MTH 102, ODE: Assignment-2

1. Find general solution of the following differential equations:

$$(\mathbf{T})(i)\ (x+2y+1)-(2x+y-1)y'=0 \quad (ii)\ y'=(8x-2y+1)^2/(4x-y-1)^2$$

Solution:

(i) Use transformation $x = X + h$, $y = Y + k$ such that $h + 2k + 1 = 2h + k - 1 = 0$. Thus $h = 1$, $k = -1$ and the ODE becomes $dY/dX = (X+2Y)/(2X+Y)$. Further substitution of $vX = Y$ leads to separable form $X dv/dX = (1-v^2)/(2+v)$. Hence,

$$\frac{3}{2}\frac{dv}{(1-v)} + \frac{1}{2}\frac{dv}{(1+v)} = \frac{dX}{X} \implies \frac{|1+v|}{|1-v|^3} = CX^2 \implies |X+Y| = C|X-Y|^3$$

Substituting X and Y we find $|x+y| = C|x-y-2|^3$.

(ii) Substituting $4x - y = v$ leads to the separable form $dv/dx = 3(1-4v)/(v-1)^2$. This can be written as

$$\frac{(v-1/4-3/4)^2}{v-1/4}dv = -12\,dx \implies v - \frac{1}{4} - \frac{3}{2} + \frac{9}{4}\frac{1}{4v-1} = -12\,dx$$

Or

$$8v^2-28v+9\ln|4v-1| = -192x+C \implies 8(4x-y)^2-28(4x-y)+9\ln|16x-4y-1|+192x = C$$

2. Find the solution of the initial value problem

$$xy' = y + \frac{2x^4}{y}\cos(x^2), \quad y(\sqrt{\pi/2}) = \sqrt{\pi}.$$

Solution: The given ODE is equivalent to

$$y' = y/x + 2x^2/(y/x)\cos(x^2).$$

Substituting $y = xv$, we find

$$vv' = 2x\cos(x^2) \implies v^2 = 2\sin(x^2) + A \implies y^2 = 2x^2\sin(x^2) + Ax^2$$

Using initial condition we find $A = 0$. Hence, solution is $y^2 = 2x^2\sin(x^2)$

3. Reduce the differential equation

$$y' = f\left(\frac{ax+by+m}{cx+dy+n}\right),\ ad-bc \neq 0$$

to a separable form. Also discuss the case of $ad = bc$.

Solution: Use transform $x = X + h$, $y = Y + k$ where h, k satisfies $ah + bk + m = ch + dk + n = 0$ (such choice of h, k is possible since $ad - bc \neq 0$). We get

$$Y' = f\left(\frac{aX+bY}{cX+dY}\right).$$

Let $vX = Y$. Then $Y' = dY/dX = v + Xv'$ ODE reduces to

$$\frac{dv}{f\left(\frac{a+bv}{c+dv}\right) - v} = \frac{dX}{X},$$

which is in seperable form.

If $ad = bc$, then $ax + by = \lambda(cx + dy)$. Use substitution $w = cx + dy$ to get $w' = c + dy'$. Substituting in the given equation,

$$\frac{w' - c}{d} = f\left(\frac{\lambda w + m}{w + n}\right) = g(w) \implies w' = c + dg(w),$$

which is an ODE in separable form.

4. Show that the following equations are exact and hence find their general solution:

(**T**)(i) $(\cos x \cos y - \cot x) = (\sin x \sin y)y'$ (ii) $y' = 2x(ye^{-x^2} - y - 3x)/(x^2 + 3y^2 + e^{-x^2})$

Solution:

(i) Comparing the given ODE with with $Mdx + Ndy = 0$ with $y' = dy/dx$, we have $M = (\cos x \cos y - \cot x)$, $N = -\sin x \sin y$. Clearly $\partial M/\partial y = \partial N/\partial x$. Hence the ode is exact.

Since the ODE is exact, we must have

$$Mdx + Ndy = du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

for some $u(x, y)$. Now $\frac{\partial u}{\partial x} = M$ implies $u = \int M\,dx = \sin x \cos y - \ln|\sin x| + f(y)$. Since $\partial u/\partial y = N$, we get $f' = 0$ and hence $f = C$. Thus the solution is $\sin x \cos y - \ln|\sin x| = C$.

(ii) Comparing the given ODE with with $Mdx + Ndy = 0$ with $y' = dy/dx$, we have $M = 2x(y + 3x - ye^{-x^2})$ and $N = x^2 + 3y^2 + e^{-x^2}$. Clearly $\partial M/\partial y = \partial N/\partial x$ and so the ode is exact.

Since the ODE is exact, we must have

$$Mdx + Ndy = du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

for some $u(x, y)$. Now $\frac{\partial u}{\partial x} = M$ implies $u = \int M\,dx = x^2y + 2x^3 + ye^{-x^2} + f(y)$. Since $\partial u/\partial y = N$, we get $f' = 3y^2$ and hence $f(y) = y^3 + C$. Thus the general solution is $x^2y + 2x^3 + ye^{-x^2} + y^3 = C$. (Implicit solution.)

5. Show that if the differential equation is of the form

$$x^a y^b(my\,dx + nx\,dy) + x^c y^d(py\,dx + qx\,dy) = 0,$$

where $a, b, c, d, m, n, p, q \in \mathbb{R}$ $(mq \neq np)$ are constants, then there exits suitable $h, k \in \mathbb{R}$ such that $x^h y^k$ is an integrating factor. Hence find a general solution of $(x^{1/2}y - xy^2) + (x^{3/2} + x^2y)y' = 0$.

Solution:

Multiplying by $x^h y^k$ we find $M = mx^{a+h}y^{b+k+1} + px^{h+c}y^{d+k+1}$ and $N = nx^{a+h+1}y^{b+k} + qx^{c+h+1}y^{d+k}$. Using the condition for exactness, we get

$$\begin{aligned} nh - mk &= m(b+1) - n(a+1) \\ qh - pk &= p(d+1) - q(c+1) \end{aligned}$$

Since $np \neq mq$, we can solve for h, k.

The given ODE can be written as $x^{1/2}(ydx + xdy) + xy(-ydx + xdy) = 0$. Hence, $a = 1/2, b = 0$ and $c = d = 1$ and $m = n = q = 1, p = -1$. The equations for h and k becomes

$$h - k = -1/2 \qquad h + k = -4 \implies h = -9/4,\ k = -7/4.$$

The given ODE becomes $(x^{-7/4}y^{-3/4} - x^{-5/4}y^{1/4})dx + (x^{-3/4}y^{-7/4} + x^{-1/4}y^{-3/4})dy = 0$. Here $M = x^{-7/4}y^{-3/4} - x^{-5/4}y^{1/4}$ and hence $u = \int M\,dx = -4x^{-3/4}y^{-3/4}/3 + 4x^{-1/4}y^{1/4} + f(y)$. Since $\partial u/\partial y = N$, we find $f' = 0$ and hence $f = C$. Thus the solution is $x^{-3/4}y^{-3/4} - 3x^{-1/4}y^{1/4} = C$.

6. (**T**) Given that the equation $(3y^2 - x) + 2y(y^2 - 3x)y' = 0$ admits an integrating factor which is a function of $(x + y^2)$. Hence solve the differential equation.

Solution:

Assume that $F(x + y^2)$ is an integrating factor. Multiplying by $F(x + y^2)$ we find $M = (3y^2 - x)F(x + y^2)$ and $N = 2y(y^2 - 3x)F(x + y^2)$. Using the condition of exactness $\partial M/\partial y = \partial N/\partial x$, we get

$$6yF(x + y^2) + (3y^2 - x)F'(x + y^2).2y = -6yF(x + y^2) + 2y(y^2 - 3x)F'(x + y^2).$$

Simplifying,

$$z\frac{dF}{dz} = -3F \implies F = \frac{1}{z^3}, \quad z = x + y^2.$$

Hence

$$u = \int M\,dx = \int \frac{3y^2 - x}{(x + y^2)^3}dx = \int \frac{4y^2 - t}{t^3}dt = -\frac{2y^2}{t^2} + \frac{1}{t} + f(y),$$

by substituting $t = x + y^2$. Hence

$$u = \frac{x - y^2}{(x + y^2)^2} + f(y).$$

Since $\partial u/\partial y = N$, we find $f' = 0$ and hence f =constant. Hence the solution is

$$\frac{x - y^2}{(x + y^2)^2} = C$$

7. Consider first order ODE $M(x,y)dx+N(x,y)dy=0$ with M,N are C^1 functions on $\mathbb{R}^2$. Show that

(**T**)(i). If $(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x})/N=f(x)$ depends on x only then, $\exp(\int f(x)dx)$ is an integrating factor for the given ODE.

(ii). If $(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x})/M=g(y)$ depends on y only then, $\exp(-\int g(y)dy)$ is an integrating factor for the given ODE.

Solution: (i) Let $\mu=e^{\int f(x)dx}$. Then $\frac{\partial}{\partial y}(\mu M)=\mu\frac{\partial M}{\partial y}$ and

$$\frac{\partial}{\partial x}(\mu N)=\mu\frac{\partial N}{\partial x}+N\frac{\partial \mu}{\partial x}=\mu\frac{\partial N}{\partial x}+N\mu f=\mu(\frac{\partial N}{\partial x}+Nf)=\mu\frac{\partial M}{\partial y}=\frac{\partial(\mu M)}{\partial y}$$

(ii) Similarly.

□

8. Find integrating factor and solve the following.

(**T**) (i) $2\sin(y^2)+xy\cos(y^2)y'=0$.

(ii) $xy-(x^2+y^4)y'=0$.

Solution:

(i) Comparing with $Mdx+Ndy=0$, we have $M=2\sin(y^2),\ N=xy\cos(y^2)$. Then $\frac{\partial M}{\partial y}=4y\cos(y^2)$ and $\frac{\partial N}{\partial x}=y\cos(y^2)$. So $(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x})/N=3/x$ which is a function of x only. So an integrating factor is $\int\exp(3/x)dx=x^3$.

Multiplying by x^3, we have exact equation $Mdx+Ndy=0$ with $M=2x^3\sin(y^2),\ N=x^4y\cos(y^2)$. If we assume that the $Mdx+Ndy=du(x,y)$, then,

$$u=\int M\,dx=\frac{x^4\sin(y^2)}{2}+f(y)$$

Since $\partial u/\partial y=N$, we find $f'=0$ and hence $f=$constant. Hence the solution is

$$\frac{x^4\sin(y^2)}{2}=A \quad \text{or} \quad x^4\sin(y^2)=C.$$

(ii) Comparing with $Mdx+Ndy=0$, we have $M=xy,\ N=-(x^2+y^4)$. Then $\frac{\partial M}{\partial y}=x,\ \frac{\partial N}{\partial x}=-2x$. So $(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x})/M=3x/xy=3/y$ which is a function of y only. So an integrating factor is $\int\exp(-3/y)dy=1/y^3$.

Multiplying by $1/y^3$, we find $M=x/y^2$ and

$$u=\int M\,dx=\frac{x^2}{2y^2}+f(y)$$

Since $\partial u/\partial y=N$, we find $f'=-y$ and hence $f=-y^2/2+A$ Hence the solution is

$$\frac{x^2}{2y^2}-\frac{y^2}{2}=B \quad \text{implies} \qquad x^2-y^4=Cy^2.$$

9. (**T**) Show that the set of solutions of the homogeneous linear equation, $y' + P(x)y = 0$ on an interval $I = [a, b]$ form a vector subspace W of the real vector space of continuous functions on I. What is the dimension of W?

Solution:

The zero function $\mathbf{0}(x) \equiv 0$ satisfies $y' + P(x)y = 0$. Hence, W is nonempty. Let $u(x), v(x) \in W$ are two arbitrary solutions of $y' + P(x)y = 0$. Consider $w(x) = \alpha u(x) + v(x)$, where α is a real number. Now, $w' + P(x)w = 0 \Rightarrow w(x) \in W$ and hence W is a subspace. We also note that any solution is of the form $y = Ce^{-\int P(x)dx}$. Thus W is spanned by $e^{-\int P(x)dx}$ and so dim(W)=1.

(Remark: Solutions of non-homogeneous or non-linear equations may not form a vector space.)

10. Solve the linear first order linear IVP $y' + y \tan x = \sin 2x, \quad y(0) = 1$.

[Recall: For $y' + p(x)y = r(x)$, the left hand side becomes exact if we multiply by $\mu(x) = \exp(\int p(x)dx)$. We get $\frac{d}{dx}(\mu(x)y) = \mu(x)r(x)$. Integrating, general solution is

$$y(x) = \mu(x)^{-1} \int \mu(x)r(x)dx + c.$$

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Solution: Comparing with $y' + p(x)y = r(x)$, we get $p(x) = \tan x$, $r(x) = \sin 2x$. Then,

$$\int p(x)dx = \ln(\sec x), \ \ \mu(x) = \sec x, \quad \int \mu(x)r(x)dx = \int \sec x \, 2 \sin x \cos x dx = -2\cos x + c.$$

So general solution is $y(x) = c \cos x - 2\cos^2 x$. Initial condition gives $c = 3$. □

11. (**T**) Let ϕ_i be a solution of $y' + ay = b_i(x)$ for $i = 1, 2$. Show that $\phi_1 + \phi_2$ satisfies $y' + ay = b_1(x) + b_2(x)$.

Solve $y' + y = x + 1$, $y' + y = \cos 2x$. Hence solve $y' + y = 1 + x/2 - \cos^2 x$

Solution:

Verification is easy.

For $y' + y = x + 1$, $y_1 = C'e^{-x} + x$ and for $y' + y = \cos 2x$ is $y_2 = C''e^{-x} + (\cos 2x + 2\sin 2x)/5$.

(Integrating by parts $\int xe^x dx = xe^x - e^x$. Also

$$\int e^{ax} \cos bx dx = \frac{e^{ax}}{a^2 + b^2}(a \cos bx + b \sin bx).$$

)

Now $y' + y = 1 + x/2 - \cos^2 x = \frac{1+x}{2} - \frac{\cos 2x}{2}$. Since, the equation is linear, the solution of $y' + y = 1 + x/2 - \cos^2 x$ is $y = (C' + C'')e^{-x} + x/2 - (\cos 2x + 2\sin 2x)/10 = Ce^{-x} + x/2 - (\cos 2x + 2\sin 2x)/10$.

12. Using appropriate substitution, reduce the following differential equations into linear form and solve:
(**T**) (i) $y^2y' + y^3/x = x^{-2}\sin x$ (**T**) (ii) $y'\sin y + x\cos y = x$ (iii) $y' = y(xy^3 - 1)$

Solution:

[Recall that Bernoulli equation is of the form $y' + P(x)y = Q(x)y^n$. To solve it, we have to change variable to $z = y^{1-n}$. Then it reduces to linear ODE $z' + (1-n)P(x)z = (1-n)Q(x)$.

More generally, for $\frac{d}{dy}(f(y))\frac{dy}{dx} + P(x)y = Q(x)$, substite $v = f(y)$ to reduce it to linear $\frac{dv}{dx} + P(x)v = Q(x)$. Bernoulli equation is a special case with $f(y) = y^{1-n}$.]

(i) Substitute $u = y^3$ and the ODE transform to linear form $u' + 3u/x = 3x^{-2}\sin x$. Using integrating factor x^3, we write

$$\frac{d}{dx}(ux^3) = 3x\sin x \implies ux^3 = 3(-x\cos x + \sin x) + C$$

Thus, the solution is $x^3y^3 + 3(x\cos x - \sin x) = C$.

(ii) Substitute $-\cos y = u$ which leads to the linear form $u' - xu = x$. Using integrating factor $e^{-x^2/2}$, we write

$$\frac{d}{dx}(ue^{-x^2/2}) = xe^{-x^2/2} \implies ue^{-x^2/2} = -e^{-x^2/2} + C \implies u = -1 + Ce^{x^2/2}$$

Hence, the solution is $\cos y = 1 - Ce^{x^2/2}$.

(iii) $u = 1/y^3$ leads to $u' - 3u = -3x$. Using integrating factor e^{-3x}, we write

$$\frac{d}{dx}(ue^{-3x}) = -3xe^{-3x} \implies ue^{-3x} = \frac{1+3x}{3}e^{-3x} + C \implies u = \frac{1+3x}{3} + Ce^{3x}.$$

Hence, the solution is $1/y^3 = Ce^{3x} + x + 1/3$.

13. (**T**) A radioactive substance A decays into B, which then further decays to C.

a) If the decay constants of A and B are respectively λ_1 and λ_2, and the initial amounts are respectively A_0 and B_0, set up an ODE for determining $B(t)$, the amount of B present at time t, and solve it. (Assume $\lambda_1 \neq \lambda_2$.)

b) Assume $\lambda_1 = 1,\ \lambda_2 = 2$. When B(t) reaches a maximum?

Solution: $dA/dt = -\lambda_1 A$ and

$dB/dt =$ rate at which B produced by decay of A$-$ rate at which B is lost by decay of B to C

$$= \lambda_1 A - \lambda_2 B.$$

From the first equation $A = A_0e^{-\lambda_1 t}$. So we get $dB/dt + \lambda_2 B = \lambda_1 A_0 e^{-\lambda_1 t}$. This is a linear first order ODE with initial condition $B(0) = B_0$. Solving, we get

$$B(t) = 1/(\lambda_2 - \lambda_1)[\lambda_1 A_0 e^{-\lambda_1 t} + (B_0 - \lambda_1 A_0)e^{-\lambda_2 t}].$$

For $\lambda_1 = 1,\ \lambda_2 = 2$, we have $B(t) = A_0 e^{-t} + (B_0 - A_0)e^{-2t}$. For maximum, we solve $0 = B'(t) = -A_0 e^{-t} - 2(B_0 - A_0)e^{-2t}$. This gives $e^t = 2(A_0 - B_0)/A_0$. We must get time $t \geq 0$, i.e, $e^t \geq 1$, for $A_0 \geq 2B_0$ and in this case $t = \ln(\frac{2(A_0 - B_0)}{A_0})$. Otherwise B is maximum initially at $t = 0$.

14. According to Newton's Law of Cooling, the rate at which the temperature T of a body changes is proportional to the difference between T and the external temperature. At time $t = 0$, a pot of boiling water is removed from the stove. After five minutes, the water temperature is $80C$. If the room temperature is $20C$, when will the water have cooled to $60C$?

    Solution: By Newton's cooling law $\frac{dT}{dt} = k(T - 20)$ where k is the constant of proportionality. Solving it we get $T(t) = ce^{kt} + 20$. The initial condition $T(0) = 100$ gives $c = 80$. Thus $T(t) = 80e^{kt} + 20$.

    Now after 5 minutes, the temperature of water is $80C$. So $T(5) = 80$ gives $k = 1/5 \ln(3/4)$

    Now the time t to cooled down the water to 60 is given by $T(t) = 60$, implies $kt = \ln(1/2)$, $t = 5\ln(2)/\ln(4/3)$ (≈ 12) minutes.