Recall: $\det(A)$ $\qquad A_{n\times n}$

$$= \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma)\, a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}$$

$A(2|1)$

Leibniz.

$$M_{ij} = \det(A(i|j))$$

$$c_{ij} = (-1)^{i+j} M_{ij}$$

$$C = (c_{ij})$$

$$\operatorname{Adj}(A) = C^T$$

$$\det A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det A(i|j) \quad \text{for every } i$$

$$= \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det A(i|j) \quad \text{for every } j$$

Laplace.

$$A \cdot \operatorname{Adj}(A) = \operatorname{Adj} A \cdot A = \det A\, I$$

Cramer's Rule

$A$ invertible

$$AX = d$$

$$x_i = \frac{\det A_i}{\det A}$$

$$A_i = \begin{pmatrix} & & d & & \end{pmatrix}$$

$$X_i = \begin{pmatrix} 1 & 0 & \cdots & 0 & x_1 & 0 & \cdots & 0 \\ 0 & 1 & & 0 & x_2 & 0 & & 0 \\ \vdots & & \ddots & & \vdots & & & \vdots \\ & & & 1 & & 0 & & \\ 0 & \cdots & & & x_n & & & 1 \end{pmatrix}$$

$$A \cdot X_i = A_i$$

$$\det(A \cdot X_i) = \det A_i$$

$$\det A \cdot x_i = \det A_i$$

Vector Space: A nonempty set $V$ over a field $F$ with a binary operation $\oplus : V \times V \to V$ and an operation $\odot : F \times V \to V$ satisfying the following properties is called a vector space

(1) $v_1 \oplus v_2 \in V \quad \forall v_1, v_2 \in V$

(2) $v_1 + (v_2 + v_3) = (v_1 + v_2) + v_3 \quad \forall v_1, v_2, v_3 \in V$

(3) $\exists\, 0 \in V$ s.t $0 + v = v + 0 = v \quad \forall v \in V$

(4) For every $v \in V$ $\exists\, -v \in V$ s.t $v + (-v) = 0$

(5) $v_1 + v_2 = v_2 + v_1 \quad \forall v_1, v_2 \in V$

(6) $a \cdot v \in V \quad \forall a \in F$ and $v \in V$

(7) $1 \cdot v = v \quad \forall v \in V$

(8) $(ab) \cdot v = a \cdot (b \cdot v) \quad a, b \in F, \; v \in V.$

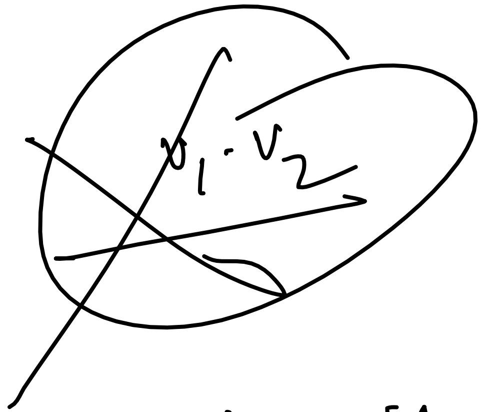
(9) $(a+b) \cdot v = a \cdot v + b \cdot v \quad \forall a, b \in F, \; v \in V$

(10) $a \cdot (v_1 + v_2) = a \cdot v_1 + a \cdot v_2 \quad \forall a \in F, \; v_1, v_2 \in V$

The elements of $V$ are called vectors

" of $F$ are called scalars.

$a \cdot v \quad a \in F, v \in V$ $\quad \cancel{v \cdot a}$



$v_1, v_2 \in V$

$V$ is a vector space over $F$.

$a + v. \quad a \in F \quad v \in V$

* $\underline{V = \mathbb{Q}}$ $\quad \mathbb{F} = \mathbb{Q}$

$+ : \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$

* $\mathbb{R}$ over $\mathbb{Q}$
* $\mathbb{R}$ over $\mathbb{R}$
* $\mathbb{C}$ over $\mathbb{R}$, $\mathbb{C}$ over $\mathbb{Q}$

$\mathbb{F} = \mathbb{R} \quad V = \mathbb{Q}$ ✗

$\mathbb{R} \times \mathbb{Q} \to \mathbb{Q}$ ?

* $V = \mathbb{R}^n \quad \mathbb{F} = \mathbb{Q}, \mathbb{R}$

$= \{(a_1, a_2, \ldots a_n) : a_i \in \mathbb{R}\}$

$(a_1, a_2, \ldots a_n) + (b_1, b_2, \ldots b_n)$

$= (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n)$

$a \in \mathbb{Q}$ $(a_1, a_2, \dots a_n)$

$a \cdot (a_1, a_2, \dots a_n) = (a a_1, \dots a \cdot a_n)$

$\in \mathbb{R}^n$

* $\mathbb{C}^n$ over $\mathbb{Q}$ or $\mathbb{R}$ or $\mathbb{C}$
* $V_2$ Set of all polynomials in $x$ over $\mathbb{Q}$, $\mathbb{R}$, or $\mathbb{C}$

$f(x) = a_0 + a_1 x + \dots + a_n x^n$

$g(x) = b_0 + b_1 x + \dots + b_m x^m$

$= (a_0 + b_0) + (a_1 + b_1) x + \dots$

$a \in$

$(V, +, \cdot)$

* Set of all $m \times n$ matrices with entries in $\mathbb{Q}$ ($\mathbb{R}$ or $\mathbb{C}$)
* Set of all functions from $\mathbb{R} \to \mathbb{R}$

$f : \mathbb{R} \to \mathbb{R}$ $g : \mathbb{R} \to \mathbb{R}$

$f+g : \mathbb{R} \to \mathbb{R}$

$(f+g)(x) = f(x) + g(x)$

$a \in \mathbb{R}$ $a \cdot f : \mathbb{R} \to \mathbb{R}$

$a \cdot f(x) = a \cdot f(x)$

$$S = \{ (x_1, x_2, \ldots x_n) \in \mathbb{R}^n : x_1 = 1 \}$$

**Subspace** : Let $(V, \boxed{F}, +, \cdot)$ be a vector space. A subset $W$ of $V$ is called a subspace of $V$ if $W$ is also a vector space w.r.t $+$ and $\cdot$.

**Examples** : $V = \mathbb{R}$ $\quad F = \mathbb{Q}$

$W = \mathbb{Q}$

$V = \mathbb{C}$ $\quad F = \mathbb{R}$ $\quad W = \mathbb{R}$

$+ : V \times V \to V$

$\cdot : F \times V \to V$

$+\big|_{W \times W} : W \times W \to W$

$\cdot : F \times W \to W.$

$\ast$ $V = \mathbb{R}^n$ $\quad F = \mathbb{R}$

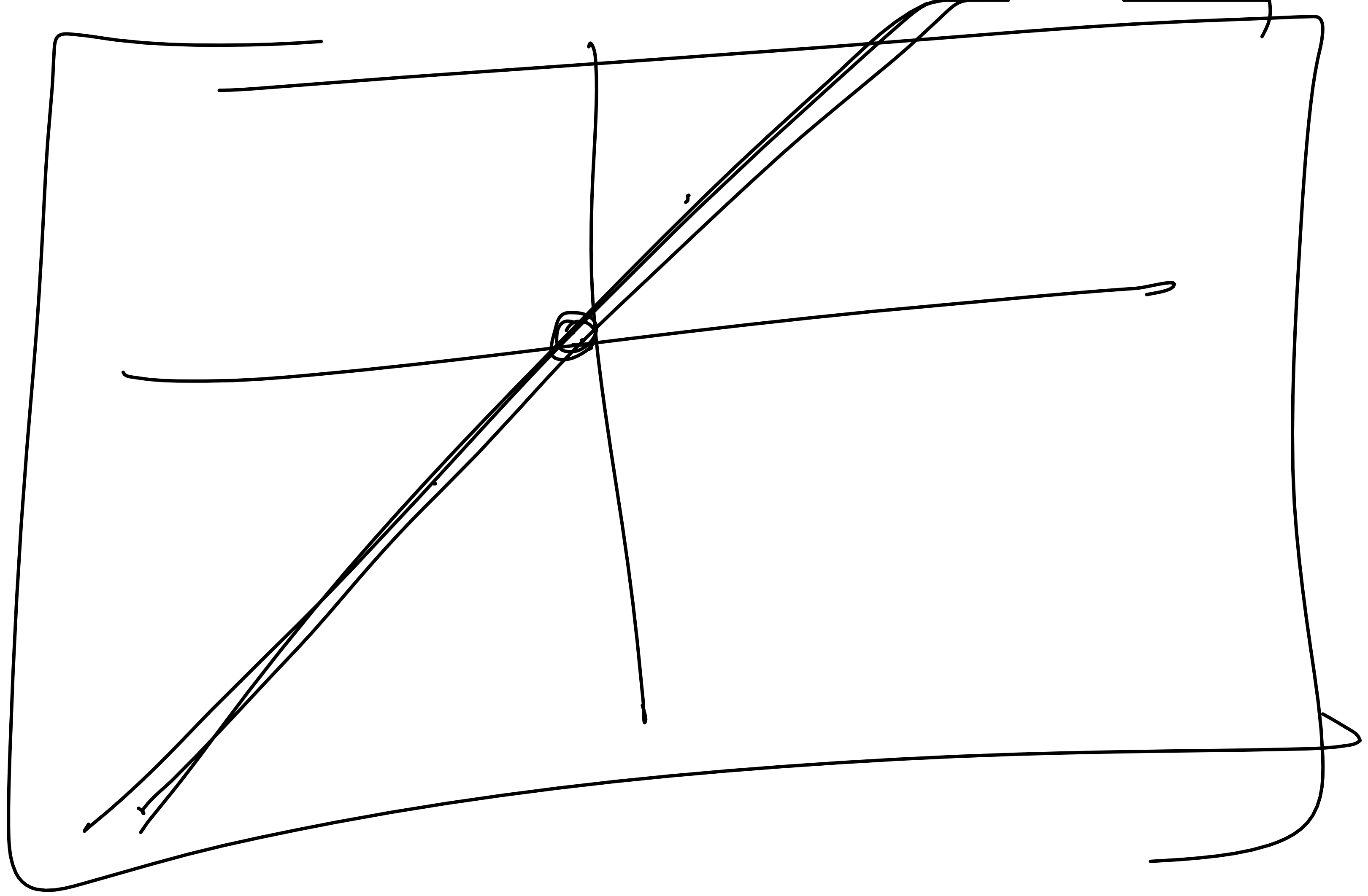
$W = \{ (x_1, x_2, \ldots x_n) : x_i = 0 \}$

$\ast$ $V =$ Set of all polynomials in $x$ over $\mathbb{R}$.

$W =$ Set of all polynomials of degree less than equal to $n$

* Set of all polynomials of degree $= n$ is not a subspace.

* $V = \mathbb{R}^2$



**Proposition:** Let $W$ be a nonempty subset of a vector space $V$. Then $W$ is a subspace iff $\boxed{a \cdot w_1 + w_2 \in W}$ $\forall w_1, w_2 \in W$ and $a \in F$

**Proof:** $a \cdot w_1 \in W$ $\quad w_2 \in W$

$\Rightarrow a \cdot w_1 + w_2 \in W$ $\quad \forall w_1, w_2 \in W \quad a \in F$

$(\Leftarrow)$ $\quad a \cdot w_1 + w_2 \in W$

Claim: $0 \in W$

$(-1) \cdot w_1 + w_1 \in W$

$= 0 \in W$

$1 w_1 + w_2 \in W$

$-w \in W$ ?

$(-1) \cdot w + 0 \in W$

$\Rightarrow -w \in W$

* $W_1 \subseteq V$ subspace $\quad W_2 \subseteq V$ subspace

$W_1 \cup W_2$ ?

* $W_1 \cup W_2$ is a subspace iff either $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$

* If $W_1 \subseteq W_2$ then $W_1 \cup W_2 = W_2$ is a subspace

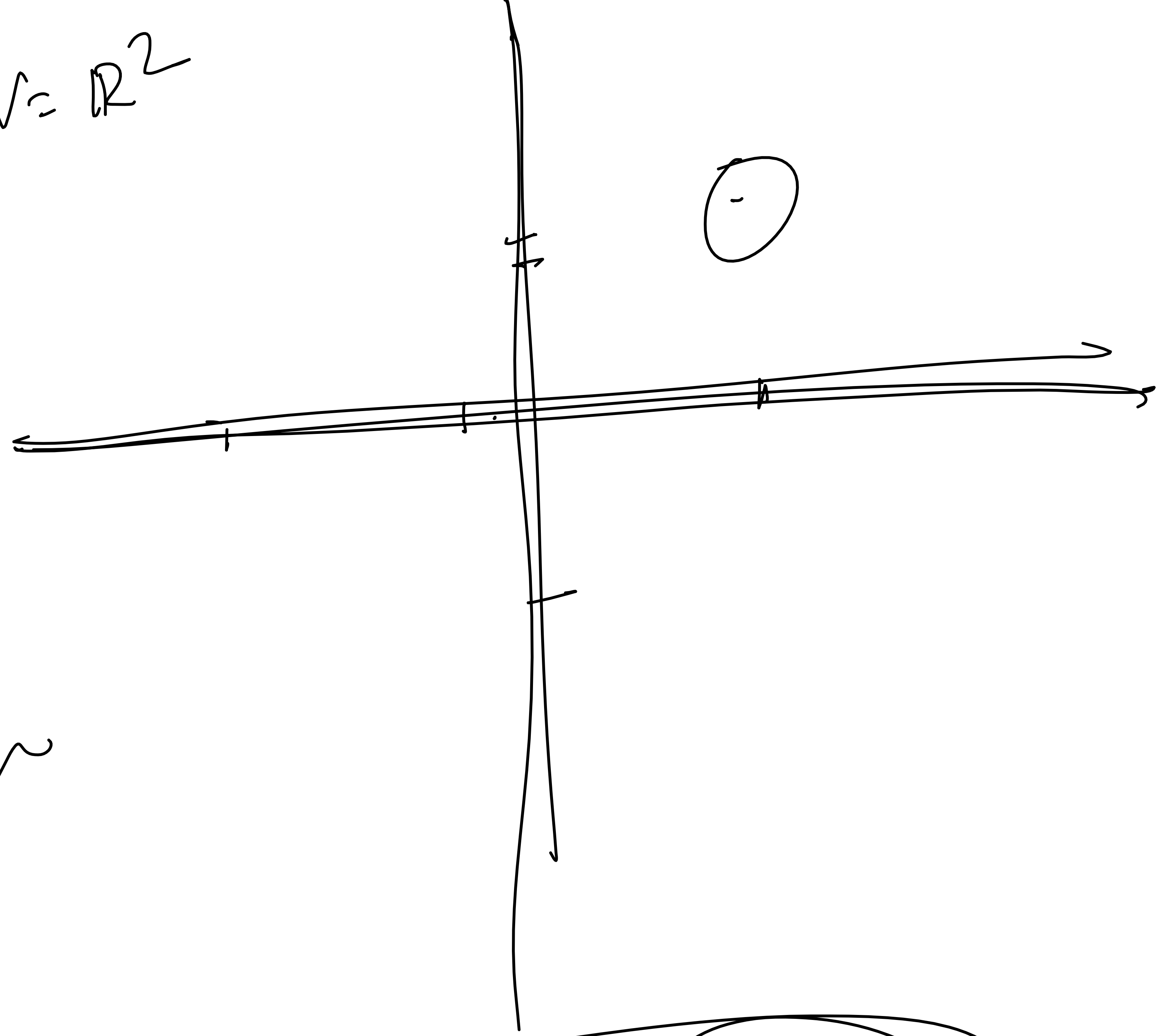
$W_2 \subseteq W_1$

($\Leftarrow$) Suppose $W_1 \cup W_2$ is a subspace

Suppose $W_1 \not\subseteq W_2$ and $W_2 \not\subseteq W_1$

$\exists\, w_1 \in W_2 \setminus W_1 \qquad \exists\, w_2 \in W_1 \setminus W_2$

$V = \mathbb{R}^2$



$w_1 + w_2 \in W_1 \cup W_2$

Is $w_1 + w_2 \in W_1$ ?

$\Rightarrow (w_1 + w_2) - w_2 \in W_1 \Rightarrow w_1 \in W_2$

Contradiction!