Recall: $(V, F, +, \cdot)$

$\mathbb{Z}_p = \{0, 1, 2, \cdots p-1\}$

$+$ ⊗ $(\mathbb{Z}_p, +, \times)$

⊗ $n = 1$

$\mathbb{Z}_4 = \{0, 1, 2, 3\}$

$a, b \in \mathbb{Z}_p$ $\quad a+b =$

$2+3 = 0$

$3 \cdot 2 = 1$

$\mathbb{Z}_5 = \{0, 1, 2, 3, 4\}$

$2+3 = 0$

$3+4 = 2$

$2 \times 3 = 1$

$x \neq -x$

- $x+y \in \mathbb{Z}_p \quad \forall x, y \in \mathbb{Z}_p$
- $xy \in \mathbb{Z}_p \quad \forall x, y \in \mathbb{Z}_p$
- $(x+y)+z = x+(y+z)$
- $x+0 = x = 0+x$
- $x+y = y+x$
- $x(y \times z) = (x \times y) \times z$

$x \neq 0$ ∃ y s.t. $x \cdot y = 1$

$a \times (b+c) = a \times b + a \cdot c$

$$\mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$$

$$\left\{ \frac{f(x)}{g(x)} : f, g \text{ are polys } g(x) \neq 0 \right\}$$

Recall $(V, F, +, \cdot)$

Subspace $W \subseteq V$

$W \subseteq V$ is a subspace iff $a \cdot w_1 + w_2 \in W$ $\forall a \in F$, $w_1, w_2 \in W$

$W_1 \cup W_2$ is a subspace iff either $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$

$\bigcap_{\alpha \in I} W_\alpha$ is a subspace

$W_1 + W_2 + \cdots + W_n$

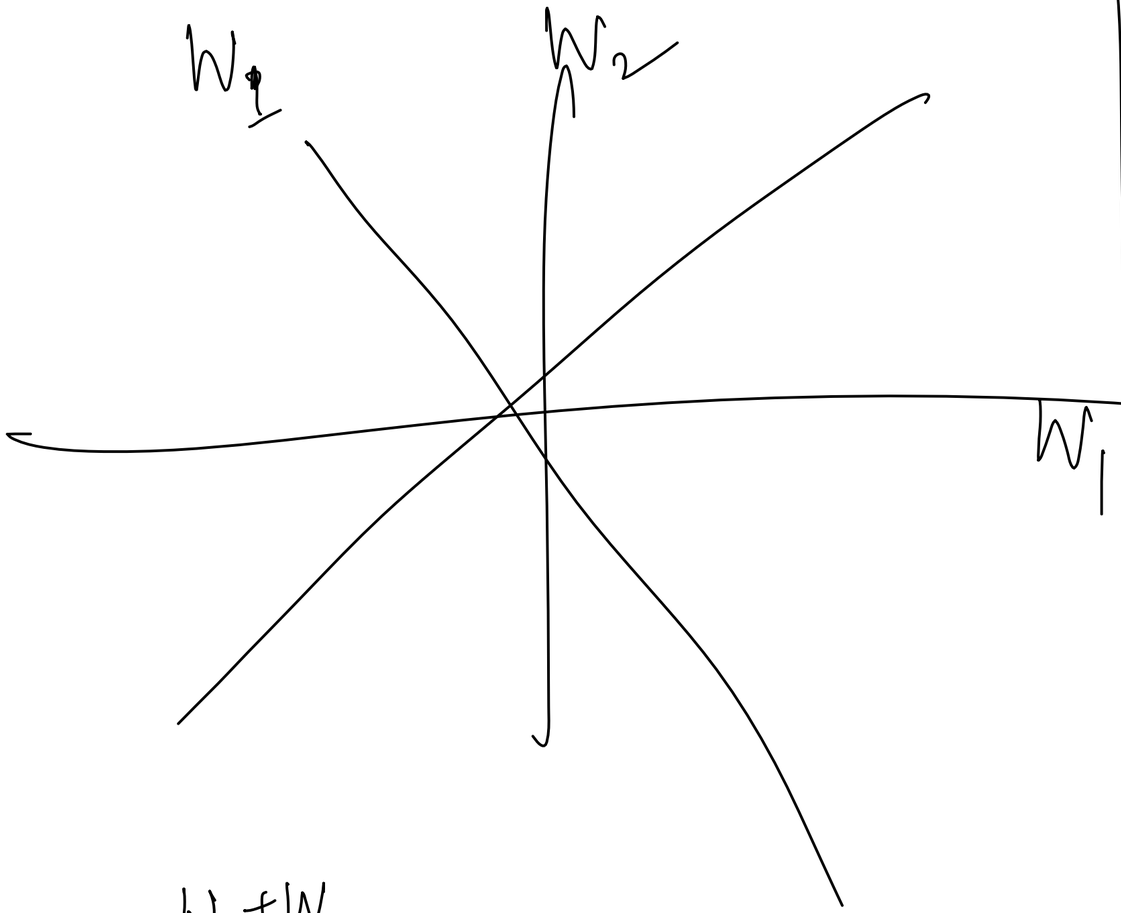
$= \{w_1 + w_2 + \cdots + w_n : w_i \in W_i\} \subseteq V$

$w_1 + w_2 + \cdots + w_n$ $\qquad$ $w_1' + w_2' + \cdots + w_n'$

$a(w_1 + w_2 + \cdots + w_n) + (w_1' + w_2' + \cdots + w_n')$

$= \underbrace{(aw_1 + w_1')}_{\in W_1} + \underbrace{(aw_2 + w_2')}_{\in W_2} + \cdots + \underbrace{(aw_n + w_n')}_{\in W_n}$

$V = \mathbb{R}^2$



$(x, 0) + (0, y) = (x, y)$

$S \subseteq V$

Subset

$\subseteq V$

$\text{Span } S = \{ c_1 v_1 + c_2 v_2 + \cdots + c_n v_n : c_i \in F, v_i \in S \}$

Linear Combinations

Finite

Claim: Span S is a subspace of V

$W_1 \in \text{Span } S$ $\quad W_2 \in \text{Span } S$

$W_1 = c_1 v_1 + c_2 v_2 + \cdots + c_m v_m$ $\quad W_2 = d_1 v_1' + d_2 v_2' + \cdots + d_n v_n'$

$aW_1 + W_2 = a(c_1 v_1 + c_2 v_2 + \cdots + c_m v_m) + d_1 v_1' + \cdots + d_n v_n'$

$S \subseteq V$. We say S is linearly dependent if $\exists\ v_1, v_2, \ldots v_n \in S$ and $c_1, c_2, \ldots c_n \in F$ not all $c_i = 0$ s.t $c_1v_1 + c_2v_2 + \cdots + c_nv_n = 0$

$$S = \{(1,0), (2,0)\}$$

$$(2)(1,0) - (1)(2,0) = 0$$

A set S which is not LD is called linearly independent

$$\{(1,0,0)\ (0,1,0),\ (1,1,0)\}$$

$$\{(1,0), (0,1)\}$$

$$a(1,0) + b(0,1) = (0,0)$$

$$\{1, x, x^2, \ldots\}$$

$$a \cdot 1 + b \cdot x + c \cdot x^2 + \ldots = 0$$

If $0 \in S$

$1 \cdot 0 = 0$

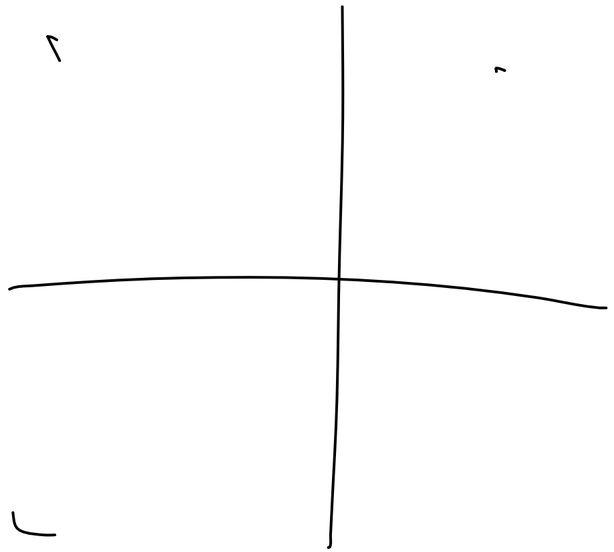
* A set containing a LD set is LD.

$S \subset T$

* A subset of a LI set is LI.

Theorem: Let V be a v.s. which is spanned by a finite set S. Then any set containing more than $|S|$ no of elements is LD.

$\text{Span } S = V$



Proof: $S = \{v_1, v_2, \cdots v_m\}$

$T = \{w_1, w_2, \cdots w_n\}$ $\quad n > m$

Claim: T is LD.

$\text{Span } S = V$

$w_j = \sum_{i=1}^{m} a_{ij} v_i \quad \forall j$

$A = (a_{ij})$

$AX = 0$

Let $c_1, c_2, \cdots c_n$ be a sol^n (non-trivial)

just $c_k \neq 0$

$c_1 w_1 + c_2 w_2 + \cdots + c_n w_n$

$\sum_{j=1}^{n} c_j w_j = \sum_{j=1}^{n} c_j \sum_{i=1}^{m} a_{ij} v_i$

$= \sum_{i=2}^{} \left(\sum_{j} a_{ij} c_j \right) v_i = 0$

~~Euro~~

Definition: A subset B of V is a basis of

(1) B spans V $\quad \text{Span } B = V$

(2) B is LI

Eg: $V = \mathbb{R}^n$

$(1, 0, 0 \cdots), (0, 1, 0, 0 \cdots), (0, 0, 1 \cdots)$

$(0 0 \cdots 1)$

$V = P(n)$

$\{1, x, x^2, \cdots$

$M_{m \times n}$

$M_{ij} = \begin{pmatrix} 0 & 0 & \cdots & & \\ 0 & 0 & \cdots & & \\ 0 & 0 & 1 & 0 & \\ & & 0 & \ddots & \\ & & & & 0 \end{pmatrix} : i,j$

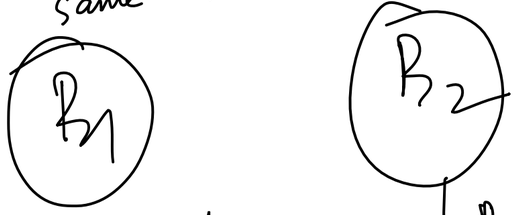
Definition: A V.S is called finite dimensional if $\exists$ a basis consisting of finitely many elements.

Definition: Let V be a V.S. The dimension of V is defined as the no of elements of a basis

dimension

Corollary: If V is finite dimensional then any two bases have same number of elements.

Proof:



B_1 spans V $\Rightarrow |B_1| \geq |B_2|$

$|B_2| \geq |B_1|$

Corollary: If V is finite dimensional and $\dim V = n$. Any set containing more than n elements is L.D.

$\dim V = n$