ODE: Assignment-3

1. (**T**) A surface $z = y^2 - x^2$ in the shape of a saddle is lying outdoors in a rainstorm. Find the paths along which raindrops will run down the surface.

   Solution:

   A curve on the surface is determined by a curve $y = y(x)$ on the xy-plane. The raindrop will take the path where z decreases at maximum rate. We know that for a real valued differentiable function $f(x, y)$, f will have maximum increase rate in direction ∇f and maximum decrease rate in direction $-\nabla f$ (This comes from the fact that directional derivative of f in direction v, $|v| = 1$, is given by $(\nabla f).v$).

   So the required curve in xy-plane will have slope $-\nabla f = (2x, -2y)$. So its differential equation is $dy/dx = -2y/2x$. Solving we get $xy = c$. Thus the curve on the surface is the intersection of the saddle $z = x^2 - y^2$ with hyperbolic cylinder $xy = c$.

2. (**T**) Does $f(x, y) = xy^2$ satisfies Lipschitz condition (LC) on any rectangle $[a, b] \times [c, d]$? What about on an infinite strip $[a, b] \times \mathbb{R}$?

   [A function $f(x, y)$ is said to satisfy Lipschitz condition on a domain $D \subseteq \mathbb{R}^2$, if there exists $L > 0$ such that $|f(x, y_1) - f(x, y_2)| \leq L|y_1 - y_2|$ for all $(x, y_1), (x, y_2) \in D$.]

   Solution:

   On closed rectangle $[a, b] \times [c, d]$, the partial derivative f_y is continuous and hence bounded and hence f satisfies LC. Alternatively,

   $$\frac{|f(x, y_1) - f(x, y_2)|}{|y_1 - y_2|} = |x||y_1 + y_2| \leq \max\{|a|, |b|\} \times 2\max\{|c|, |d|\}$$

   On the vertical strip, $|x|$ is bounded but $|y_1 + y_2|$ can be made arbitrarily large for large choices of y_1 and y_2. So f does not satisfy LC there.

3. (**T**) Let (x_0, y_0) be an arbitrary point in the plane and consider the initial value problem (IVP)

   $$y' = y^2, \quad y(x_0) = y_0.$$

   Explain why Picard theorem guarantees that this problem has a unique solution on some interval $|x - x_0| \leq h$. Since $f(x, y) = y^2$ and $\partial f/\partial y$ are continuous on the entire plane, it is tempting to conclude that this solution is valid for all x. But considering the solutions through the points $(0, 0)$ and $(0, 1)$, show that this consideration is sometime true and sometime false, and that therefore the inference is not legitimate.

   [Remark: Compare the above with the fact that if f is continuous and Lipschitz on $[a, b] \times \mathbb{R}$, then the IVP $y' = f(x, y), \quad y(x_0) = y_0, \quad x_0 \in [a, b]$ has solution over $[a, b]$. Simmons book Theorem B in chapter 'The Existence and Uniqueness of Solutions'.]

   Solution:

   Since $f(x, y) = y^2$ and $\partial f/\partial y$ are continuous on the entire plane, they are continuous on any closed rectangle containing (x_0, y_0). Hence Picard theorem guarantees unique solution on some interval $|x - x_0| \leq h$.

Solving the equation we get $y = -\frac{1}{x+c}$. Initial condition $y(0) = 1$ gives us $y = \frac{1}{1-x}$ which is valid for $(-\infty, 1)$. For initial condition $y(0) = 0$, we cant find value of c. But we observe that $y(x) = 0$ satisfies the equation with $y(0) = 0$. So it is valid for all $\mathbb{R}$.

4. (**T**) Consider the IVP $y' = 2\sin(3xy),\ \ y(0) = y_0$. Show that it has unique solution in $(-\infty, \infty)$.

   Solution:

   It suffices to show that it has unique solution on every interval $[-L, L]$. This is because if we have a unique solution on $[-L_1,\ L_1]$ and a unique solution on $[-L_2, L_2]$ with $L_2 > L_1$, then by uniqueness part the two solution has to agree on the smaller interval $[-L_1, L_1]$.

   Now fix L. Define $R = [-L, L] \times [y_0 - b,\ y_0 + b$ for some large $b > 0$. Note that the function $f(x, y) = 2\sin(3xy)$ satisfies $|f \leq 2$ and $|f_y| \leq 6L$ on the rectangle R. So by Picard theorem, unique solution exist on the interval $[-h, h]$ where $h = \min\{L, b/2\}$. We can choose $b > 2L$ so that $h = L$. Thus we get a unique solution on $[-L,\ L]$.

5. (**T**) Given

$$y' = \frac{e^{y^2} - 1}{1 - x^2y^2}, \quad y(-2) = 1.$$

   Find an interval on which solution exist.

   Solution:

   Here our function f is defined by $f = \frac{e^{y^2}-1}{1-x^2y^2}$ and $x_0 = -2, y_0 = 1$. Thus we need to pick a rectangle R which is centered at $(-2, 1)$. In this rectangle we need to have f, f_y continuous and so we certainly have to choose R so small that it contains no points at which the denominator $1 - x^2y^2$ vanishes. The exact choice of the rectangle is up to you.

   We choose $a = 1/2,\ \ b = 1/4$ so that

$$R = [x_0 - a, x_0 + a] \times [y_0 - b, y_0 + b] = [-5/2, -3/2] \times [3/4, 5/4],$$

   so that R is disjoint from the hyperbolas $xy = \pm 1$.

   On R, we have $x^2 \geq 9/4,\ y^2 \geq 9/16$ and therefore $|1 - x^2y^2| \geq (81/64) - 1 = 17/64 > 1/4$. Also $|e^{y^2} - 1| \leq e^{9/16} < e < 3$. Thus $|f(x, y)| \leq 3 \times 4 = 12 = M$ on R. This is a legitimate (but non-optimal) bound.

   Since R is disjoint from $xy = \pm 1$, clearly f_y will be continuous on R. So by Picard theorem unique solution will exist on $[-2 - h, -2 + h]$ for $h = \min\{a, b/M\} = \min\{1/2, 1/48\} = 1/48$.

6. (**T**) Consider the ode $y' = \frac{2xy}{x^2-y^2}$. Solve it. Sketch the solutions. Verify Picard theorem for initial values in $\mathbb{R}^2 - \{(x, y) :\ \ x^2 = y^2\}$. What is your solution passing through $(1, 0)$?

   Solution:

   Comparing with $Mdx + Ndy = 0$, we have $M = 2xy,\ N = x^2 - y^2$. So $\frac{1}{M}(M_y - N_x) = 2/y$. So integrating factor is $e^{-\int 1/y dy} = 1/y^2$. We get solution $x^2 + y^2 = cy$.

   (Also we can solve it as homogeneous equation.)

   Solution curves are circles with centre on the y-axis and touching the x axis at the origin.

The function $f(x,y) = \frac{2xy}{x^2-y^2}$ and f_y is continuous on $D = \mathbb{R}^2 - \{(x,y): \quad x^2 = y^2\}$. So Picard theorem tells us: given any $(x_0, y_0) \in D$ there passes through a unique solution curve.

Given initial condition $(x_0, y_0), \quad x_0 \neq 0$ there is circle as above passing though that point.

For point $(x_0, 0), x_0 \neq 0$ we can not find a circle like that. But we observe that $y(x) = 0$ is also a solution of the equation and so this must be the unique solution passing through $(x_0, 0), x_0 \neq 0$.

7. A function $f(x,y)$ is said to satisfy Lipschitz condition on a domain $D \subseteq \mathbb{R}^2$, if there exists $L > 0$ such that $|f(x,y_1) - f(x,y_2)| \leq L|y_1 - y_2|$ for all $(x,y_1), (x,y_2) \in D$.

   (i) Show that if $f(x,y)$ satisfies Lipschitz condition (LC)with respect to y on a rectangle D, then for each fixed x, the resulting function of y is continuous function of y.

   (ii) Let $f(x,y) = y + [x]$. Then how that f satisfies LC on $\mathbb{R}^2$ but not continuous on $\mathbb{R}^2$.

   (iii) Let $f(x,y) = xy$. Then show that f is continuous on $\mathbb{R}^2$ but not LC on $\mathbb{R}^2$.

   Solution:

   (i) Follows from definition.

   (ii) Let $f(x,y) = y + [x]$. Clearly $|f(x,y_1) - f(x,y_2)| = |y_1 - y_2|$, so LC is satisfied on the entire plane. But f is not continuous for any integral x.

   (iii) Let $f(x,y) = xy$. It is continuous on entire plane, being polynomial. But $\frac{|f(x,y_1) - f(x,y_2)|}{|y_1 - y_2|} = |x|$ can be made arbitrarily large on $\mathbb{R}^2$. So LC not satisfied on $\mathbb{R}^2$.

8. (**T**) What does Picard theorem says about existence and uniqueness of solution of the IVP $y' = (3/2)y^{1/3}, \quad y(0) = 0$? Show that it has uncountably many solutions.

   Solution:

   Here $f(x,y) = (2/3)y^{1/3}$ is continuous on the plane. So Picard theorem (Peano existence) tells us that it has at least one solution. But f_y is not continuous in any rectangle containing $(0,0)$ and also f does not satisfy Lipschitz condition on any rectangle containing $(0,0)$. So we can not say anything about uniqueness of the solution from the theorem.

   Solving the equation we get $y^2 = x^3$. Also $y(x) = 0$ satisfies the IVP. Moreover, $y(x) = (x-a)^{3/2}$ for $x \geq a$ and $y(x) = 0$ for $x \leq a$ also satisfies the IVP for any $a \geq 0$ (just need check derivative at $x = a$ exists and equal to 0). Thus we get uncountably many solutions.

9. Consider the IVP $y' = \sqrt{y} + 1, \quad y(0) = 0, \quad x \in [0,1]$. Show that $f(x,y) = \sqrt{y} + 1$ does not satisfy Lipschitz condition in any rectangle containing origin, but still the solution is unique.

   (Remark: It is fact that if an IVP, with f is continuous (not necessarily Lipschitz), has more than one solution, then it has uncountably many solutions. This is known as Kneser's Theorem. The previous exercise illustrates this phenomenan.)

   Solution:

   Consider any rectangle $R = [0,a] \times [0,d]$ containing origin We have

$$\frac{|f(x,y_1) - f(x,y_2)|}{|y_1 - y_2|} = \frac{|\sqrt{y_1} - \sqrt{y_2}|}{|y_1 - y_2|} = 1/\sqrt{\delta}, \quad \text{for } y_1 = \delta > 0, \quad y_2 = 0.$$

For δ arbitrary small, we can make $\frac{|f(x,y_1)-f(x,y_2)|}{|y_1-y_2|}$ arbitrarily large on R. Hence f does not satisfy Lipschitz condition in any rectangle containing origin.

Let $g_1(x),\ g_2(x)$ be two solutions of the IVP. Consider $z(x) = (\sqrt{g_1} - \sqrt{g_2})^2$. Then $z'(x) = -\frac{z(x)}{\sqrt{g_1}\sqrt{g_2}} \leq 0$. Thus $z(x)$ is a decreasing function. Further $z(x)$ is non negative and $z(0) = 0$. Then $z(x) = 0$ for all $x \geq 0$. Hence $g_1 = g_2$.

10. (**T**) (i) Let $f(x,y)$ be continuous on the closed rectangle $R : |x - x_0| \leq a,\ |y - y_0| \leq b$. Show that y is a solution of the initial value problem $y' = f(x,y),\ y(x_0) = y_0$ iff

$$y(x) = y_0 + \int_{x_0}^{x} f\big(t, y(t)\big)\, dt.$$

(ii) Let $|f(x,y)| \leq M$ on the closed rectangle R and $y_n(x) = y_0 + \displaystyle\int_{x_0}^{x} f\big(t, y_{n-1}(t)\big)\, dt$, with $y_0(x) = y_0$. Use induction to show that $y_{n+1}(x)$ is well defined for $I :\ |x - x_0| \leq h$, where $h = \min\{a, b/M\}$; that is $|y_n(x) - y_0| \leq b$ for $x \in I$.

(Remark: The sequence of functions $y_n(x)$ are called Picard's Iterates. Precisely because of this step, the solution exist in possibly smaller interval in Picard theorem.)

Solution:

(i) Let $y(x)$ is the solution to $y' = f(x,y),\ y(x_0) = y_0$. Then $y' = f\big(x, y(x)\big),\ y(x_0) = y_0$. Integrating from x_0 to x we get $y(x) - y_0 = \int_{x_0}^{x} f\big(t, y(t)\big)\, dt$.

Conversely, let $y(x) = y_0 + \displaystyle\int_{x_0}^{x} f\big(t, y(t)\big)\, dt$. Then $y(x_0) = y_0$ and from fundamental theorem of integral calculus, $y' = f\big(x, y(x)\big) = f(x,y)$.

(ii) For $n = 0$, $y_0(x) \equiv y_0$ and the relation is obvious. For $n = 1$, $|y_1(x) - y_0| = |\int_{x_0}^{x} f\big(t, y_0(t)\big)\, dt| \leq \int_{x_0}^{x} |f\big(t, y_0(t)\big)|\, dt \leq Mh \leq b$. Let it be true for $n = m$ and so $|y_m(x) - y_0| \leq b$. So for $a \leq x \leq b$, $(x, y_m(x))$ lies in the rectangle R and hence $|f\big(x, y_m(x)\big)| \leq M$. Therefore, $|y_{m+1} - y_0| \leq \int_{x_0}^{x} |f\big(t, y_m(t)\big)|\, dt \leq Mh \leq b$. Hence proved.

11. Use Picard's method of successive approximation to solve the following initial value problems and compare these results with the exact solutions:
(i) (**T**) $y' = 2\sqrt{x},\ y(0) = 1$ (ii) $y' + xy = x,\ y(0) = 0$ (iii) $y' = 2\sqrt{y}/3,\ y(0) = 0$

Solution:

Picard iteration is $y_{n+1}(x) = y_0 + \int_{x_0}^{x} f\big(t, y_n(t)\big)\, dt$ with $y_0(x) \equiv y_0$.

(i) $y_0 = 1,\ y_n(x) = 1 + 2\int_0^x \sqrt{t}\, dt = 1 + (4/3)x^{3/2},\ n \geq 1$ (since f is independent of y). Here $y_n(x)$ $(n \geq 1)$ coincides with the exact solution.

(ii) For exact solution

$$\frac{dy}{1-y} = x\, dx \implies -\ln(1-y) = \frac{x^2}{2} + C$$

Using $y(0) = 0$ we find $C = 0$. So,

$$1 - y = e^{-x^2/2} \implies y = 1 - e^{-x^2/2}.$$

Now we calculate the Picard iterates. Here $f(x,y) = x(1-y)$ and $y_0 = 0$. Thus $y_1(x) = \int_0^x t(1-0)\, dt = x^2/2$. Using y_1, we get $y_2(x) = \int_0^x t(1 - t^2/2)\, dt = x^2/2 - (x^2/2)^2/2$. $y_3(x) =$

$x^2/2 - (x^2/2)^2/2 + (x^2/2)^3/3!$. By induction, we get $y_n(x) = \sum_{m=1}^{n}(-1)^{m-1}(x^2/2)^m/m!$. Thus as $n \to \infty$, $y_n(x) \to -\sum_{m=0}^{\infty}(-x^2/2)^m/m! + 1 = 1 - e^{-x^2/2}$, which is the exact solution.

(iii) Here $y_0 = 0$ and $f(x,y) = 2\sqrt{y}/3$. If we take $y_0(x) \equiv y_0 = 0$, then $y_n(x) = 0,\ n \geq 1$. Here $y_n(x)$, $\forall n$ coincides with the analytical solution $y(x) = 0$. The other solution $y(x) = (x/3)^2$ is not reachable from here.

Note: However, if we start with $y_0(x) = 1$, then

$$y_1(x) = \frac{2}{3}x, \quad y_2(x) = \left(\frac{2}{3}\right)^{5/2} x^{3/2}, \quad y_3(x) = \left(\frac{2}{3}\right)^{9/4} \frac{4}{7} x^{7/4}$$

$$y_4(x) = \left(\frac{2}{3}\right)^{17/8} \left(\frac{4}{7}\right)^{1/2} x^{15/8}$$

Clearly, $y_n(x) = a_n x^{b_n}$ where $a_1 = 2/3, a_2 = (2/3)^{5/2}, a_3 = (2/3)^{9/4}(4/7), \cdots$ and $b_n = (2^n - 1)/2^{n-1}$. The sequence $b_n \to 2$ and a_n is a decreasing sequence bounded below. Hence, $y_n(x) \to Ax^2$. To find we substitute in the integral relation and find

$$Ax^2 = 2/3\sqrt{A}x^2/2 \implies A = 1/3^2 \implies y_n(x) \to (x/3)^2.$$

12. Consider the initial value problem (IVP) $xy' - y = 0,\ \ y(x_0) = y_0$. Solve it for different values of x_0 and y_0. Does the result contradict Picard theorem ?

    Solution:

    We have $xdy - ydx = 0$. Dividing by x^2, we have $d(y/x) = 0$ Integrating we get $y = cx$ for arbitrary c. If $x_0 \neq 0$, then we have unique solution for any y_0. If $x_0 = 0$ and $y_0 = 0$ then initial condition is satisfied for any c and so there are infinite solutions. If $x_0 = 0$ and $y_0 \neq 0$, there is no solution.

    Here $f(x,y) = y/x$ which is not even defined on y-axis. So Picard theorem does not apply there. At other points conditions of Picard theorem is satisfied and also we have unique solution.

13. Solve $y' = (y - x)^{2/3} + 1$. Show that $y = x$ is also a solution. What can be said about the uniqueness of the initial value problem consisting of the above equation with $y(x_0) = y_0$, where (x_0, y_0) lies on the line $y = x$.

    Solution:

    Put $u = y - x \implies u' = u^{2/3}$. Solving we get $y = x + [(x + C)/3]^3$. Also $y = x$ is a solution by direct verification. If $y(x_0) = y_0$ and $x_0 = y_0$, then $C = -x_0$. Thus the solutions $y = x + [(x - x_0)/3]^3$ and $y = x$ both satisfy the initial conditions $y(x_0) = y_0$ with $x_0 = y_0$. Clearly the solution to the IVP is nonunique.

14. Discuss the existence and uniqueness of the solution of the initial value problem

    $$(x^2 - 2x)y' = 2(x - 1)y, \qquad y(x_0) = y_0.$$

    Solution:

Here $f(x,y) = 2(x-1)y/(x^2-2x)$ and $\partial f/\partial y = 2(x-1)/(x^2-2x)$. The existence and uniqueness theorem guarantees the existence of unique solution in the vicinity of (x_0, y_0) where f and $\partial f/\partial y$ are continuous and bounded. Thus, existence of unique solution is guaranteed at all x_0 for which $x_0(x_0-2) \neq 0$. Hence, unique solution exists when $x_0 \neq 0, 2$.

When $x_0 = 0$ or $x_0 = 2$, nothing can be said using the existence and uniqueness theorem. However, since the equation is separable, we can find the general solution to be $y = Cx(x-2)$. Using initial condition we get $y_0 = Cx_0(x_0-2)$. Clearly the IVP has no solution if $x_0(x_0-2) = 0$ and $y_0 \neq 0$. If $x_0(x_0-2) = 0$ and $y_0 = 0$ then $y = \alpha x(x-2)$ is a solution to the IVP for any real α. Hence, in summary
(i) No solution for $x_0 = 0$ or $x_0 = 2$ and $y_0 \neq 0$;
(ii) Infinite number of solutions for $x_0 = 0$ or $x_0 = 2$ and $y_0 = 0$;
(iii) Unique solution for $x_0 \neq 0, 2$.

15. (**T**) Consider the IVP $y' = x - y,\ \ y(0) = 1$. Show that for Euler method, $y_n = 2(1-h)^n - 1 + nh$ where h is the step size. ($x_n = nh$ with $x_0 = 0,\ y_0 = y(0) = 1$). Deduce that if we take $h = 1/n$, then the limit of y_n converges to actual value of $y(1)$.

    Solution:

    The inductive formula of Euler method is

    $$y_n = y_{n-1} + hf(x_{n-1}, y_{n-1}) = y_{n-1} + h(x_{n-1} - y_{n-1}) = (1-h)y_{n-1} + h^2(n-1).$$

    (Using $x_n = nh$.)

    We now use induction to prove the required formula for y_n. Clearly it is true for $n = 0$. Assume the formula is true for n. Then $y_{n+1} = (1-h)y_n + h^2 n = 2(1-h)^{n+1} - 1 + (n+1)h$.

    Taking $h = 1/n$, we have $x_n = 1$. Thus approximate value of $y(1)$ is given by $y_n = 2(1-1/n)^n$ which converges to $2e^{-1}$.

    Exact solution of the equation is $y = 2e^{-x} - 1 + x$. So $y(1) = 2e^{-1}$.

16. Use Euler method and step size .1 on the IVP $y' = x + y^2,\ \ y(0) = 1$ to calculate the approximate value for the solution $y(x)$ when $x = .1, .2, .3$. Is your answer for $y(.3)$ is higher or lower than the actual value ?

    Solution:

    We have $x_0 = 0, y_0 = 1$. Using the Euler iterative formula with $h = .1$ (see previous exercise), we get $y_1 = 1.1,\ y_2 = 1.231,\ y_3 = 1.403$.

    Using graphical method, we see that the solution curve through $(0,1)$ is convex. So Euler method approximate value is lower than actual value.

17. Verify that $y = x^2 \sin x$ and $y = 0$ are both solution of the initial value problem (IVP)

    $$x^2y'' - 4xy' + (x^2+6)y = 0, \quad y(0) = y'(0) = 0.$$

    Does it contradict uniqueness of solution of IVP?

Solution: It is easy to verify that they satisfies the equation. For second order ode $y'' + p(x)y' + q(x)y = r(x)$, with initial condition $y(x_0) = a,\ y'(x_0) = b$, the existence and uniqueness theorem assets unique solution when p, q, r are continuous on an interval containing x_0. Here $p(x) = -4/x$ and $q(x) = (x^2 + 6)/2$ are not continuous at $x = 0$.

18. (**T**) (i)The differential equation of the form $y = xy' + f(y')$ is called a *Clairaut equation.* Show that the general solution of this equation is the family of straight lines $y = cx + f(c)$. In addition to these show that it has a special solution given by $f'(p) = -x$ where $p = y'$. This special solution which does not (in general) represent one of the straight lines $y = cx + f(c)$, is a singular solution.

    (Hint. Differentiate the given equation w.r.t. x.)

    (Recall: A General Solution of an n-th order differential equation is one that involves n arbitrary constants. A singular solution of a differential equation is a solution that is not obtainable by specifying the values of the arbitrary constant in general solution)

    (**T**) (ii) Solve the equation: $y'^2 - xy + y = 0$.

    Solution:

    (i)The given equation is $y = xp + f(p), \quad p = y'$. Differentiating with respect to x, we get $\frac{dp}{dx}(x + f'(p)) = 0$. For $\frac{dp}{dx} = 0$ implies that $p = const = c_1$. Further integrating, we have $y = c_1x + c_2$. Substituting in the given equation we have $c_1x + c_2 = xc_1 + f(c_1)$, i.e. $c_2 = f(c_1)$. Hence the general solution is

    $$y = c_1x + c_2 = c_1x + f(c_1).$$

    (ii) The given equation is $y = xy' - y'^2$ which is in Clairaut form with $f(p) = -p^2$. General solution is the family of st lines $y = mx + m^2$. For singular solution, $-2p = f'(p) = -x$. Putting $y' = p = x/2$ in the given equation, we have singular solution $y = x^2/2 - x^2/4 = x^2/4$.