Matrices

Operations

System of Linear Equation. $AX = d$

$(A|d)$

REF
RREF } - Row operation (elementary)

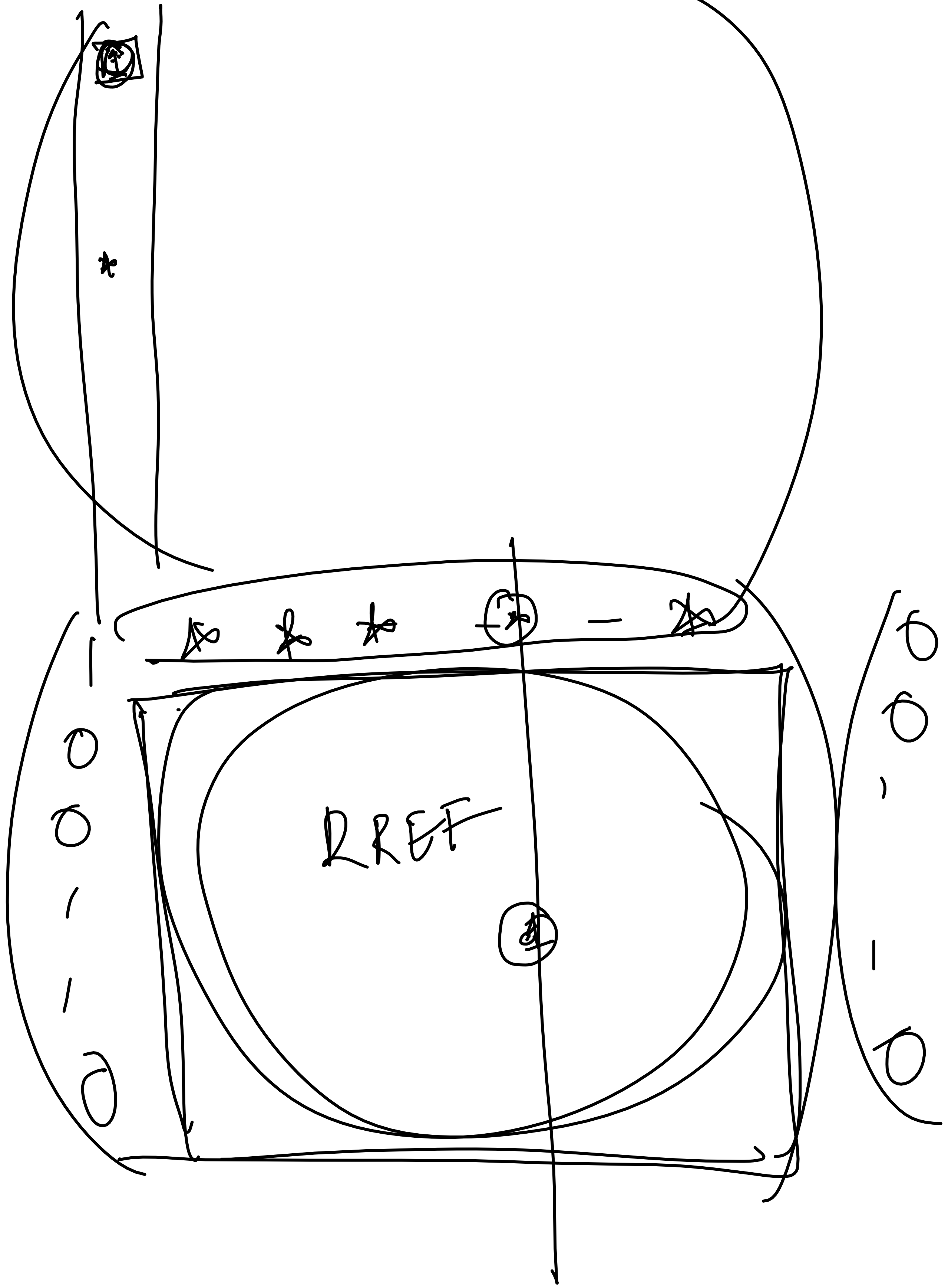
R_{ij} = ____ E_{ij}

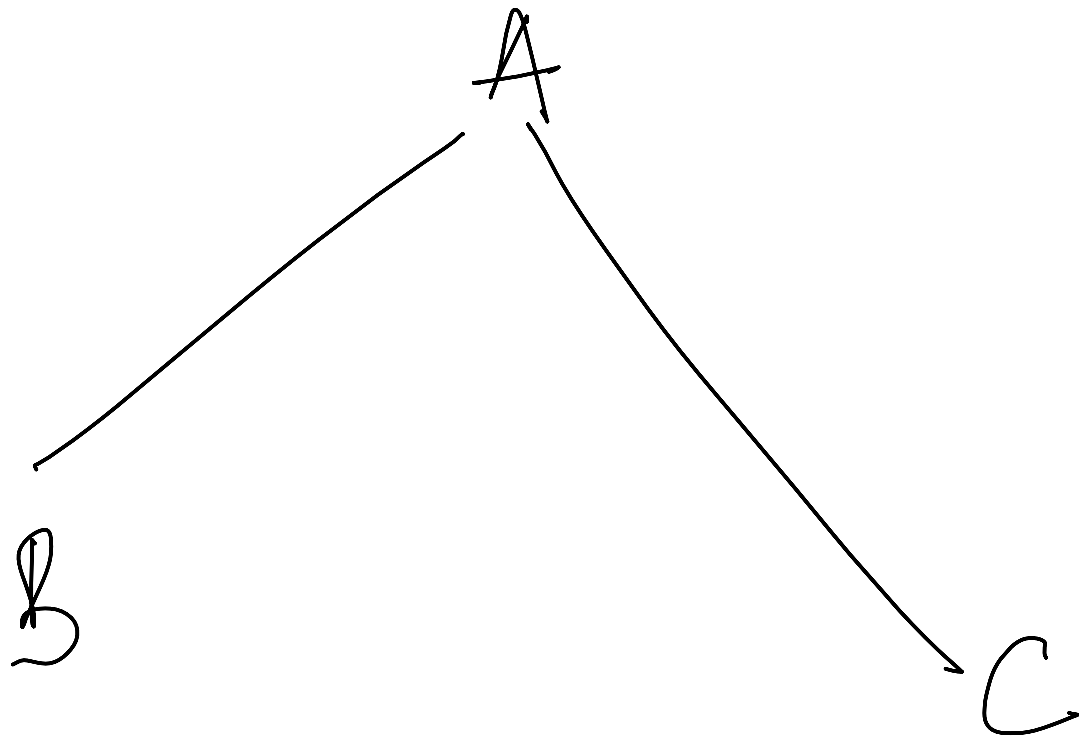
$R_i(c)$ ____ $E_i(c)$

$R_{ij}(c)$ ____ $E_{ij}(c)$

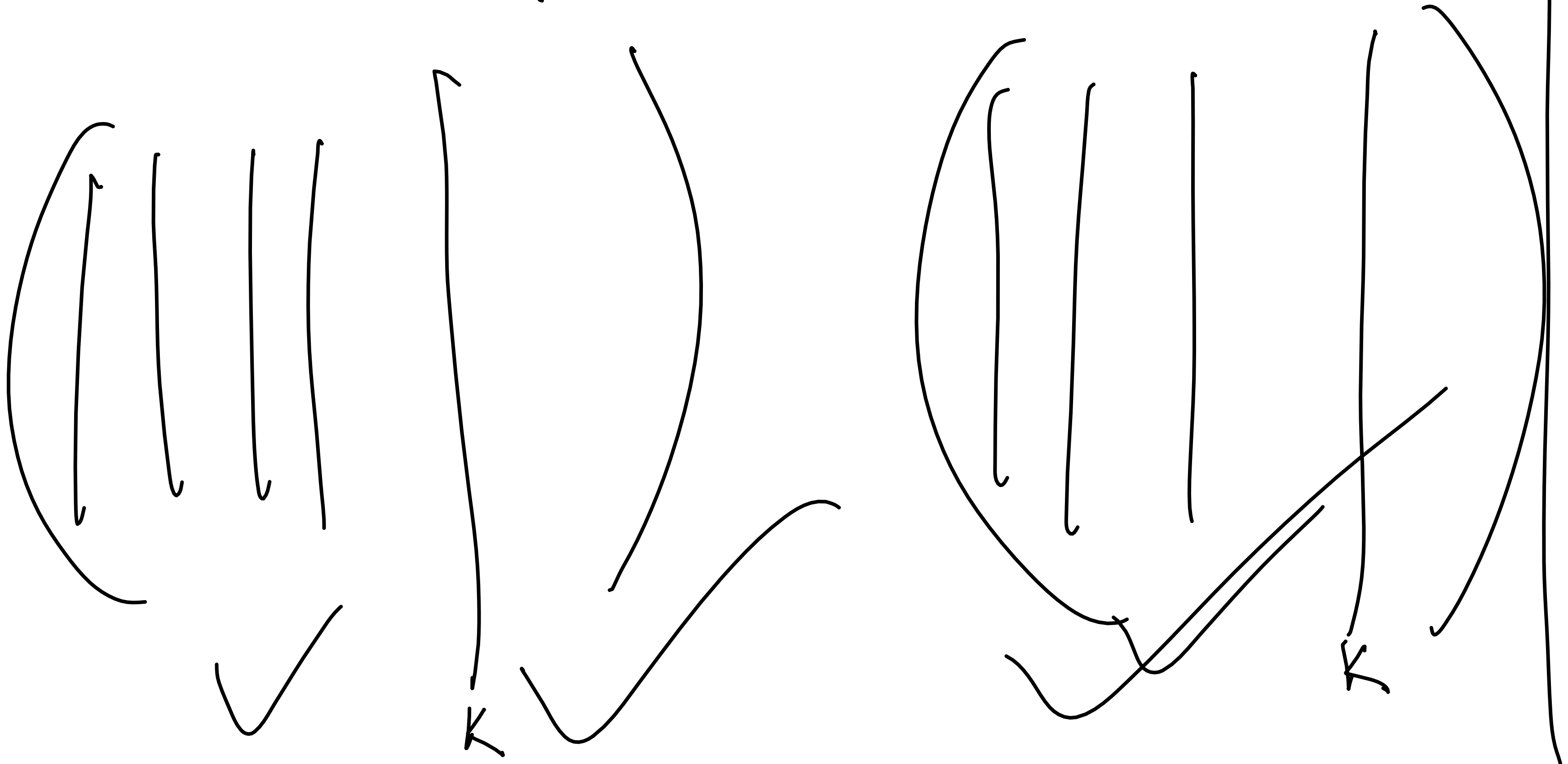
} - elementary Matrices.

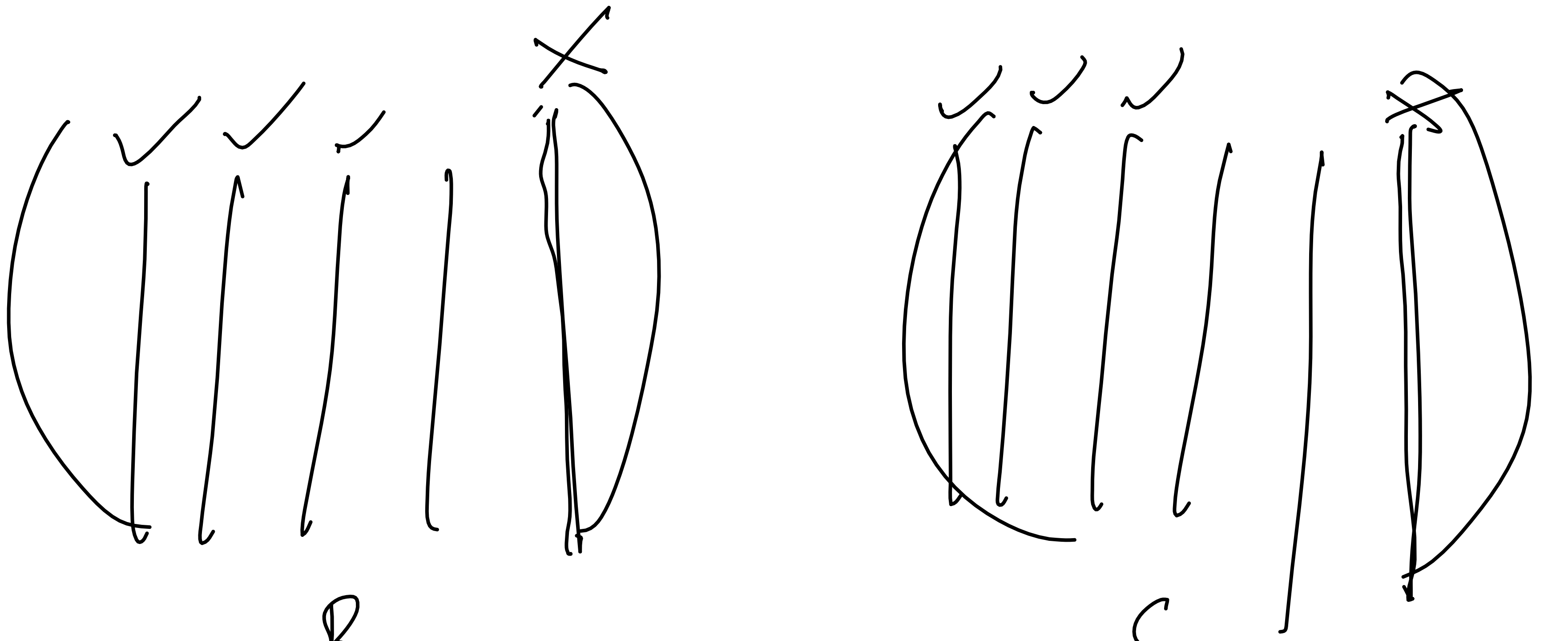
Theorem: Every matrix has a RREF





$B \neq C$





R

S

$$\begin{pmatrix} I_{r\times r} & 0 \\ 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} I_{r\times r} & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$

C_{ij} = Interchange ith column and jth column

$C_i(c)$ = Multiply c to the ith column

$C_{ij}(c)$ = Replace ith column by ith column + c times jth column

F_{ij}

$F_i(c)$

$F_{ij}(c)$

$E_{ij} = F_{ij}$ $\quad$ $E_i(c) = F_i(c)$

$E_{ij}(c) = F_{ji}(c)$

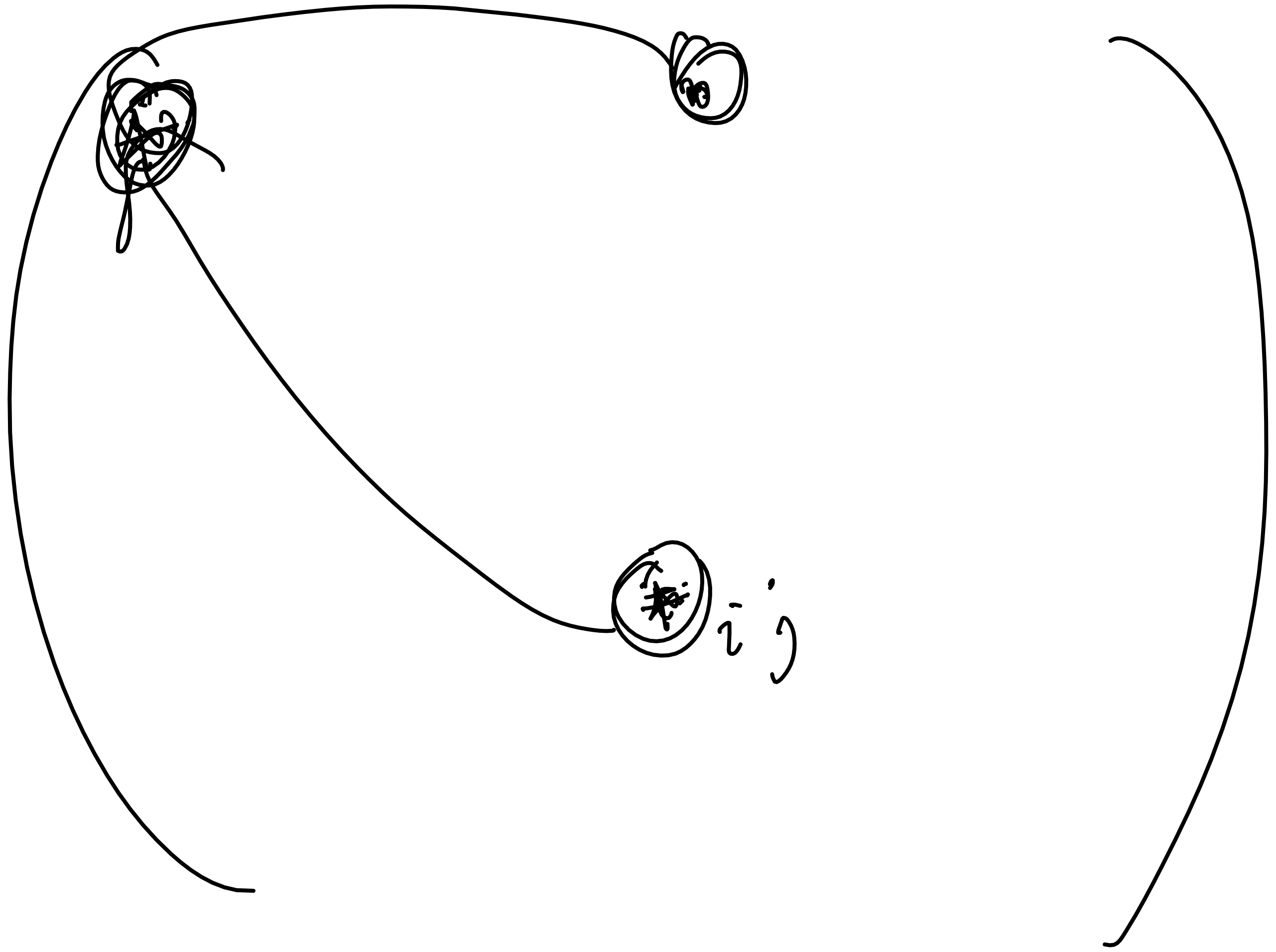
A

Theorem: Any matrix can be reduced to a matrix of the form

$$\left(\begin{array}{c|c} \begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix} & 0 \\ \hline 0 & 0 \end{array}\right)$$

by applying a sequence of row and column operations

Proof =



$$\begin{pmatrix} 1 & * & * & * & 0 & 0 & \cdots & 0 \\ * & & & & & & & \\ * & & & & & & & \\ 0 & & & & & & & \\ 0 & & & & & & & \\ \vdots & & & & & & & \\ 0 & & & & & & & \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} & & 0 & \\ 0 & & & & \\ \vdots & & & & \\ 0 & & 0 & & 0 \end{pmatrix}$$

$$A_{m\times n}$$

$$\left(\begin{array}{c|c} I_{r\times r} & 0 \\ \hline 0 & 0 \end{array}\right)$$

$$E_k \cdots E_2 E_1 \cancel{A} F_1 F_2 \cdots F_l = \underline{\left(\begin{array}{c|c} I_{r\times r} & 0 \\ \hline 0 & 0 \end{array}\right)}$$

$$\left(\begin{array}{c|c} I_{r\times r} & 0 \\ \hline 0 & 0 \end{array}\right)_{n\times n}$$

$$C = \left(\begin{array}{cccc|c} 1 & & & & \\ & 1 & & & 0 \\ & & 1 & & \\ & & & 1 & \\ \hline & 0 & & & 0 \end{array}\right)_{n\times n}$$

Invertible iff $n = r$

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$E_p \cdots E_2 E_1 A F_1 F_2 \cdots F_q = \left(\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right)$$

$n = r$

Suppose A is invertible

$\Rightarrow E_p \cdots E_1 A F_1 F_2 \cdots F_q = I_{n\times n}$

$\Rightarrow A = E_1^{-1} \cdots E_{p-1}^{-1} E_p^{-1} I_{n\times n} F_q^{-1} F_{q-1}^{-1} \cdots F_1^{-1}$

$= E_1^{-1} \cdots E_p^{-1} F_q^{-1} F_{q-1}^{-1} \cdots F_1^{-1}$

$= E_1^{-1} \cdots E_p^{-1} E_{p+1} \cdots E_k$

$$A = E_1 E_2 \cdots E_l I_{n\times n}$$

* A is invertible iff A is a product of elementary matrices.

$$E_l^{-1} \cdots E_2^{-1} E_1^{-1} A = I_{n\times n}$$

* Given any invertible matrix by applying elementary row operation we can always reduce it to the identity matrix

Application

$$\boxed{E_q \cdots E_1 A} = I$$

$$A^{-1} =$$

$$A^{-1} E_1^{-1} E_2^{-1} \cdots E_q^{-1} = I$$

$$\Rightarrow A^{-1} = E_q \cdots E_1 I$$

A

$$\left(E_q \cdots E_2 E_1 A \,\middle|\, E_q \cdots E_1 I \right)$$

$$\downarrow$$

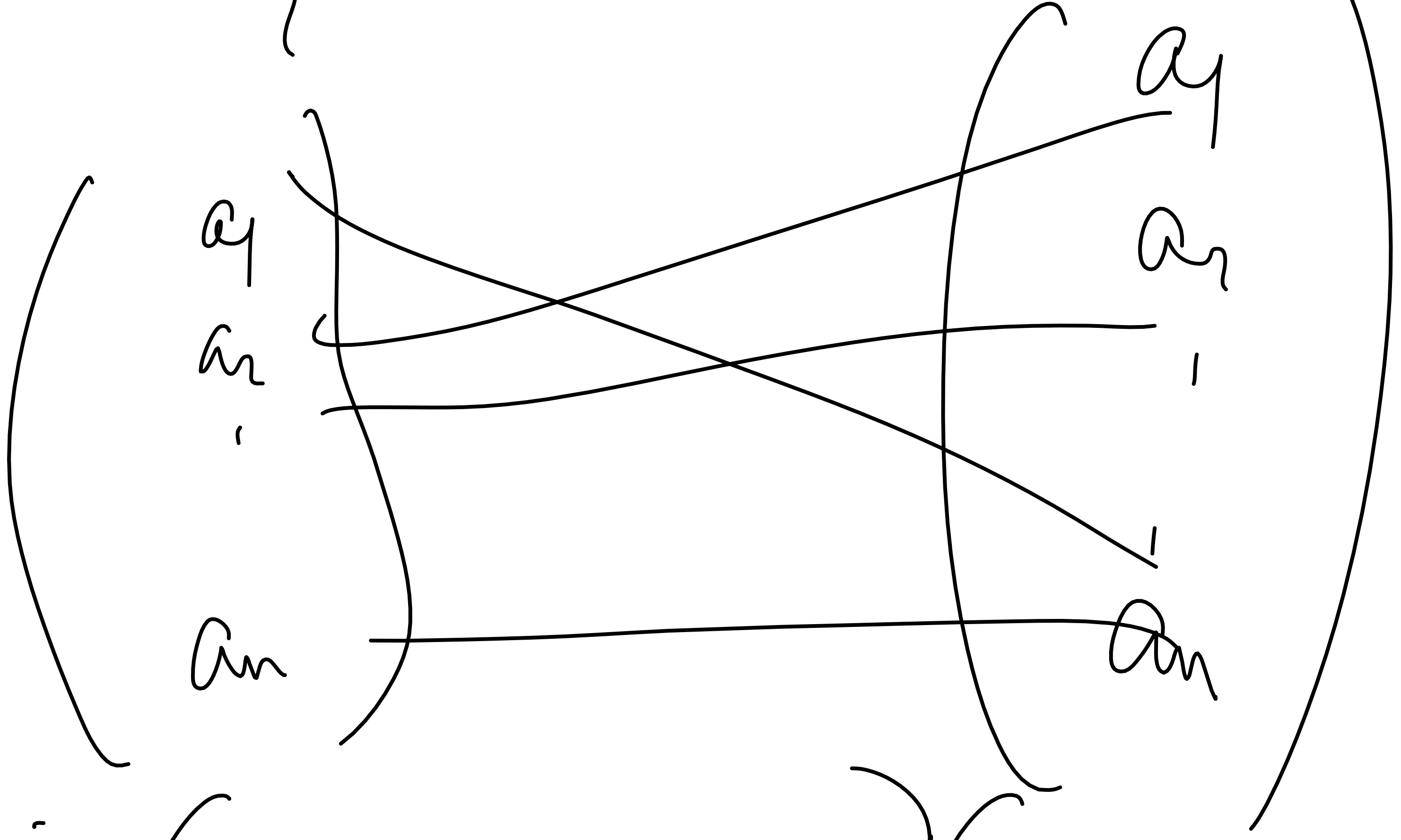
$$\left(I \,\middle|\, A^{-1} \right)$$

$$\left(\begin{array}{c|c} A & I \end{array}\right) \quad n\times 2n$$

$$\downarrow \qquad \downarrow$$

$$\left(\begin{array}{c|c} I & A^{-1} \end{array}\right)$$

$S = \{a_1, a_2, \ldots, a_n\}$



$f : S \longrightarrow S$

$S = \{1, 2, \ldots, n\}$

S_n = Set of all bijections fr S to S

$$|S_n| = n!$$

$$S \xrightarrow{g} S \xrightarrow{f} S$$

$g \circ f \in S_n \quad f \circ g \in S_n$

$f^{-1} \in S_n$