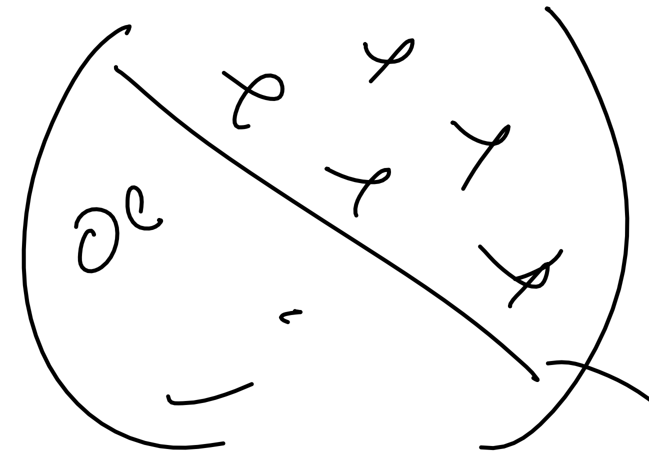
Recall

$A_{n\times n} \longrightarrow \det(A)$ — Leibniz.

$$\det A = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}$$



$= a_{11} a_{22} \cdots a_{nn}$

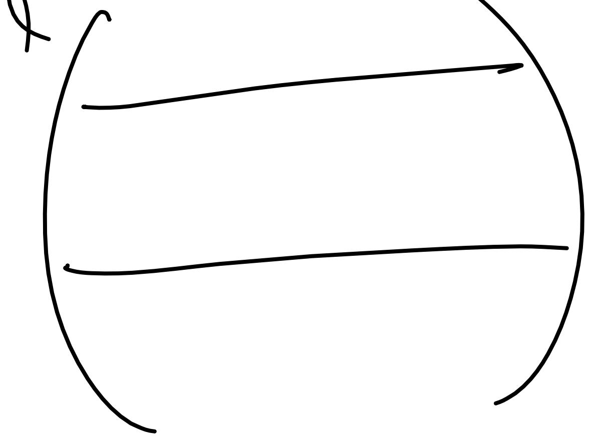
$= a_{11} a_{22} \cdots a_{nn}$

$\det(I_{n\times n}) = 1$

$\det(R_{ij} A) = -\det A$

$\det(R_i(c) A) = c \det A$

$\det(R_{ij}(c) A) = \det A$

$A =$ 

$\det A = 0.$

$\det(E_i A) = \det E_i \cdot \det A$

$\det(AB) = \det A \cdot \det B$

$\det(A^T) = ?\ \det A$ ✓

If A is invertible.

$A = E_1 E_2 \cdots E_k$

$A^T = E_k^T E_{k-1}^T \cdots E_2^T E_1^T$

$\Rightarrow \det A = \det E_1 \det E_2 \cdots \det E_k$

$\Rightarrow \det A^T = \det(E_k^T) \det(E_{k-1}^T) \cdots \det(E_1^T)$

If A is not invertible $\Rightarrow \det A = 0$

$\Rightarrow \det A^T = 0$

E_{ij} = Symmetric

$E_i(c)$ ✓

$$\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & c & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$$

✓ $\det E_{ij}(c) = 1$

✓ $\det(E_{ij}(c))^T = 1$

$\det(E_i) = \det(E_i^T)$

$$A_{n\times n} = \begin{pmatrix} a_{11} & a_{12} & - & - & a_{1n} \\ a_{21} & a_{22} & & & a_{2n} \\ \vdots & & & & \\ a_{n1} & a_{n2} & - & - & a_{nn} \end{pmatrix}$$

$$M_{ij} = \det(A(i|j))$$

$(i,j)^{th}$ Minor

$$A(i|j) = \begin{pmatrix} & \\ & \end{pmatrix}$$

$(n-1)\times(n-1)$

$$C_{ij} = (-1)^{i+j} M_{ij}$$

$$\boxed{C = (c_{ij})}_{n\times n}$$

Cofactor Matrix

$$Adj(A) = C^T$$

Adjugate of A

Adjoint

$A_{n\times n}$

$$\det A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(A(i|j))$$ for each i

$$= \sum_{j} (-1)^{i+j} a_{ij} M_{ij}$$

$$= \sum_{j} a_{ij} C_{ij}$$

$$\det A = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det(A(i|j))$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$= a_{11} \det\begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} - a_{12} \det\begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix}$$

$$+ a_{13} \det\begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$$

$\boxed{i=1}$

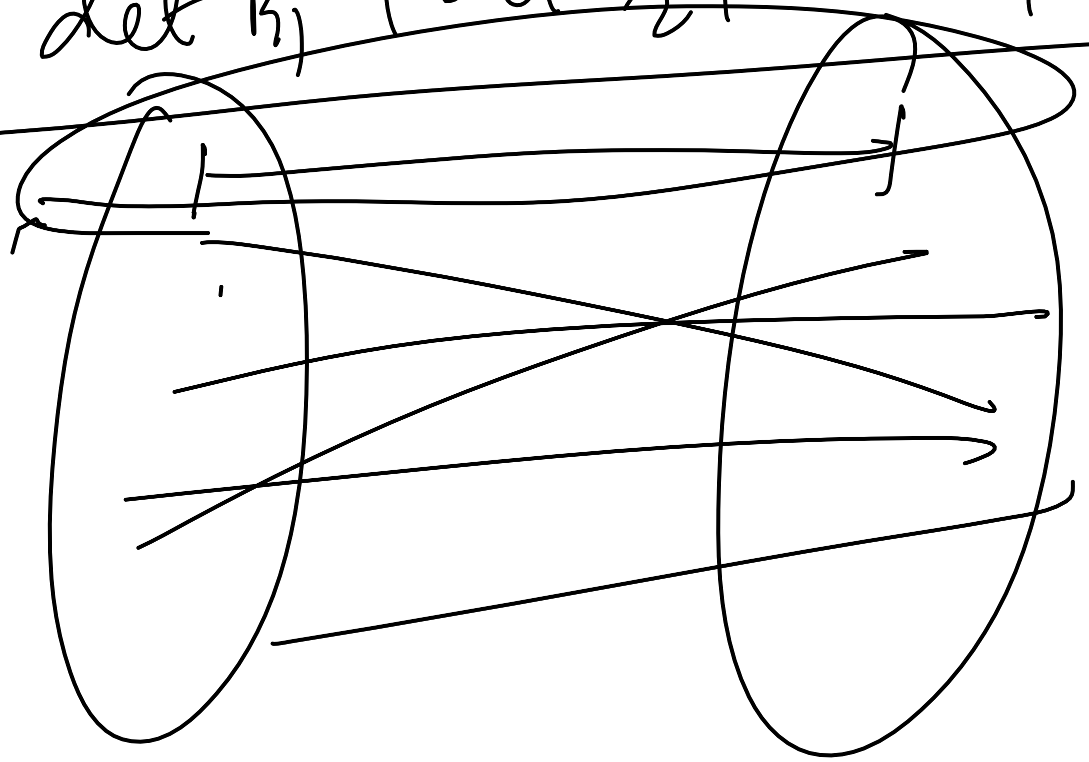
$$\det A = \sum_j (-1)^{j+1} a_{1j} \det A(1|j)$$

A_2 $a_{ij})$

$$B_j = \begin{pmatrix} 0 & 0 & \cdots & a_{1j} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & & & & & a_{2n} \\ \vdots & & & & & & & \\ \vdots & & & & & & & \\ a_{n1} & a_{n2} & \cdots & & & & & a_{nn} \end{pmatrix}$$

$j = 1, 2, \dots, n$

$$\det A = \det B_1 + \det B_2 + \cdots + \det B_n$$



$$C = \begin{pmatrix} a & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & & a_{2n} \\ \vdots & & & & \\ a_{n1} & \cdots & & \cdots & a_{nn} \end{pmatrix}$$

$\det C = ?$

$$\det C = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}$$

$$= \sum_{\substack{\sigma \in S_n \\ \sigma(1)=1}} a\, a_{2\sigma(2)} \cdots a_{n\sigma(n)}$$

$$= a \sum_{\substack{\sigma \in S_n \\ \sigma(1)=1}} a_{2\sigma(2)} \cdots a_{n\sigma(n)}$$

$$a \sum_{\sigma \in S_{n-1}} a_{2\sigma(2)} \cdots a_{n\sigma(n)} = a \det(A(1|1))$$

$$B_1 = \begin{pmatrix} a_{11} & 0 & 0 & \cdots \\ a_{21} & & & \\ a_{22} & & & \\ \vdots & & & \\ \vdots & & & \end{pmatrix}$$

$$\det B_1 = a_{11} \det(A(1|1))$$

$\det B_2$

$$B_2 = \begin{pmatrix} 0 & a_{12} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & & \cdots \\ \vdots & & & & \\ a_{n1} & \cdots & & & a_{nn} \end{pmatrix}$$

$$\begin{pmatrix} a_{12} & 0 & \cdots & \cdots & 0 \\ a_{22} & a_{21} & & & a_{2n} \\ \vdots & \vdots & & & \vdots \\ a_{n2} & a_{n1} & & & a_{nn} \end{pmatrix}$$

$$-a_{12} \det(A(1|2))$$

$\mathrm{Adj}(A) = C^T$

$C = (c_{ij})$

$c_{ij} = (-1)^{i+j} M_{ij}$

$A \cdot \mathrm{Adj}(A) = \det A \, I$

$$\begin{pmatrix} \det A & & 0 \\ & \det A & \\ 0 & & \ddots \\ & & & \det A \end{pmatrix}$$

$A \cdot \mathrm{Adj}\,A = B$

$\mathrm{Adj}\,A = C^T$

If $i \neq j$

$= \sum_{j=1}^{n} a_{ij} c_{ij} =$

If $i = j$ $\quad b_{ij} = \det A$

If $i \neq j$ $\quad b_{ij} = 0$

$AX = d$

$$(A \mid d)$$

$$\downarrow$$

$$(\mathrm{RREF}(A) \mid C)$$

A is invertible

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = X = A^{-1}_{n\times n} \cdot d_{n\times 1} = \begin{pmatrix} \ \end{pmatrix}_{n\times 1}$$

$$x_i = \frac{\det A_i}{\det A}$$

$$A_i = \begin{pmatrix} C_1 & C_2 & \cdots & C_{i-1} & \boxed{d} & C_{i+1} & \cdots & C_n \end{pmatrix}$$

Cramer's Rule :

$$AX = d$$

$$x_i = \frac{\det A_i}{\det A}$$

$$X_i = \begin{pmatrix} 1 & 0 & & x_1 & 0 & - & 0 \\ 0 & 1 & & x_2 & 0 & & 0 \\ 0 & 0 & & \vdots & \vdots & & 1 \\ \vdots & \vdots & & \vdots & 1 & & \\ & & & & 0 & & \\ 0 & 0 & & x_n & \vdots & & 1 \end{pmatrix}$$

$\cancel{x_i}$

$$\det X_i = x_i$$

$$A X_i = \left(A \begin{pmatrix} 1 \\ 0 \\ \vdots \\ d \end{pmatrix} \quad A \begin{pmatrix} 0 \\ 1 \\ \vdots \\ d \end{pmatrix} \; - - \right)$$

$$= \left(c_1 \; c_2 \; \cdots \; \boxed{AX} \; c_{i+1} \cdots c_n \right)$$

$$= \left(c_1 \; c_2 \; \cdots \; d \; c_{i+1} \cdots c_n \right)$$

$$\det (AX_i) = \det A_i$$

$$\det A \, \det X_i = \det A_i$$

$$x_i = \frac{\det A_i}{\det A}$$

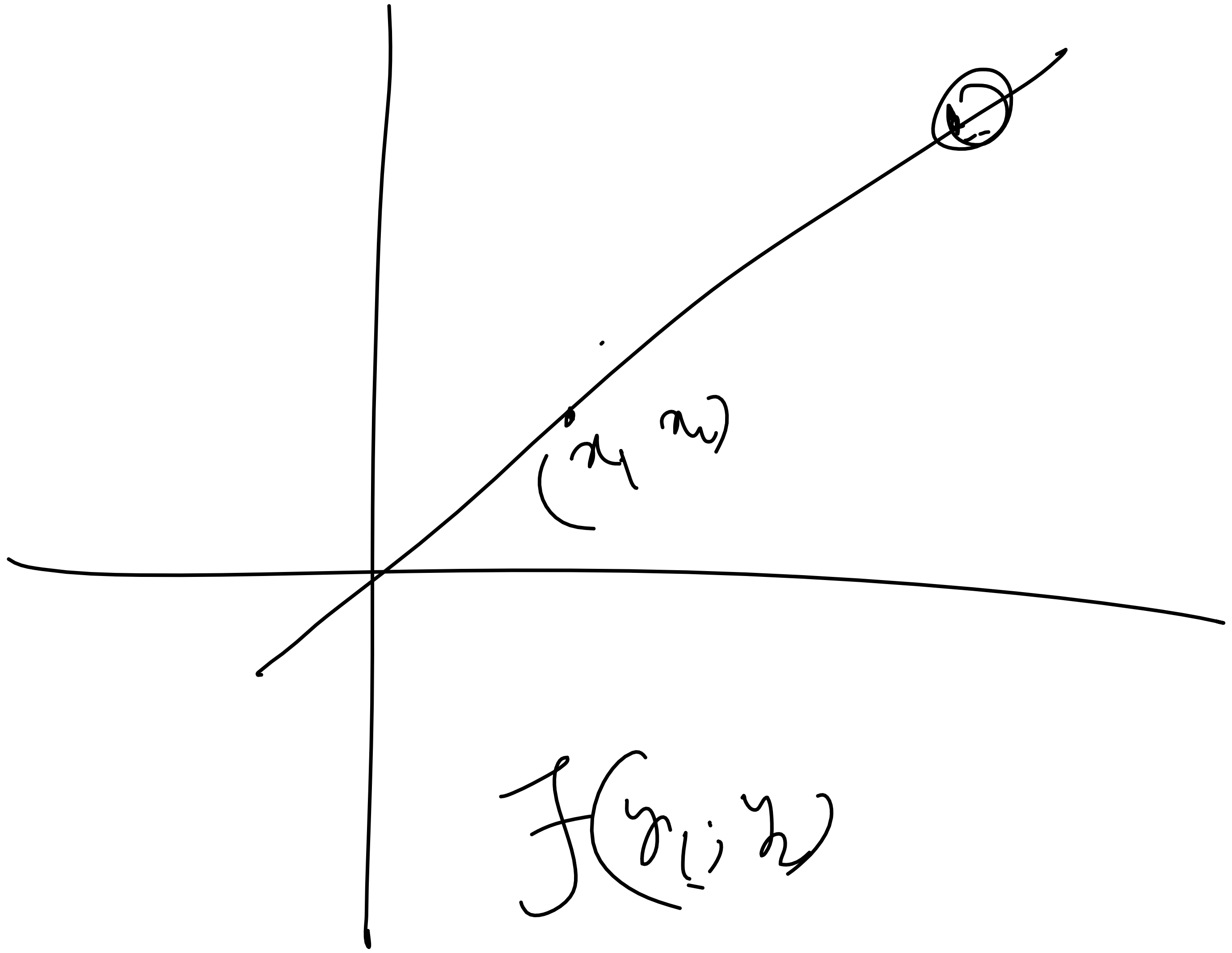
$(x_1, x_2) \cdot \mathbb{R}^2$

$(x_1, x_2) + (y_1, y_2)$

$= (x_1 + y_1,\ x_2 + y_2)$

$\alpha \in R$

$\alpha \cdot (x_1, x_2) = (\alpha x_1, \alpha x_2)$



$(x_1, x_2) + (y_1, y_2) = (0, 0)$

$(x_1, x_2) + (0, 0) = (x_1, x_2)$

Vector Space: A nonempty set V over a field with a binary operation

$\mathbb{Q}, \mathbb{R}, \mathbb{C}$

$+ : V \times V \to V$ and an operation $F \times V \to V$ satisfying the following is called a ~~field~~ vector space

(1) v_1, v_2 & $v_1 + v_2 \in V$ — closure w.r.t $+$

(2) $v_1 + (v_2 + v_3) = (v_1 + v_2) + v_3$ — associative w.r.t $+$

(3) $\exists\, 0$ s.t $v + 0 = 0 + v = v$ $\forall v \in V$

(4) for every $v \in V$ $\exists\, (-v)$ $v + (-v) = 0$

(5) $v_1 + v_2 = v_2 + v_1$

$\mathbb{R}$

$v_1 + v_2 = +(v_1, v_2)$

(6) $a \cdot v \in V$ $a \in F$, $v \in V$

(7) $1 \cdot v = v$ $\exists\, 1$

(8) $(a \cdot b) \cdot v = a \cdot (b \cdot v)$ $\forall a, b \in F, v \in V$

(9) $(a + b) \cdot v = a \cdot v + b \cdot v$

(10) $a(v_1 + v_2) = a \cdot v_1 + a \cdot v_2$

$\mathbb{R}^2$ —