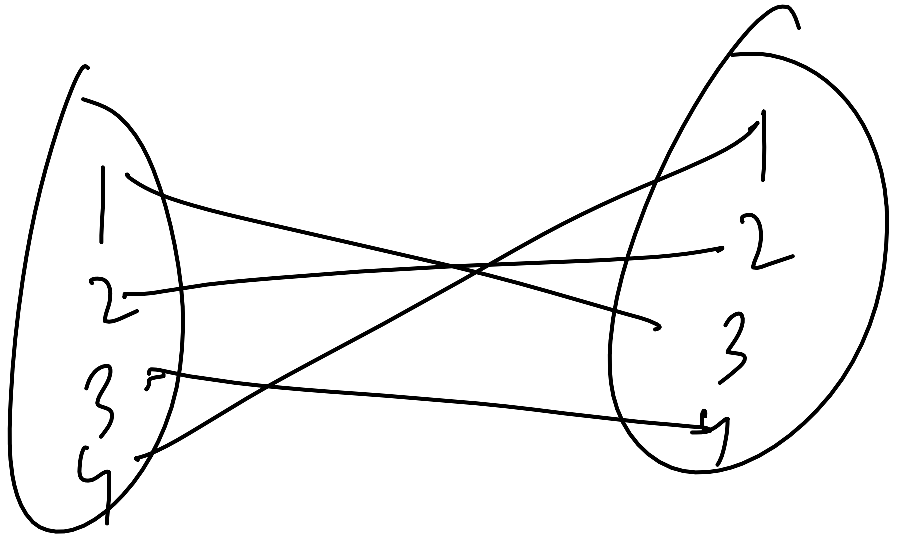
Recall

$f : S \to S$

S_n = Set of all bijections from S to S when $S = \{1, 2, 3, \cdots n\}$

= Set of all permutations on S

$$\begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & n \\ f(1) & f(2) & f(3) & f(4) & \cdots & f(n) \end{pmatrix}$$



$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix}$$

$$S \xrightarrow{f} S \xrightarrow{g} S$$

$f \circ g \in S_n \quad g \circ f \in S_n \quad f^{-1} \in S_n$

$\boxed{\sigma \cdot S_n} = \{\sigma \circ \tau : \tau \in S_n\} \subset S_n$

$\sigma^{-1} \cdot \sigma \circ \tau_i = \sigma^{-1} \cdot \sigma \circ \tau_j$

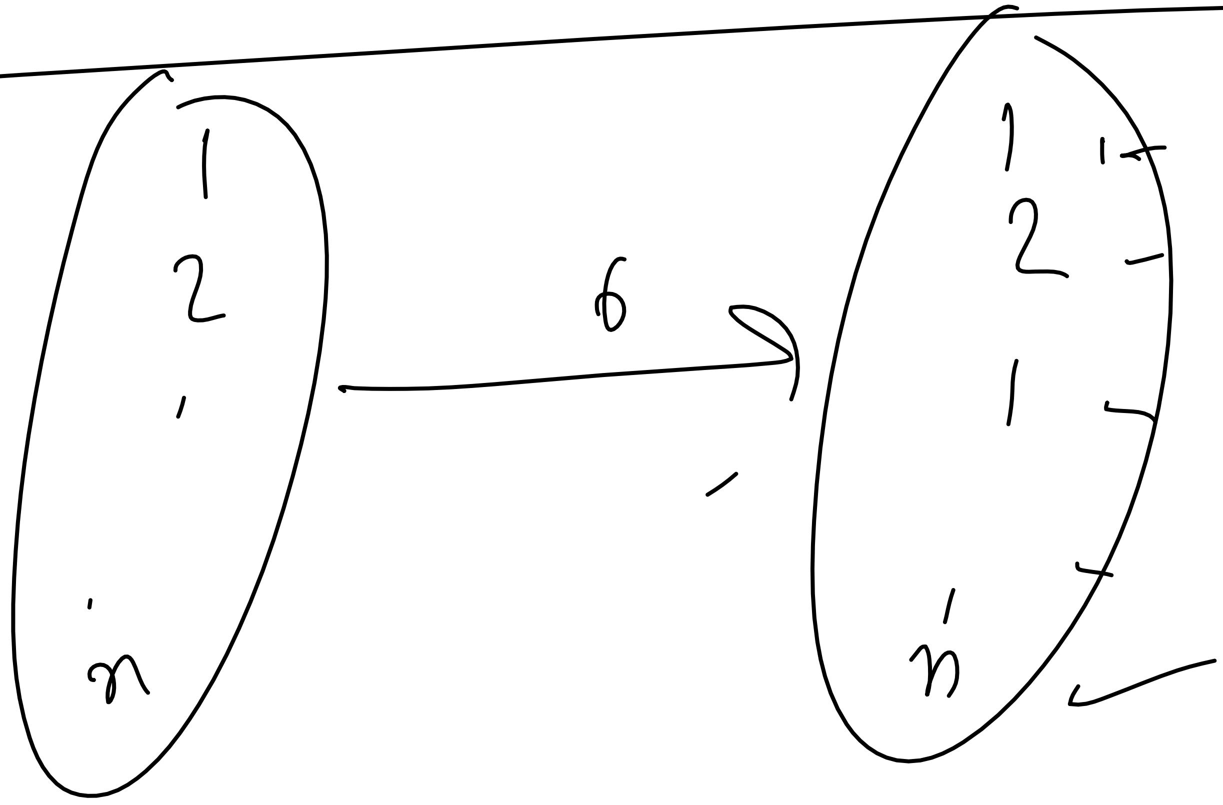
$\Rightarrow \mathrm{id} \cdot \tau_i = \mathrm{id} \circ \tau_j$

$\Rightarrow \tau_i = \tau_j \qquad \sigma \cdot S_n = S_n$

$S_n \cdot \sigma = S_n.$

$\{\sigma^{-1} : \sigma \in S_n\} \subseteq S_n$

$\tau = (\tau^{-1})^{-1}$



$(1\ 2\ 3) = (1\ 2)(2\ 3)$
$= (1\ 3)(1\ 2)$

$(1\ \sigma(1)\ \sigma^2(1)\ \cdots\ \sigma^k(1))(\ \cdots\ )$

* Every permutation is a product of disjoint cycles.

transpositions.

* $(a_1, a_2, \ldots, a_n)$

$= \boxed{(a_1\ a_n)} \circ (a_1\ a_{n-1}) \cdots (a_1\ a_2)$

$= (a_1\ a_2)(a_2\ a_3) \cdots (a_{n-1}\ a_n)$

* Every cycle is a product of transpositions.
* Every permutation is a product of transpositions.

$\sigma = t_1 t_2 \cdots t_m$

$= t_1' t_2' \cdots t_s'$

$\sigma^{-1} = t_s'^{-1} t_{s-1}'^{-1} \cdots t_1'^{-1}$

$id = \sigma \circ \sigma^{-1} = t_1 t_2 \cdots t_m t_s'^{-1} t_{s-1}'^{-1} \cdots t_1'^{-1}$

$= t_1 t_2 \cdots t_m t_s' t_{s-1}' \cdots t_1'$

$m+s$

$(c\,d)(a\,b) = (a\,b)(c\,d)$

$(a\,b)^{-1} = (a\,b)$

Theorem: Identity permutation is even.

Proof: a

$id = t_1 t_2 \cdots t_{m-1} t_m$

$(a\,b)$

$t_{m-1} t_m = (a\,b)(a\,b)$

$(a\,b)(c\,d) = (c\,d)(a\,b)$

$(a\,c)(b\,c) = (b\,c)(a\,b)$

$(a\,c)(b\,c) = (a\,c)(a\,b)$

$(a\,b)(a\,b)$

$$\operatorname{Sgn} : S_n \longrightarrow \{\pm 1\}$$

$$\operatorname{Sgn}(\sigma) = \begin{cases} 1 & \text{if } \sigma \text{ is even} \\ -1 & \text{if } \sigma \text{ is odd} \end{cases}$$

$$\operatorname{Sgn}(\sigma \circ \tau) = \operatorname{Sgn}(\sigma) \cdot \operatorname{Sgn}(\tau)$$

$$A_{n\times n} = (a_{ij})$$

$\det A$ - determinant

$$= \sum_{\sigma \in S_n} \operatorname{Sgn}(\sigma)\, a_{1\sigma(1)}\, a_{2\sigma(2)} \cdots a_{n\sigma(n)}$$

$$A = \begin{pmatrix} a_{11} & a_{12} & - & - & a_{1n} \\ a_{21} & a_{22} & - & & a_{2n} \\ | & & & & \\ | & & & & \\ a_{n1} & a_{n2} & - & - & a_{nn} \end{pmatrix}_{n\times n}$$

$\sigma \qquad \mathrm{sgn}(\sigma)\, a_{1\sigma(1)} - \cdot a_{n\sigma(n)}$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$\det A = \sum_{\sigma \in S_2} \mathrm{sgn}(\sigma)\, a_{1\sigma(1)} a_{2\sigma(2)}$$

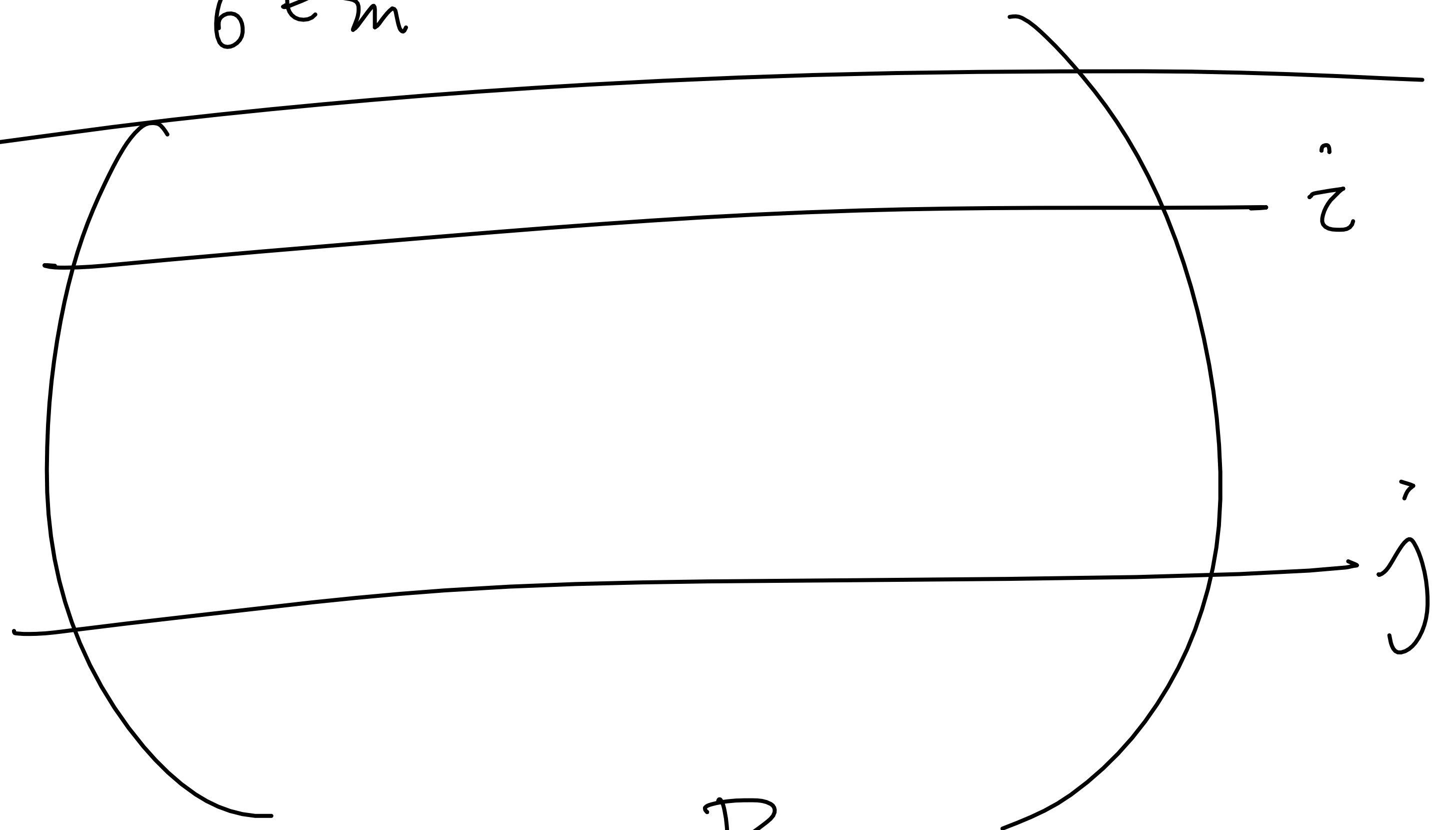
If $\sigma = id$ $\quad a_{1\sigma(1)} a_{2\sigma(2)}$
$= a_{11} a_{22}$

If $\sigma = (12)$ $\quad a_{1\sigma(1)} a_{2\sigma(2)}$
$= a_{12} a_{21}$

$$\det A = \mathrm{sgn}(id)\, a_{11} a_{22} + \mathrm{sgn}(12)\, a_{12} a_{21}$$
$$= a_{11} a_{22} - a_{12} a_{21}$$

$$\det A = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma)\, \underline{a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}}$$

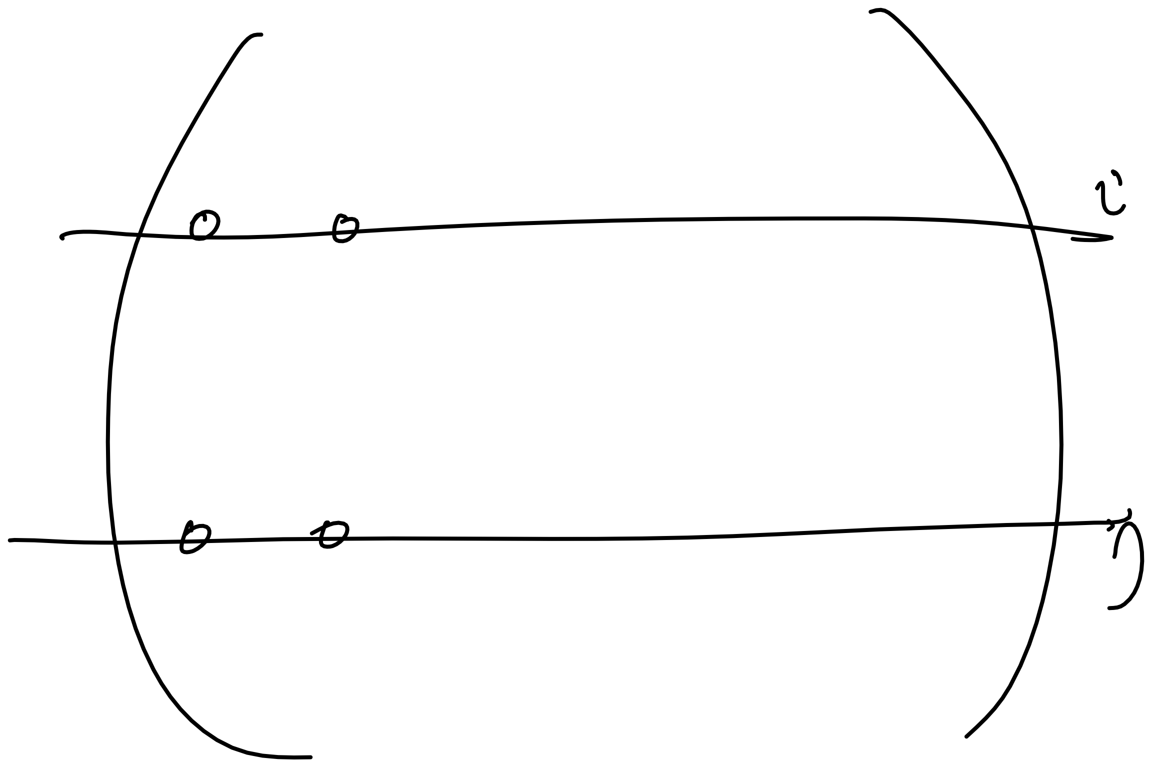
$\cancel{A}$

$A =$ 

B

$$\det B = \det(E_{ij} A)$$

$$\det B = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, b_{1\sigma(1)}\, b_{2\sigma(2)} \cdots b_{n\sigma(n)}$$



let $\tau = (i\ j)$

$$S_n = S_n \circ \tau$$

$$= \sum_{\sigma \in S_n} \underline{\operatorname{sgn}(\sigma\circ\tau)}\, b_{1\,\sigma\circ\tau(1)}\, b_{2\,\sigma\circ\tau(2)} \cdots b_{i\,\sigma\circ\tau(i)} \cdots b_{j\,\sigma\circ\tau(j)} \cdots b_{n\,\sigma\circ\tau(n)}$$

$$= -\sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, \underline{b_{1\sigma(1)}}\, b_{2\sigma(2)} \cdots b_{i-1\,\sigma(i-1)}\, \underbrace{b_{i\sigma(j)}}_{a_{j\sigma(j)}} \cdots \underbrace{b_{j\sigma(i)}}_{a_{i\sigma(i)}} \cdots b_{n\sigma(n)}$$

$$= -\det(A)$$

$$A \xrightarrow{R_i(c)} \begin{pmatrix} a_{11} & a_{12} & - & a_{1n} \\ a_{21} & - & \cdot & a_{2n} \\ ca_{i1} & ca_{i2} & \cdot & -ca_{in} \\ a_{i+1\,1} & - & & \\ a_{n1} & - & - & a_{nn} \end{pmatrix} = B$$

$$\det B = c \det(A)$$

$$A = \begin{pmatrix} \\ \\ \\ \end{pmatrix} \xrightarrow{R_{ij}} \begin{pmatrix} \\ \\ \\ \end{pmatrix} B$$

$$\det A = -\det A$$

$$\Rightarrow \det(A) = 0$$

$b_{i\sigma(i)}$

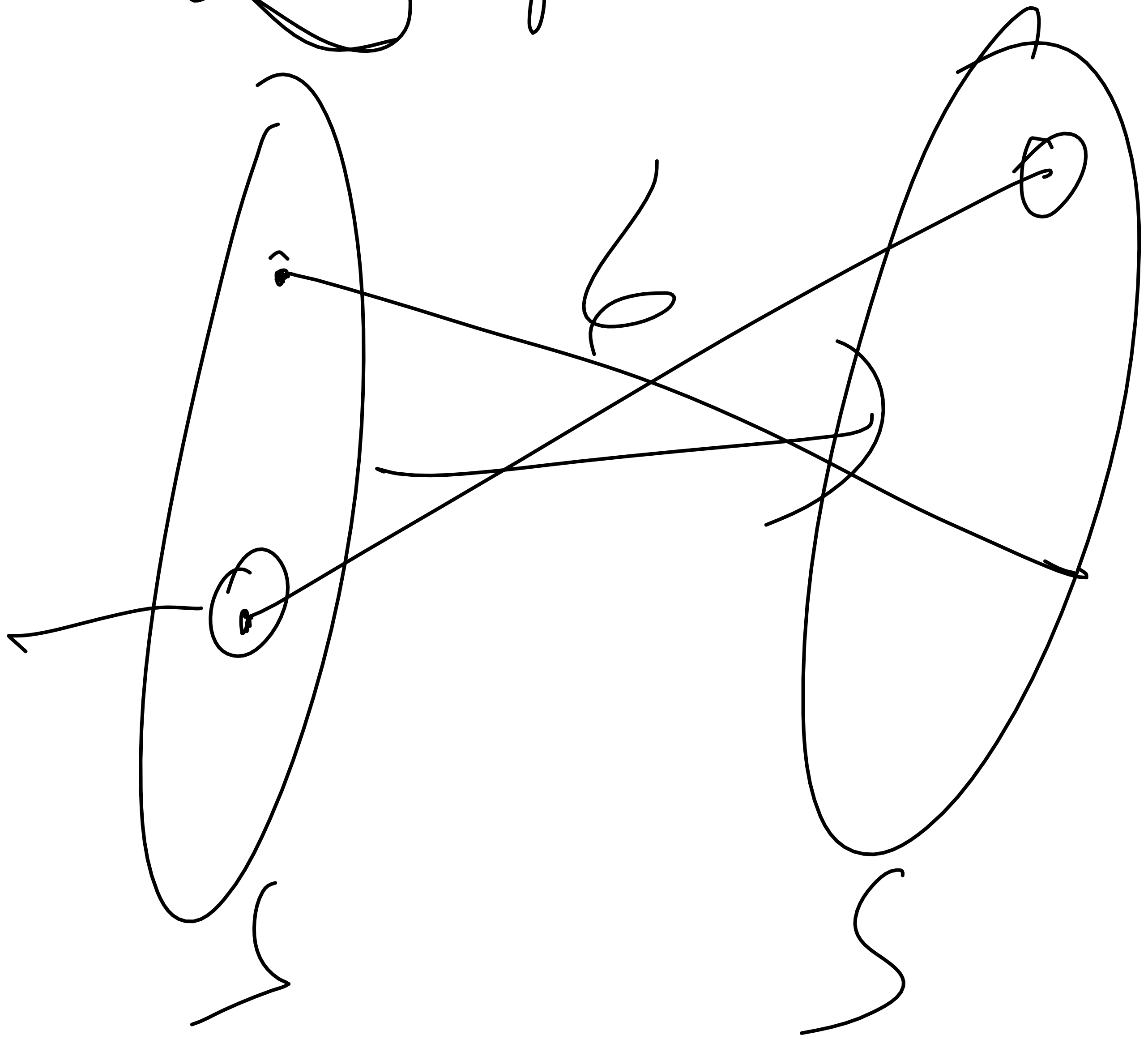
$$\det B = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma)\, b_{1\sigma(1)} b_{2\sigma(2)} \cdots b_{n\sigma(n)}$$

$$= \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma)\, a_{1\sigma(1)} \cdots (a_{i\sigma(i)} + c\,a_{j\sigma(j)}) \cdots a_{n\sigma(n)}$$

$$= \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma)\, a_{1\sigma(1)} \cdots a_{i\sigma(i)} \cdots a_{n\sigma(n)}$$

$$+\, c \sum \mathrm{sgn}(\sigma)\, a_{1\sigma(1)} \cdots a_{j\sigma(j)} \cdots a_{j\sigma(j)} \cdots a_{n\sigma(n)}$$

$\det A$



$$\det A = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, a_{1\sigma(1)} \cdots a_{n\sigma(n)}$$

$\sigma = id \quad a_{11} a_{22} \cdots a_{nn}$

If $\sigma \neq id \quad a_{1\sigma(1)} \cdots a_{n\sigma(n)} = 0$

If $\sigma \neq id \quad \exists\, 1 \leq m \leq n-1$

s.t. $m > \sigma(m)$

$a_{m\sigma(m)} = 0$

$\det(E_{ij}) = -1$

$\det(E_i(c)) = c$

$\det(E_{ij}(c)) = 1$

If A is invertible

$AB = E_1 E_2 \cdots E_q B$

$\det(AB) = \det E_1 \det E_2 \cdots \det E_q \cdot \det B$

$= \det(E_1 E_2 \cdots E_q) \det B$

$= \det A \cdot \det B$

$\det(E_k A) = \det E_k \cdot \det A$

$\det(AB) = \det A \cdot \det B$

* A is invertible iff $\det A \neq 0$

Suppose A is not invertible

A is invertible iff

$A = E_1 E_2 \cdots E_k$

$\det A = \det E_1 \det E_2 \cdots \det E_k$