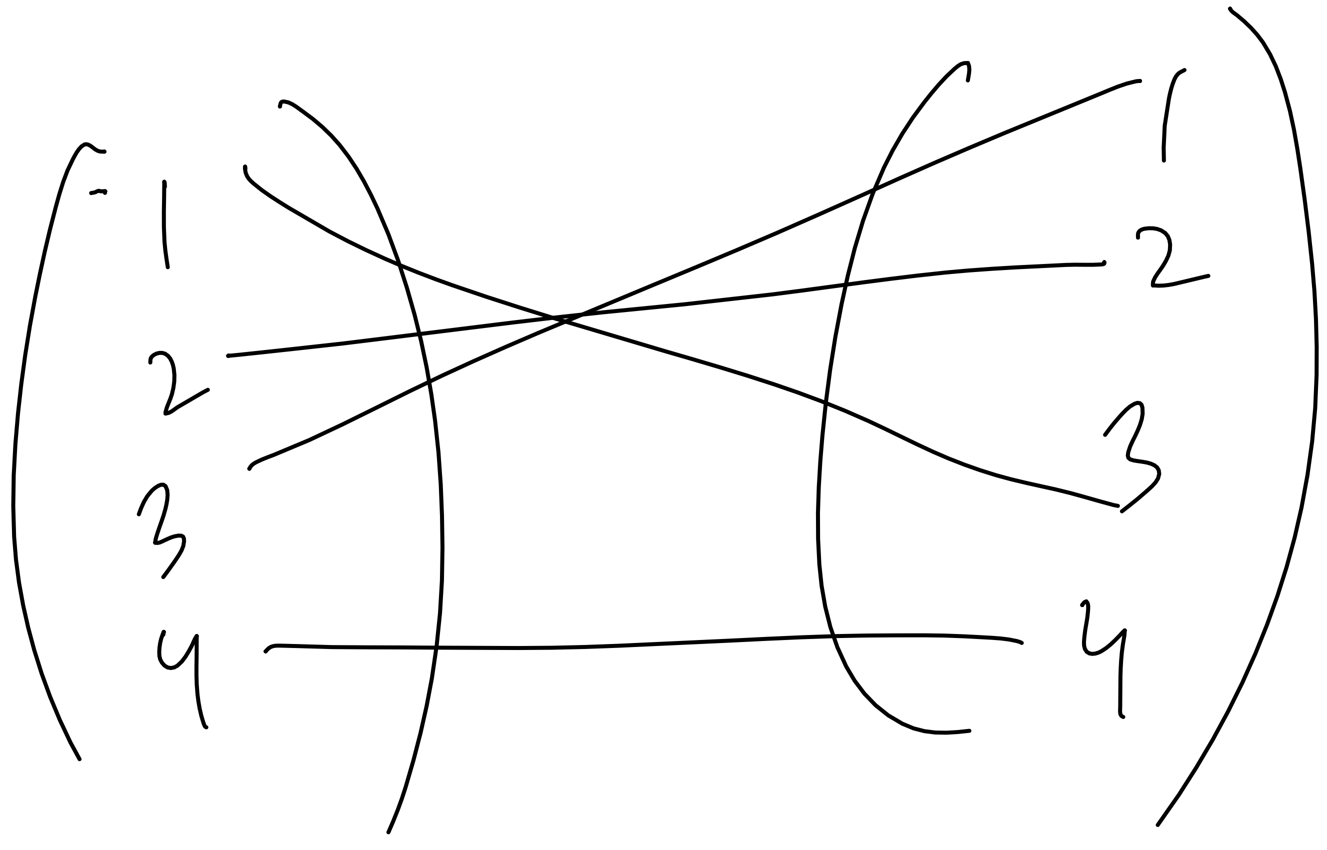
S_n = Set of all permutations

If $\sigma, \tau \in S_n$ then $\sigma \circ \tau \in S_n$

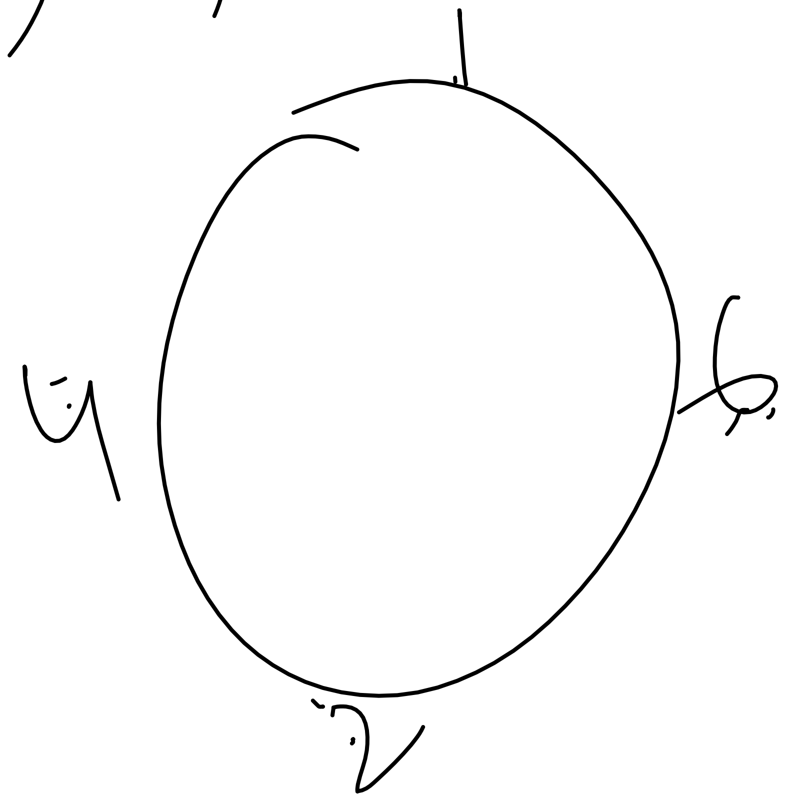
If $\sigma \in S_n$ then $\sigma^{-1} \in S_n$

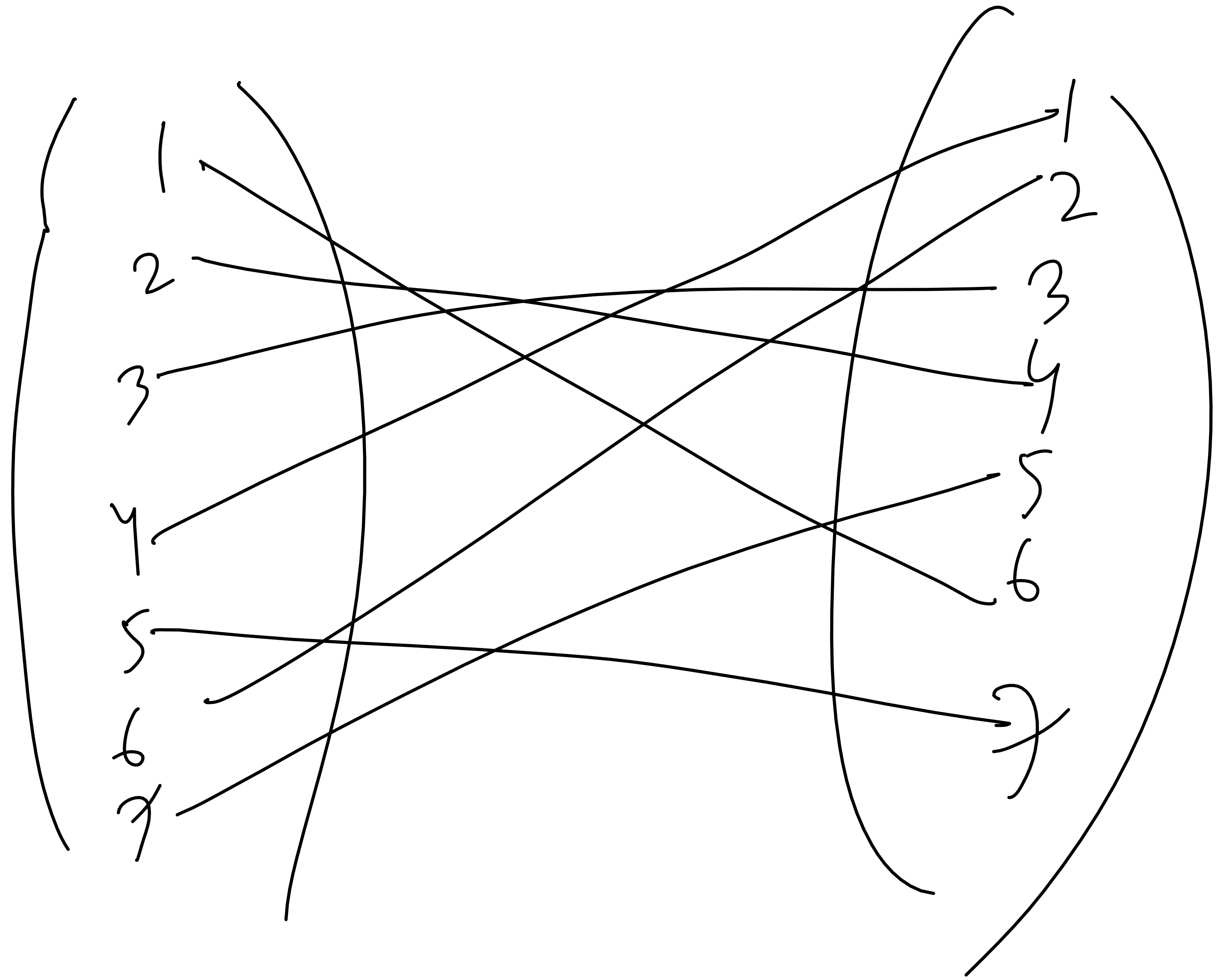
$\mathrm{Id} \in S_n$





$(1\,3)(2)(4)$ $\quad$ $(1\,3)$





$(1\,6\,2\,4)(3)(5\,7)$