ODE: Assignment-3

1. (**T**) A surface $z = y^2 - x^2$ in the shape of a saddle is lying outdoors in a rainstorm. Find the paths along which raindrops will run down the surface.

2. (**T**) Does $f(x, y) = xy^2$ satisfies Lipschitz condition (LC) on any rectangle $[a, b] \times [c, d]$? What about on an infinite strip $[a, b] \times \mathbb{R}$?

   [A function $f(x, y)$ is said to satisfy Lipschitz condition on a domain $D \subseteq \mathbb{R}^2$, if there exists $L > 0$ such that $|f(x, y_1) - f(x, y_2)| \leq L|y_1 - y_2|$ for all $(x, y_1), (x, y_2) \in D$.]

3. (**T**) Let (x_0, y_0) be an arbitrary point in the plane and consider the initial value problem (IVP)

$$y' = y^2, \quad y(x_0) = y_0.$$

   Explain why Picard theorem guarantees that this problem has a unique solution on some interval $|x - x_0| \leq h$. Since $f(x, y) = y^2$ and $\partial f/\partial y$ are continuous on the entire plane, it is tempting to conclude that this solution is valid for all x. But considering the solutions through the points $(0, 0)$ and $(0, 1)$, show that this consideration is sometime true and sometime false, and that therefore the inference is not legitimate.

   [Remark: Compare the above with the fact that if f is continuous and Lipschitz on $[a, b] \times \mathbb{R}$, then the IVP $y' = f(x, y), \quad y(x_0) = y_0, \quad x_0 \in [a, b]$ has solution over $[a, b]$. Simmons book Theorem B in chapter 'The Existence and Uniqueness of Solutions'.]

4. (**T**) Consider the IVP $y' = 2\sin(3xy), \quad y(0) = y_0$. Show that it has unique solution in $(-\infty, \infty)$.

5. (**T**) Given

$$y' = \frac{e^{y^2} - 1}{1 - x^2y^2}, \quad y(-2) = 1.$$

   Find an interval on which solution exist.

6. (**T**) Consider the ode $y' = \frac{2xy}{x^2 - y^2}$. Solve it. Sketch the solutions. Verify Picard theorem for initial values in $\mathbb{R}^2 - \{(x, y) : \quad x^2 = y^2\}$. What is your solution passing through $(1, 0)$?

7. A function $f(x, y)$ is said to satisfy Lipschitz condition on a domain $D \subseteq \mathbb{R}^2$, if there exists $L > 0$ such that $|f(x, y_1) - f(x, y_2)| \leq L|y_1 - y_2|$ for all $(x, y_1), (x, y_2) \in D$.

   (i) Show that if $f(x, y)$ satisfies Lipschitz condition (LC)with respect to y on a rectangle D, then for each fixed x, the resulting function of y is continuous function of y.

   (ii) Let $f(x, y) = y + [x]$. Then how that f satisfies LC on $\mathbb{R}^2$ but not continuous on $\mathbb{R}^2$.

   (iii) Let $f(x, y) = xy$. Then show that f is continuous on $\mathbb{R}^2$ but not LC on $\mathbb{R}^2$.

8. (**T**) What does Picard theorem says about existence and uniqueness of solution of the IVP $y' = (3/2)y^{1/3}, \quad y(0) = 0$? Show that it has uncountably many solutions.

9. Consider the IVP $y' = \sqrt{y} + 1, \quad y(0) = 0, \quad x \in [0, 1]$. Show that $f(x, y) = \sqrt{y} + 1$ does not satisfy Lipschitz condition in any rectangle containing origin, but still the solution is unique.

(Remark: It is fact that if an IVP, with f is continuous (not necessarily Lipschitz), has more than one solution, then it has uncountably many solutions. This is known as Kneser's Theorem. The previous exercise illustrates this phenomenan.)

10. (**T**) (i) Let $f(x,y)$ be continuous on the closed rectangle $R : |x-x_0| \leq a,\ |y-y_0| \leq b$. Show that y is a solution of the initial value problem $y' = f(x,y),\ y(x_0) = y_0$ iff
$$y(x) = y_0 + \int_{x_0}^{x} f\big(t, y(t)\big)\, dt.$$

(ii) Let $|f(x,y)| \leq M$ on the closed rectangle R and $y_n(x) = y_0 + \displaystyle\int_{x_0}^{x} f\big(t, y_{n-1}(t)\big)\, dt$, with $y_0(x) = y_0$. Use induction to show that $y_{n+1}(x)$ is well defined for $I :\ |x-x_0| \leq h$, where $h = \min\{a, b/M\}$; that is $|y_n(x) - y_0| \leq b$ for $x \in I$.

(Remark: The sequence of functions $y_n(x)$ are called Picard's Iterates. Precisely because of this step, the solution exist in possibly smaller interval in Picard theorem.)

11. Use Picard's method of successive approximation to solve the following initial value problems and compare these results with the exact solutions:
(i) (**T**) $y' = 2\sqrt{x},\ y(0) = 1$ (ii) $y' + xy = x,\ y(0) = 0$ (iii) $y' = 2\sqrt{y}/3,\ y(0) = 0$

12. Consider the initial value problem (IVP) $xy' - y = 0,\quad y(x_0) = y_0$. Solve it for different values of x_0 and y_0. Does the result contradict Picard theorem ?

13. Solve $y' = (y-x)^{2/3} + 1$. Show that $y = x$ is also a solution. What can be said about the uniqueness of the initial value problem consisting of the above equation with $y(x_0) = y_0$, where (x_0, y_0) lies on the line $y = x$.

14. Discuss the existence and uniqueness of the solution of the initial value problem
$$(x^2 - 2x)y' = 2(x-1)y, \qquad y(x_0) = y_0.$$

15. (**T**) Consider the IVP $y' = x - y,\ \ y(0) = 1$. Show that for Euler method, $y_n = 2(1-h)^n - 1 + nh$ where h is the step size. ($x_n = nh$ with $x_0 = 0,\ y_0 = y(0) = 1$). Deduce that if we take $h = 1/n$, then the limit of y_n converges to actual value of $y(1)$.

16. Use Euler method and step size .1 on the IVP $y' = x + y^2,\ \ y(0) = 1$ to calculate the approximate value for the solution $y(x)$ when $x = .1, .2, .3$. Is your answer for $y(.3)$ is higher or lower than the actual value ?

17. Verify that $y = x^2 \sin x$ and $y = 0$ are both solution of the initial value problem (IVP)
$$x^2 y'' - 4xy' + (x^2 + 6)y = 0, \quad y(0) = y'(0) = 0.$$
Does it contradict uniqueness of solution of IVP?

18. (**T**) (i)The differential equation of the form $y = xy' + f(y')$ is called a *Clairaut equation*. Show that the general solution of this equation is the family of straight lines $y = cx + f(c)$. In addition to these show that it has a special solution given by $f'(p) = -x$ where $p = y'$. This

special solution which does not (in general) represent one of the straight lines $y = cx + f(c)$, is a singular solution.

(Hint. Differentiate the given equation w.r.t. x.)

(Recall: A General Solution of an n-th order differential equation is one that involves n arbitrary constants. A singular solution of a differential equation is a solution that is not obtainable by specifying the values of the arbitrary constant in general solution)

(**T**) (ii) Solve the equation: $y'^2 - xy + y = 0$.