$f : S \longrightarrow S$ bijection

$S = \{1, 2, \ldots, n\}$

$$f = \begin{pmatrix} 1 & 2 & \cdots & n \\ f(1) & f(2) & & f(n) \end{pmatrix}$$

S_n = Set of all permutations on S.

$|S_n| = n!$

$f \circ g \in S_n \quad g \circ f \in S_n$

$f^{-1} \in S_n \quad id \in S_n$.

$\boxed{\sigma \cdot S_n} = \{\sigma \circ \tau : \tau \in S_n\} \underset{\subseteq}{\supseteq} S_n$

$\sigma \circ \tau_i = \sigma \circ \tau_j$

$\Rightarrow \tau_i = \tau_j$

$S_n \cdot \sigma = \{\tau \cdot \sigma : \tau \in S_n\} = S_n$

$\{\sigma^{-1} : \sigma \in S_n\} \underset{\subseteq}{\supseteq} S_n$

* Every permutation is a product of disjoint cycles.

σ

$(1\ \sigma(1)\ \sigma^2(1) - \sigma^k(1))(- - -) - -$

$(a\ b)^{-1} = (a\ b)$

$(a_1 a_2 \cdots a_k) = (a_1 a_k)\cdot(a_1 a_{k-1}) \cdots (a_1 a_2)$

$= (a_1 a_2)(a_2 a_3) \cdots (a_{k-1} a_k)$

$\sigma = t_1 t_2 \cdots t_m$

$= s_1 s_2 \cdots s_k$

$\sigma^{-1} = s_k s_{k-1} \cdots s_2 s_1$

$id = \sigma\cdot\sigma^{-1} = t_1 t_2 \cdots t_m s_k s_{k-1} \cdots s_2 s_1$

$k+m$

$id =$

* The identity permutation is even.

$$id = t_1 t_2 \cdots t_m t_m$$

$(a\,b)$

$$t_{m-1} t_m = (a\,b)(a\,b)$$
$$= (c\,d)(a\,b)$$
$$= (c\,a)(a\,b)$$
$$= (b\,c)(a\,b)$$

$id = (\ldots m)$

$$(a\,b)(c\,d) = (c\,d)(a\,b)$$

$$(c\,a)(a\,b) = (a\,c)(b\,c)$$

$$(b\,c)(a\,b) = (a\,c)(b\,c)$$

$a \to c$
$b \to a$
$c \to b$

$a \to c$
$b \to a$
$c \to b$

$$A_{n\times n} = (a_{ij})$$

$$\det A = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}$$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$\det A = \sum_{\sigma \in S_2} \operatorname{sgn}(\sigma)\, a_{1\sigma(1)} a_{2\sigma(2)}$$

$$\operatorname{sgn}(\sigma) = \begin{cases} 1 & \text{if } \sigma \text{ is even} \\ -1 & \text{if } \sigma \text{ is odd} \end{cases}$$

$$\operatorname{sgn}(\sigma \circ \tau) = \operatorname{sgn}(\sigma) \cdot \operatorname{sgn}(\tau)$$

1

id

$\sigma = id$ $\quad \operatorname{sgn}(\sigma)\, a_{1\sigma(1)} a_{2\sigma(2)}$

$= a_{11} a_{22}$

$\sigma = (1\,2)$ $\quad \operatorname{sgn}(\sigma)\, a_{1\sigma(1)} a_{2\sigma(2)}$

$= -a_{12} a_{21}$

$$\det A = a_{11} a_{22} - a_{12} a_{21}$$

$$\det A = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, a_{1\sigma(1)}\, a_{2\sigma(2)} \cdots a_{n\sigma(n)}$$

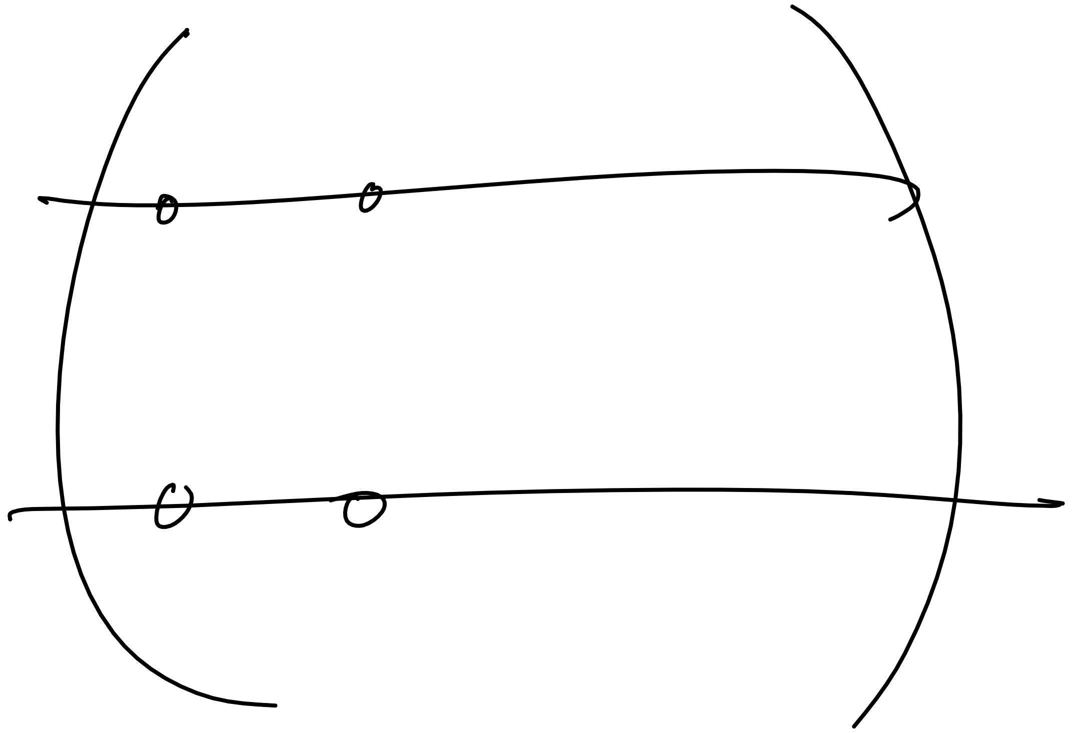
$$R_{ij} A = B$$

$$\det B = -\det A$$

$$A = \begin{pmatrix} a_{11} & a_{12} & - & - & a_{1n} \\ a_{21} & a_{22} & - & & a_{2n} \\ \vdots & & & & \\ a_{n1} & a_{n2} & - & & a_{nn} \end{pmatrix}$$

$$\tau = (i\ j)$$

$$S_n \cdot \tau = S_n$$



$$\det B = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, b_{1\sigma(1)} b_{2\sigma(2)} \cdots b_{n\sigma(n)}$$

$$= \sum_{\sigma \in S_n \cdot \tau} \operatorname{sgn}(\sigma \circ \tau)\, b_{1\,\sigma\circ\tau(1)} b_{2\,\sigma\circ\tau(2)} \cdots b_{i\,\sigma\circ\tau(i)} \cdots b_{j\,\sigma\circ\tau(j)} \cdots b_{n\,\sigma\circ\tau(n)}$$

$$= \sum_{\sigma\circ\tau \in S_n \cdot \tau} \operatorname{sgn}\sigma \cdot \operatorname{sgn}\tau\, b_{1\sigma(1)} b_{2\sigma(2)} \cdots b_{i\sigma(j)} \cdots b_{j\sigma(i)} \cdots b_{n\sigma(n)}$$

$$= -\sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{j\sigma(j)} \cdots a_{i\sigma(i)} \cdots a_{n\sigma(n)}$$

$$= -\det A$$

$R_i(c)A = B$

$\det B = c \cdot \det A$

$$\sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, b_{1\sigma(1)}\, b_{2\sigma(2)} \cdots b_{i\sigma(i)} \cdots b_{n\sigma(n)}$$

$$a_{1\sigma(1)}\, a_{2\sigma(2)} \cdot c\, a_{i\sigma(i)} \cdots a_{n\sigma(n)}$$

$$= c \det A$$

$$A = \begin{pmatrix} \\ \\ \\ \end{pmatrix} \begin{matrix} i \\ \\ j \end{matrix}$$

ith row = jth row.

$$\boxed{R_{ij}(c) \cdot A = B}$$

$B =$

$\det B = -\det A = \det A$

$$\sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, b_{1\sigma(1)} \cdots b_{i\sigma(i)} \cdots b_{n\sigma(n)} = \det B = \det A$$

$$b_{i\sigma(i)} = a_{i\sigma(i)} + c\, a_{j\sigma(j)}$$

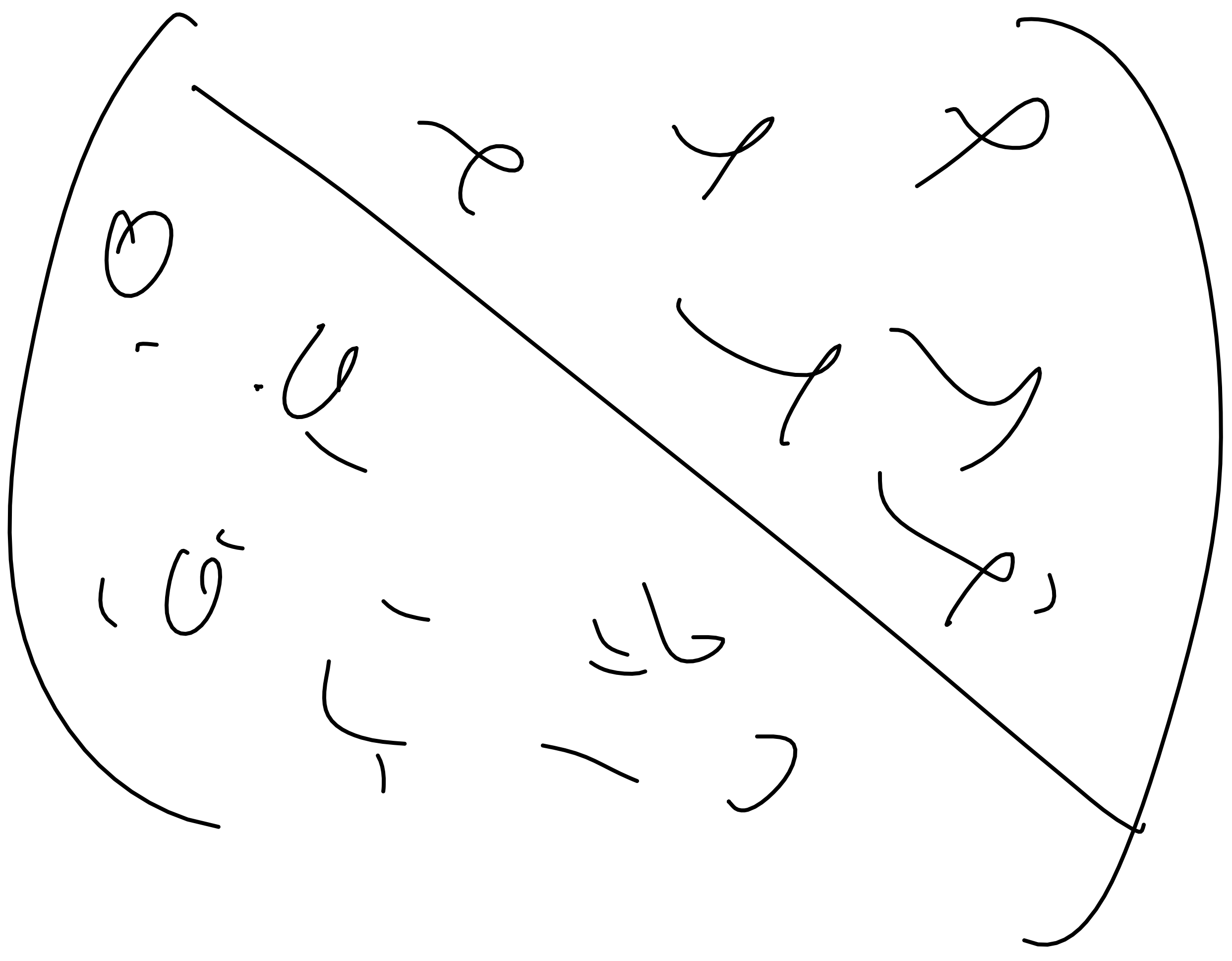
$$\det A = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, a_{1\sigma(1)} \cdots a_{i\sigma(i)} \cdots a_{n\sigma(n)} + c \underbrace{\sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, a_{1\sigma(1)} \cdots a_{j\sigma(j)} \cdots a_{j\sigma(j)} \cdots a_{n\sigma(n)}}_{0}$$

$$\det(E_{ij}A) = \det(E_{ij}) \cdot \det A$$

$$\det(E_i(c) \cdot A) = c \det A$$

$$c = \det(E_i(c))$$

$$\det(E_{ij}(c)A) = \det A$$

$A =$ 

$$\det A = a_{11} a_{22} \cdots a_{nn}$$

$$\det A = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, a_{1\sigma(1)} \cdots a_{n\sigma(n)}$$

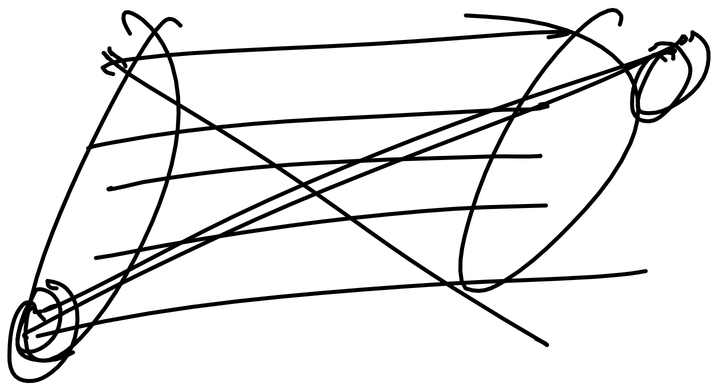
$\sigma = id$ $\qquad a_{11} a_{22} \cdots a_{nn}$

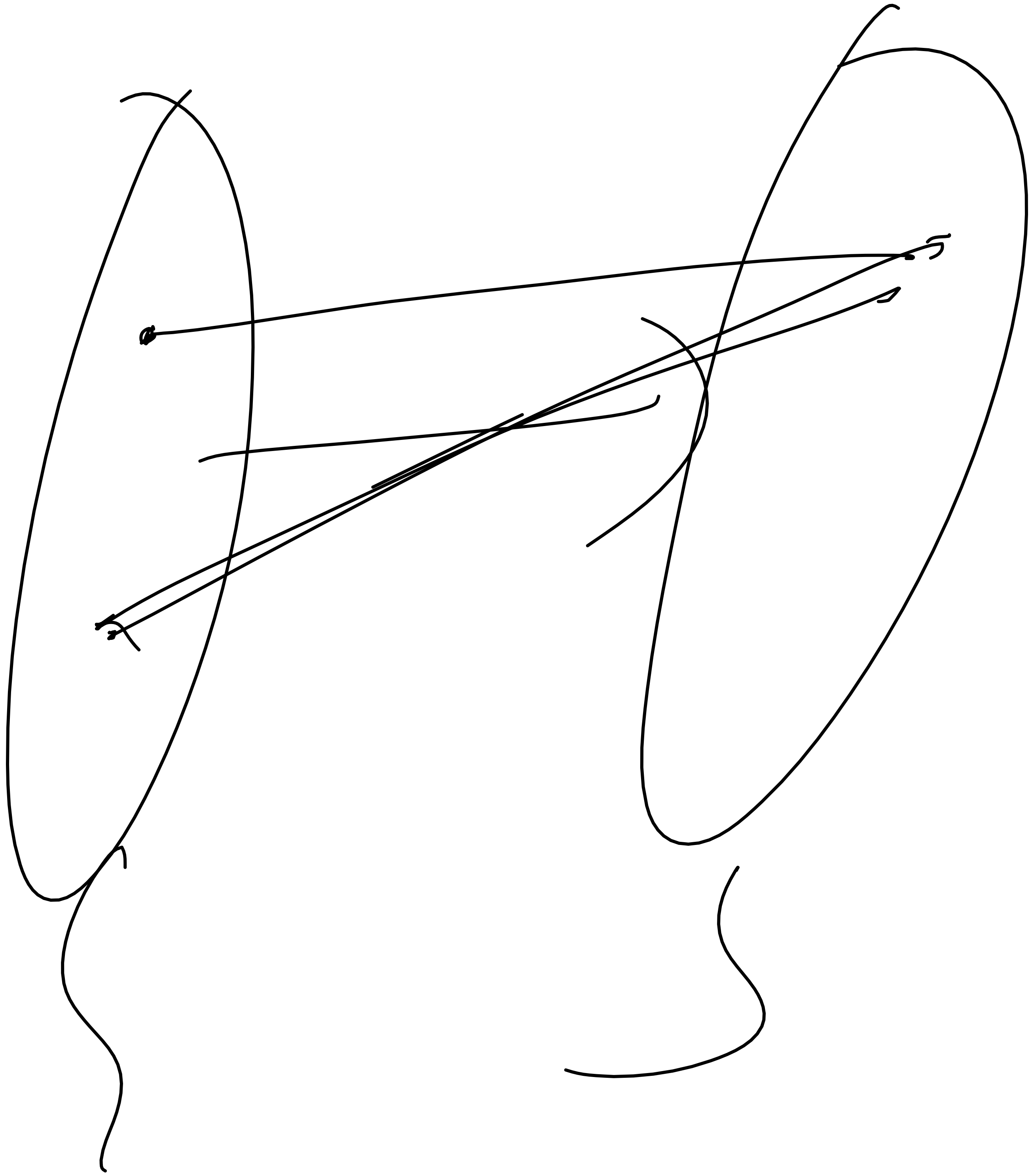
$\sigma \neq id \quad a_{1\sigma(1)} \cdots a_{n\sigma(n)} = 0$?

$\exists m \quad 1 \leq m \leq n-1$ s.t

$m > \sigma(m)$

$a_{m\sigma(m)} = 0$





* A is invertible iff $\det A \neq 0$

A is invertible

$\Rightarrow A = E_1 E_2 \cdots E_k$

$\Rightarrow \det A = \det(E_1 \cdots E_k)$

$= \det E_1 \det E_2 \cdots \det E_k$

$\neq 0$

A is not invertible.

$A \xrightarrow{\text{RREF}} R$

$\det(AB) = \det A \cdot \det B$

If A is invertible

$A = E_1 E_2 \cdots E_k$

$\det(AB) = \det(E_1 E_2 \cdots E_k B)$

$= \det E_1 \det E_2 \cdots \det E_k \cdot \det B$

$= \det(E_1 E_2 \cdots E_k) \det B$

$= \det A \cdot \det B$

If A is not invertible

$\Rightarrow AB$ is not invertible

$\det AB = 0$