$f: A = \begin{pmatrix} & \\ & \end{pmatrix}_{n\times n} \longrightarrow \det A$

$$\det A = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}$$

$\det(E_{ij} A) = -\det A$

$\det(E_i(c) A) = c \det A$

$\det(E_{ij}(c) A) = \det A$

$A = \begin{pmatrix} & * & * \\ 0 & & * \\ 0 & & \end{pmatrix}$ $\quad \det A = a_{11} a_{22} \cdots a_{nn}$

$B = \begin{pmatrix} & 0 \\ * & \end{pmatrix}$ $\quad \det B = b_{11} b_{22} \cdots b_{nn}$

$\det A = 0$

$\det(AB) = \det A \cdot \det B$

$A$ invertible $A = E_1 E_2 \cdots E_p$

$\det A = \det(E_1) \det(E_2) \cdots \det(E_p)$

$AB = (E_1 E_2 \cdots E_p B)$

$\det AB = \det E_1 \cdot \det E_2 \cdots \det E_p \cdot \det B$

$= \det A \cdot \det B$

$A$ is not invertible $\Rightarrow \det A = 0$

$\Rightarrow AB$ is not invertible.

$\Rightarrow \det(AB) = 0$

* $A$ is invertible iff $\det(A) \neq 0$.

$\det(A^T) = \det A$

$E_{ij}$ $\quad E_i(c)$ $\quad E_{ij}(c)$

$(E_{ij})^T = E_{ij}$ $\quad (E_i(c))^T = E_i(c)$

$\det(E_i^T) = \det E_i$

A invertible

2) $A = E_1 E_2 \cdots E_k$

$A^T = E_k^T E_{k-1}^T \cdots E_1^T$

$\det(A^T) = \det(E_k^T) \cdots \det(E_1^T)$

$= \det(E_k) \cdots \det(E_1)$

$= \det(E_1) \cdots \det(E_k)$

$= \det(E_1 E_2 \cdots E_k)$

$= \det A$

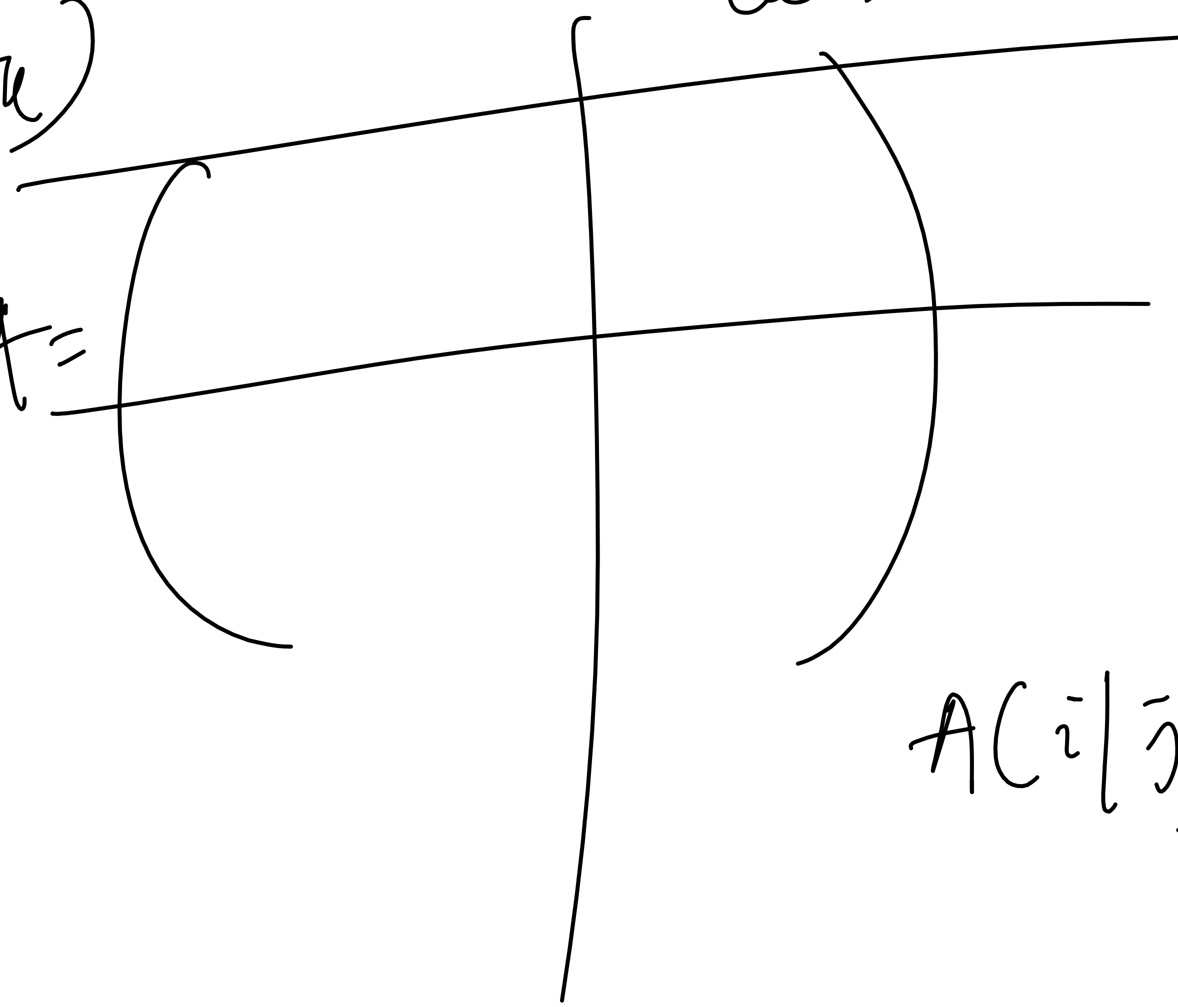
* A is not invertible iff $A^T$ is not invertible

$\det A = \det A^T = 0$

$AB = I$

$B^T A^T = I$

Leibniz

$A =$ 

$A(i|j)_{n-1 \times n-1}$

$M_{ij} = \det A(i|j)$

$(i,j)$th minor of $A$

$C_{ij} = (-1)^{i+j} M_{ij}$ $(i,j)^{th}$ co-factor of $A$

$C = (C_{ij})_{n \times n}$ — co-factor Matrix

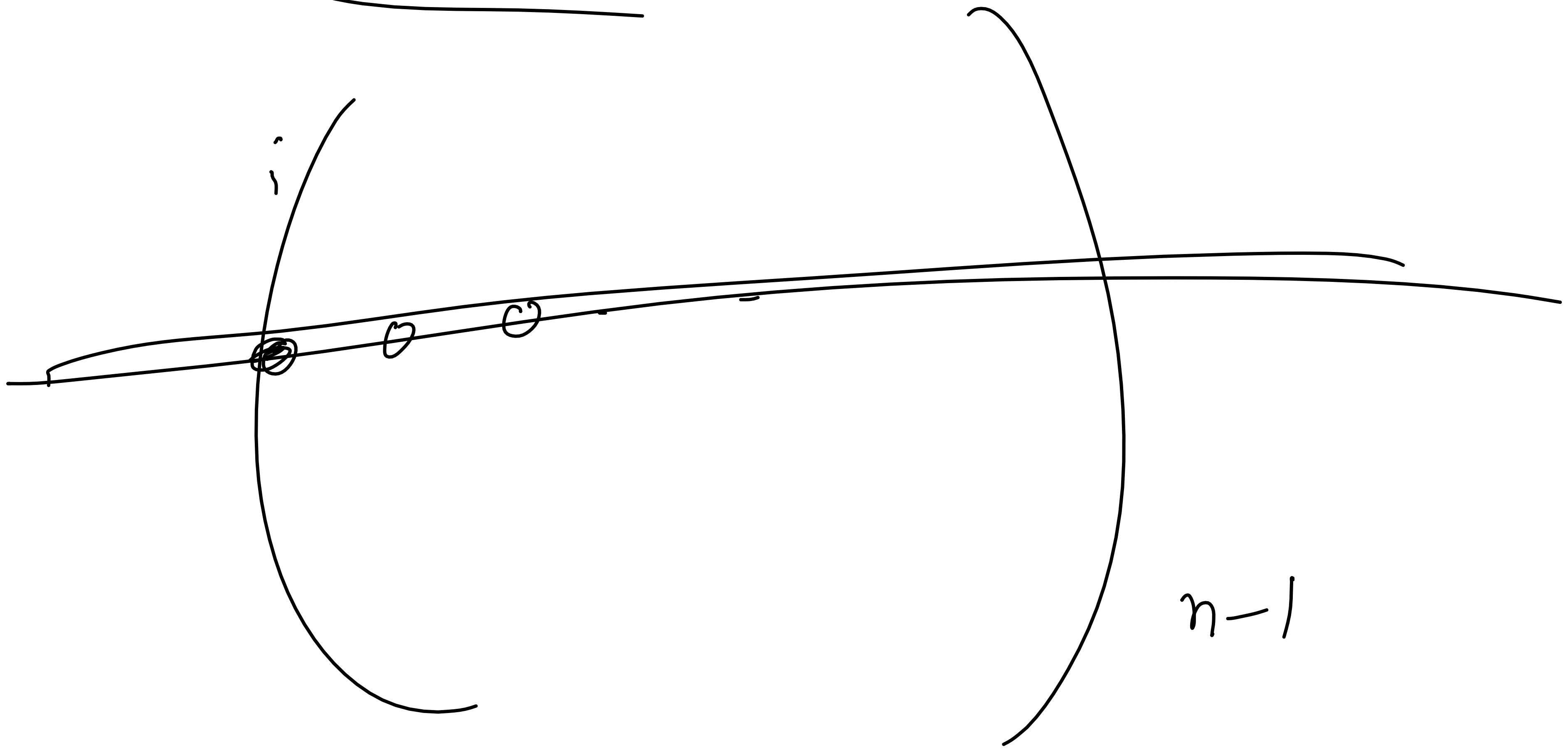
$\mathrm{Adj}\, A = C^{T}$

Adjugate, Adjoint of $A$

$A \cdot \mathrm{Adj}\, A = \det A \, I_{n \times n}$

$$\det A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(A(i|j)) \quad \forall i$$

$$= \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det(A(i|j)) \quad \forall j$$



$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$ 2nd

$$(-1)^{2+1} a_{21} \det\begin{pmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{pmatrix} + (-1)^{2+2} a_{22} \det\begin{pmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{pmatrix}$$

$$+ (-1)^{2+3} a_{23} \det\begin{pmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{pmatrix}$$

$i = 1$

$$\det A = \sum_{j=1}^{n} (-1)^{j+1} a_{1j} \det A(1|j).$$

$$B_j = \begin{pmatrix} 0 & 0 & \cdots & a_{1j} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & & \cdots & a_{2n} \\ \vdots & & & & & & & \\ a_{n1} & a_{n2} & \cdots & & \cdots & & \cdots & a_{nn} \end{pmatrix}$$

$j = 1, 2 \cdots n$

$$\det A = \det B_1 + \det B_2 + \cdots + \det B_n$$

$$B_1 = \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & & a_{nn} \end{pmatrix}$$

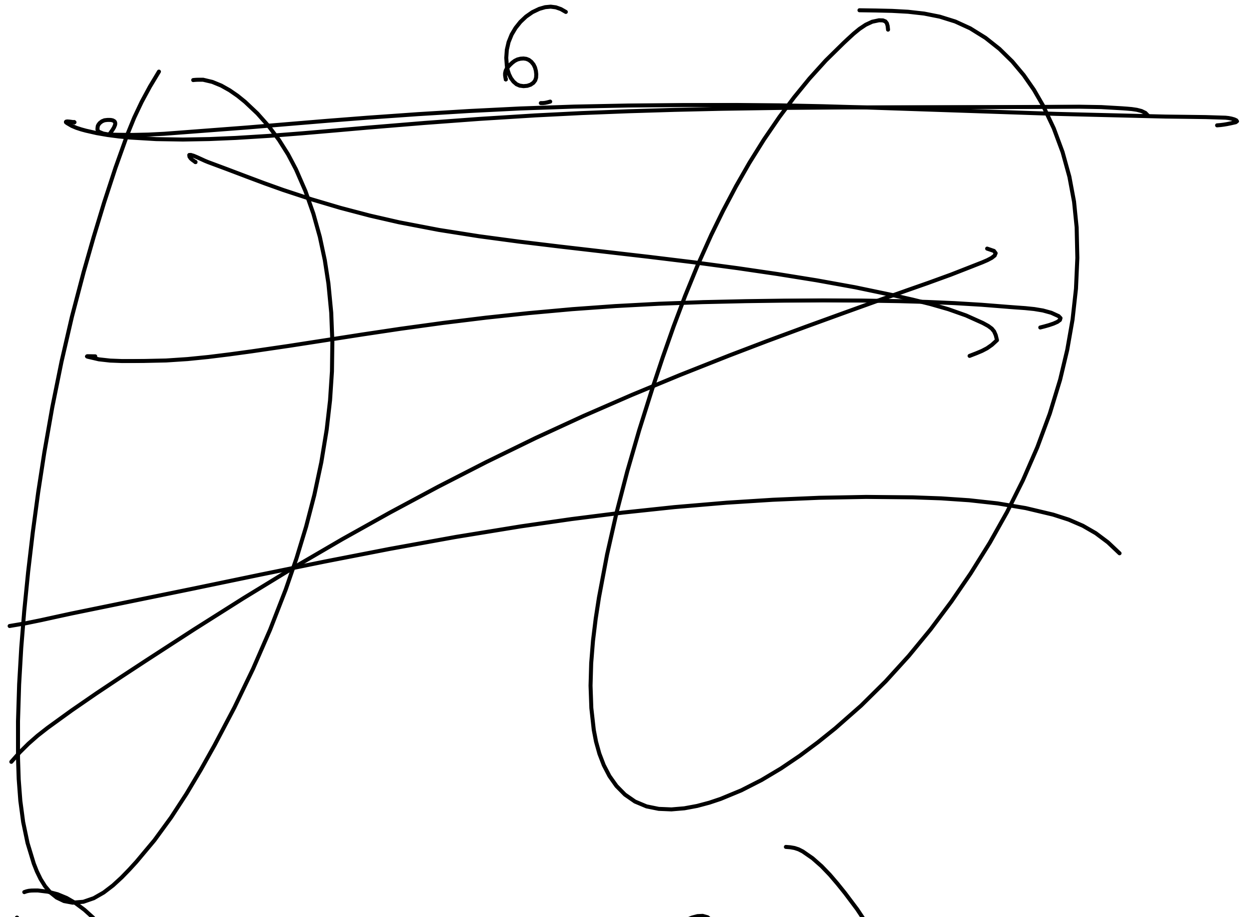
$$\det B_1 = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}$$

$$= \sum_{\substack{\sigma \in S_n \\ \sigma(1)=1}} \operatorname{sgn}(\sigma)\, a_{11}\, a_{2\sigma(2)} \cdots a_{n\sigma(n)}$$

$$= a_{11} \sum_{\substack{\sigma \in S_n \\ \sigma(1)=1}} \operatorname{sgn}(\sigma)\, a_{2\sigma(2)} \cdots a_{n\sigma(n)}$$

$$= a_{11} \sum_{\sigma \in S_{n-1}} \operatorname{sgn}(\sigma)\, a_{2\sigma(2)} \cdots a_{n\sigma(n)}$$

$$= a_{11} \det A(1|1).$$




$$B_2 = \begin{pmatrix} 0 & a_{12} & 0 & 0 & \cdots & a_{2n} \\ a_{21} & a_{22} & a_{23} & & & \\ | & & | & & & \\ a_{n1} & a_{n2} & | & & & a_{nn} \end{pmatrix}$$

$$-\det a_{12} \det(A(1|2))$$

$$\det B_{ij} = (-1)^{j-1} a_{ij} \det A(1|j)$$

$$= (-1)^{j+1} a_{1j} \det A(1|j)$$

$$AX = d$$

$$(A|d)$$

$$\downarrow$$

$$(I \mid c)$$

$$X = A^{-1} d$$

## Cramer's Rule;

If $A$ is invertible then $AX = d$ has a unique solⁿ

and it is given by

$$x_i = \frac{\det(A_i)}{\det A}$$

$$A_i = \begin{pmatrix} C_1 & C_2 & \cdots & C_{i-1} & d & C_{i+1} & \cdots & C_n \end{pmatrix}$$

$$X_i = \begin{pmatrix} 1 & 0 & & 0 & x_1 & 0 & \cdots & 0 \\ 0 & 1 & & 0 & x_2 & 0 & & 0 \\ \vdots & 0 & & \vdots & 1 & \vdots & & 1 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \\ 0 & 0 & & 0 & x_n & 0 & & 1 \end{pmatrix}_{n \times n}$$

$$AX_i = \begin{pmatrix} C_1 & C_2 & \cdots & C_{i-1} & d & C_{i+1} & \cdots & C_n \end{pmatrix} = A_i$$

$$\det(AX_i) = \det A_i$$

$$\Rightarrow \det A \cdot \det X_i = \det A_i$$

$$= \det A \cdot x_i = \det A_i$$

$$\begin{pmatrix} 1 & x_1 & 0 \\ 0 & x_2 & 0 \\ 0 & x_3 & 1 \end{pmatrix} = x_i$$

## Vector Spaces ;

A nonempty set $V$ over a field $F$ with a binary operation $+ : V \times V \to V$ and an operation $\boxed{\cdot} : F \times V \to V$ satisfying the following is called a vector space

(1) $v_1 + v_2 \in V \quad \forall v_1, v_2 \in V$ —

(2) $v_1 + (v_2 + v_3) = (v_1 + v_2) + v_3$ a

(3) $\exists\, 0$ s.t $0 + v = v + 0 = v$

(4) For any $v \in V$ $\exists\, -v \in V$ s.t $v + (-v) = 0$

(5) $v_1 + v_2 = v_2 + v_1 \quad \forall v_1, v_2 \in V$

(6) $\alpha \cdot v \in V \quad \forall v \in V, \alpha \in F$

(7) $(\alpha \cdot \beta) \cdot v = \alpha \cdot (\beta \cdot v) \quad \forall v \in V \quad \alpha, \beta \in F$

(8) $\exists\, 1$ st $1 \cdot v = v$

closure

(9) $(\alpha + \beta) \cdot v = \alpha \cdot v + \beta \cdot v \quad \forall \alpha, \beta \in F, v \in V$

(10) $\alpha \cdot (v_1 + v_2) = \alpha \cdot v_1 + \alpha \cdot v_2 \quad \alpha \in F, v_1, v_2 \in V$

$V = \mathbb{R}^2$ . $\mathbb{F} = \mathbb{R}$.

$(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$

$\alpha \cdot (x_1, x_2) = (\alpha x_1, \alpha x_2)$

$+ : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$

$\mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$

$(\alpha \quad (x_1, x_2)) \longmapsto (\alpha x_1, \alpha x_2)$