Recall: $(V, F, +, \cdot)$

Subspace: $W \subseteq V$ $\quad +, \cdot$

- V

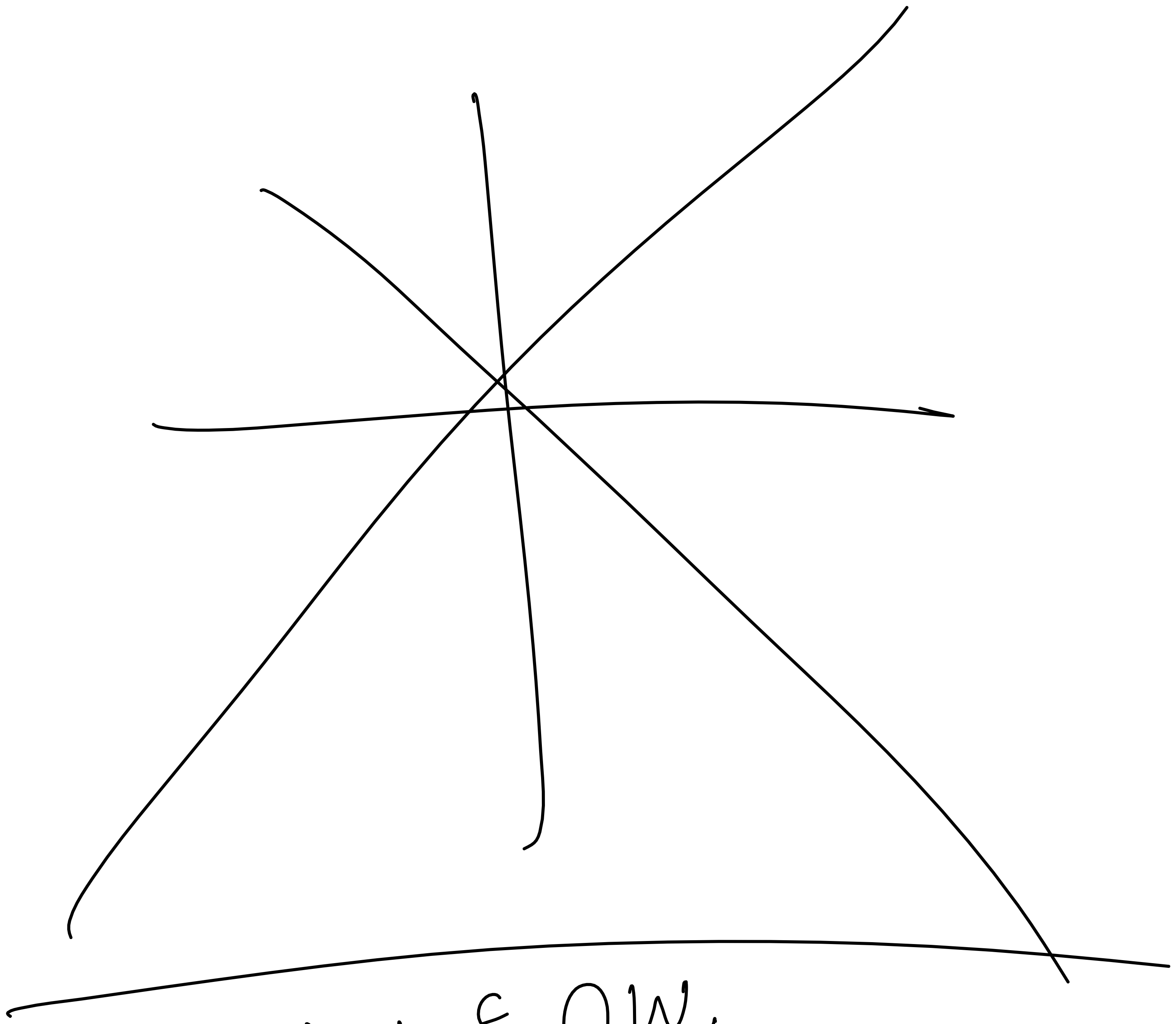
$\{0\}, V$ - trivial subspace.

$W \neq \emptyset$ $\quad$ W is subspace iff

$a \cdot w_1 + w_2 \in W \quad \forall w_1, w_2 \in W$ and $a \in F$

* $W_1 \cup W_2$ is a subspace iff either $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$

* $\bigcap_{\alpha \in \Lambda} W_\alpha$ is a subspace if all W_α's are subspaces.



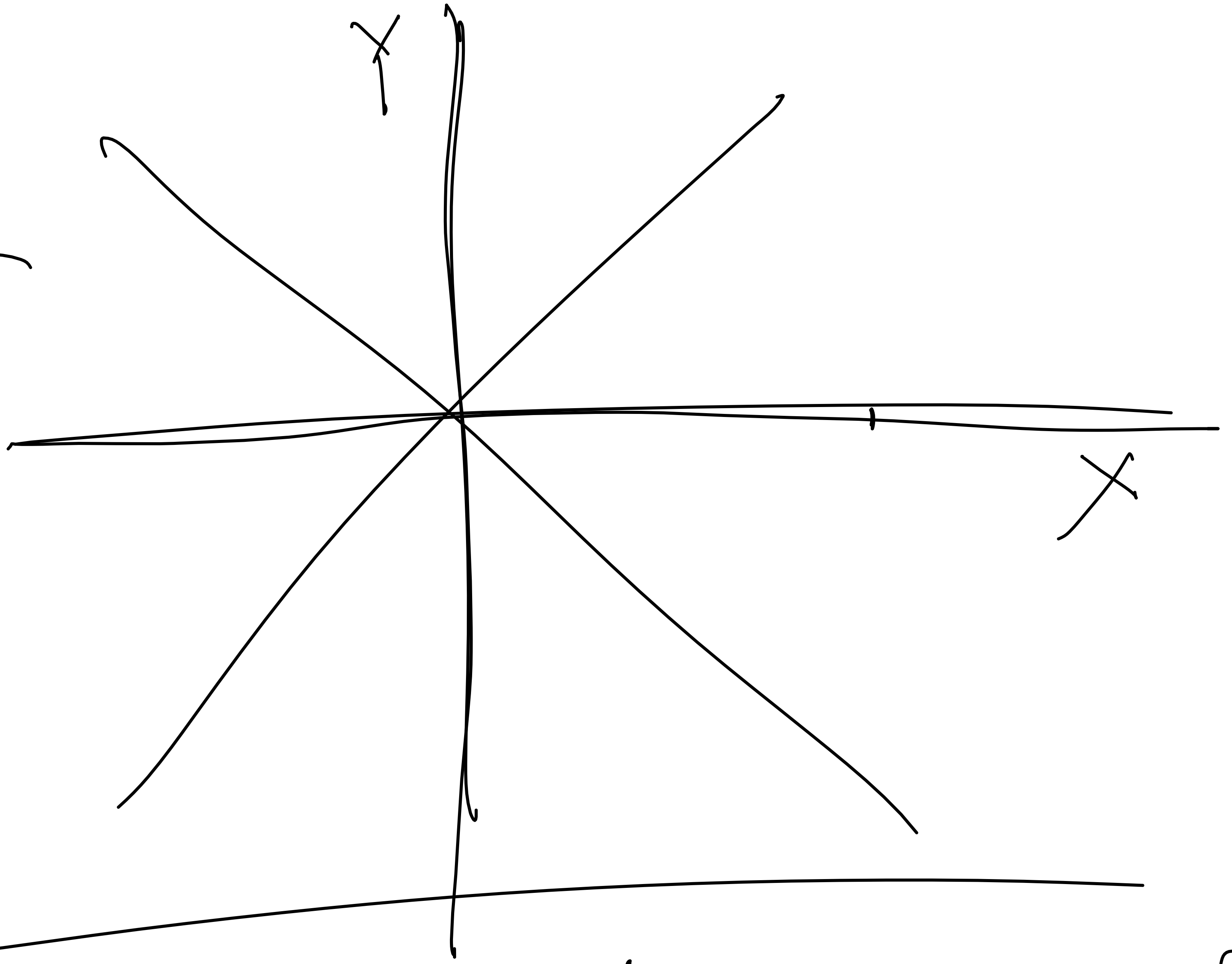
$w_1, w_2 \in \bigcap_{\alpha \in \Lambda} W_\alpha$

$\Rightarrow w_1, w_2 \in W_\alpha \quad \forall \alpha.$

$\Rightarrow a \cdot w_1 + w_2 \in W_\alpha \quad \forall \alpha$

$W_1 + W_2 = \{ w_1 + w_2 : w_1 \in W_1 \text{ and } w_2 \in W_2 \}$

$$X + Y = \{ (x, 0) + (0, y) : x \in X \text{ and } y \in Y \}$$
$$= \{ (x, y) : x \in X \text{ and } y \in Y \}$$
$$= \mathbb{R}^2$$



$w_1 + w_2, \quad w_1' + w_2' \quad a \in F$

$$a(w_1 + w_2) + (w_1' + w_2')$$
$$= \underbrace{(aw_1 + w_1')}_{\in W_1} + \underbrace{(aw_2 + w_2')}_{\in W_2}$$

$W_1 + W_2 + \cdots + W_n = \{ w_1 + w_2 + \cdots + w_n : w_i \in W_i \}$

$S \subseteq V$ a subset

$\mathrm{Span}(S) = \{ c_1 v_1 + c_2 v_2 + \dots + c_n v_n : v_i \in S \text{ and } c_i \in F \}$

$\cap|$

V

L.C of $\{v_1, v_2, \dots v_n\}$

$w_2 \in \mathrm{Span}(S)$

$w_1 \in \mathrm{Span}(S)$

$w_1 = c_1 v_1 + c_2 v_2 + \dots + c_n v_n$

$w_2 = d_1 v_1' + d_2 v_2' + \dots + d_n v_n'$

$$\begin{aligned} a w_1 + w_2 &= a(c_1 v_1 + \dots + c_n v_n) \\ &\quad + d_1 v_1' + d_2 v_2' + \dots + d_n v_n' \\ &= a c_1 v_1 + \dots + a c_n v_n \\ &\quad + d_1 v_1' + d_2 v_2' + \dots + d_n v_n' \in \mathrm{Span}(S) \end{aligned}$$

$\mathrm{Span}(S)$ is a subspace of V.

$S \subseteq V$ a subset.

We say S is linearly dependent if $\exists\, v_1, v_2, \dots v_n \in S$ and $c_1, c_2, \dots c_n \in F$ ~~not~~ all c_i's are zero, such that $c_1 v_1 + c_2 v_2 + \dots + c_n v_n = 0$

$\{(1,0), (0,1), (1,1)\}$ $V = \mathbb{R}^2$

$1\,(1,0) + 1\,(0,1) + (-1)\,(1,1) = 0$

$\{1, \sin^2 x, \cos^2 x\} \subseteq$ Set of all function fr $\mathbb{R} \to \mathbb{R}$

$1 \cdot \sin^2 x + 1 \cdot \cos^2 x - 1 \cdot 1 = 0$

Linearly Independent Set:

A subset of V which is not linearly dependent.

$\{(1,0), (0,1)\} \subseteq \mathbb{R}^2$

* A subset containing '0' is LD.

$0 \in S$

$1 \cdot 0 = 0$

* A subset containing a linearly dependent set is again LD.

$S \subset T$

LD

* A subset of a linearly independent set is also LI

* A subset $S \subseteq V$ is LI ~~iff~~ for every finite subset $T \subseteq S$ is LI

* $S \subseteq V$ is LI if for every $v_1, v_2, \dots v_n$ and $c_1, c_2, \dots c_n \in F$ s.t.

$$c_1 v_1 + c_2 v_2 + \dots + c_n v_n = 0 \Rightarrow c_i = 0 \ \forall i$$

Basis: A subset $B \subseteq V$ is called a basis if (1) B spans V. $\operatorname{Span} B = V$

(2) B is LI

Exp: $V = \mathbb{R}^3$

$B = \{(1,0,0), (0,1,0), (0,0,1)\}$

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* $V = M_{m \times n}(\mathbb{R})$

$$\left\{ E_{ij} = \begin{pmatrix} 0 & 0 & - & - \\ 0 & 0 & & - \\ 0 & & 1 & 0 \\ 0 & & - & - \end{pmatrix} : \begin{array}{l} i = 1, 2, \dots m \\ j = 1, 2, \dots n \end{array} \right\}$$

* Set of all polynomials in x over $\mathbb{R}$

$\{1, x, x^2, x^3, ---\}$

$f(x) = a_0 + a_1 x + --- + a_n x^n$

$a_0 \cdot 1 + a_1 \cdot x + --- + a_n x^n = 0$

Theorem: Let V be a vector space spanned by a finite subset S. Then every subset of V containing more than $|S|$ no of elements has to be LD.

$|S| = m$

Proof: $S = \{v_1, v_2, -- v_m\}$

$\boxed{V = \operatorname{Span}(S)}$

Let T be a subset $T = \{w_1, w_2, -- w_n\} \subseteq V$

$n > m$

Claim: T is LD

$w_j = \sum_{i=1}^{m} a_{ij} v_i$ $\quad a_{ij} \in F, \; v_i \in S$ $\quad \forall j$

$A = (a_{ij})$ $\quad X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$

$AX = 0$

m equations and n variables. $n > m$

$\Rightarrow \exists\, c_1, c_2, -- c_n$ not all $c_i = 0$ s.t

$\left.\sum_{i=1}^{m} a_{ij} c_j = 0 \right\}$

$c_1 w_1 + c_2 w_2 + --- + c_n w_n = 0$?

$$\sum_{j=1}^{n} c_j w_j = \sum_{j=1}^{n} c_j \sum_{i=1}^{m} a_{ij} v_i$$

$$= \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij} c_j \right) v_i$$

$$= 0$$

Corollary: If V is finite dimensional. Then any two bases have same number of elements.

Proof: B_1 $\quad$ B_2

B_1 spans V

$$|B_1| \geq |B_2|$$

$$|B_2| \geq |B_1|$$

$$\Rightarrow |B_1| = |B_2|$$

A vector space V is called finite dimensional if it has a basis consisting of finitely many elements.