ODE: Assignment-4

In this assignment, we will denote:

$$y'' + p(x)y' + q(x)y = r(x), \ x \in I \qquad (*)$$

$$y'' + p(x)y' + q(x)y = 0, \quad x \in I \quad (**)$$

where $I \subset \mathbb{R}$ is an interval and $p(x), q(x), r(x)$ are continuous functions on I.

1. (**T**) Let y_1 be the solution of the IVP

$$y'' + (2x-1)y' + \sin(e^x)y = 0, \quad y(0) = 1, y'(0) = -1;$$

and y_2 be the solution of the IVP

$$y'' + (2x-1)y' + \sin(e^x)y = 0, \quad y(0) = 2, y'(0) = -1.$$

Find the Wronskian of y_1, y_2. What is the general solution of $y''+(2x-1)y'+\sin(e^x)y = 0$?

Solution:

We know that if y_1, y_2 are solutions of $(**)$, then the Wronskian $W(y_1, y_2)(x) = W(x) = c\exp(-\int p(x)dx) = ce^{-x^2+x}$. From the given initial conditions we have $W(0) = 3$. So $c = 3$. Hence $W(x) = 3e^{-x^2+x}$.

Since $W(0) \neq 0$, we deduce thaty_1, y_2 are independent solutions. Therefore, the general solution is given by $c_1y_1 + c_2y_2$.

2. (**T**) Show that the set of solutions of the linear homogeneous equation $(**)$ is a real vector space. Also show that the set of solutions of the linear non-homogeneous equation $(*)$ is not a real vector space. If $y_1(x), y_2(x)$ are any two solutions of $(*)$, obtain conditions on the constants a and b so that $ay_1 + by_2$ is also its solution.

Solution:

Let S be the set of solutions of the linear homogeneous ODE (**). Clearly S is a subset of set of twice differentiable functions on I which is a real vector space. Thus it is sufficient to show that S is subspace of the above vector space of twice differentiable function. Now $\mathbf{0}(x) = 0$ satisfies (**) and hence $\mathbf{0} \in S$. Thus S is nonempty. Also if u, v both satisfies (**), then $\alpha u(x) + v(x)$ is also a solution of (**). This implies $\alpha u + v \in S$. Hence S is a subspace, i.e. a vector space.

Now $\mathbf{0}(x) = 0$ is not a solution of (*), thus zero element does not exist. Hence, the set of solution of (*) is not a real vector space.

Let $y_1(x), y_2(x)$ are any two solutions of $(*)$. Then

$$y_1'' + p(x)y_1' + q(x)y_1 = r(x), \qquad (1)$$

$$y_2'' + p(x)y_2' + q(x)y_2 = r(x). \qquad (2)$$

Multiplying (1) by a and (2) by b and adding, we find

$$(ay_1 + by_2)'' + p(x)(ay_1 + by_2)' + q(x)(ay_1 + by_2) = (a + b)r(x).$$

If $ay_1 + by_2$ is also a solution, then the LHS is $r(x)$ and hence $a + b = 1$.

3. Decide if the statements are true or false. If the statement is true, prove it, if it is false, give a counter example showing it is false.

   (i) If $f(x)$ and $g(x)$ are linearly independent functions on an interval I, then they are linearly independent on any larger interval containing I.

   If $f(x)$ and $g(x)$ are linearly independent functions on an interval I, then they are linearly independent on any smaller interval contained in I.

   (ii) If $f(x)$ and $g(x)$ are linearly dependent functions on an interval I, then they are linearly dependent on any subinterval of I.

   If $y_1(x)$ and $y_2(x)$ are linearly dependent functions on an interval I, then they are linearly dependent on any larger interval containing I.

   (iii) If $y_1(x)$ and $y_2(x)$ are linearly independent solution of $(**)$ on an interval I, they are linearly independent on any interval contained in I.

   (iv) If $y_1(x)$ and $y_2(x)$ are linearly dependent solutions of $(**)$ on an interval I, they are linearly dependent on any interval contained in I.

   Solution:

   (i) True, follows from the definition of linear independence. Flase: take $f(x) = x^2$ and $g(x) = x|x|$. Then f, g linearly independent over $[-1, 1]$ but dependent over $[0, 1]$.

   (ii)True, follows from definition.

   (iii) True, follows from the fact that, in this case y_1, y_2 is linearly independent on I iff $W(y_1, y_2) \neq 0$ on all I.

   (iv) True, follows from the fact that, in this case y_1, y_2 is linearly dependent on I iff $W(y_1, y_2) = 0$ on all I.

4. Can x^3 be a solution of $(**)$ on $I = [-1, 1]$? Find two 2nd order linear homogeneous ODE with x^3 as a solution.

   Solution:

   No. Putting $y = x^3$ in the given equation, we get $6x + p(x)3x^2 + q(x)x^3 = 0$ for all $x \in [-1, 1.]$ Cancelling x, we get $6 + p(x)3x + q(x)x^2 = 0$ for all $[-1, 1] \ni x \neq 0$. That is $p(x)3 + q(x)x = -6/x$ for all $x \in [-1, 1.]$. We see that LHS is continuous at 0 but RHS is not continuous at 0. This cant not happen.

   Two ODEs with x^3 as solution are: $xy'' = 2y'$ and $x^2y'' = 6y$. Note that here p, q are not continuous at 0.

5. (**T**) Can $x \sin x$ be a solution of a second order linear homogeneous equation with constant coefficients?

Solution:

No, putting $x \sin x$ in $y'' + py' + qy = 0$, we get $(q-1)x \sin x + p(\sin x + x \cos x) = 0$ for all $x \in \mathbb{R}$. Here p, q are constants. This is clearly not possible.

6. (**T**) Find the largest interval on which a unique solution is guaranteed to exist of the IVP. $(x+2)y'' + xy' + \cot(x)y = x^2 + 1, \quad y(2) = 11, \quad y'(2) = -2$.

Solution:

Comparing with $(*)$, we have

$$p(x) = \frac{x}{x+2}, \quad q(x) = \frac{\cos(x)}{(x+2)\sin x}, \quad r(x) = \frac{x^2+1}{x+2}.$$

The discontinuities of p, q, r are $x = -2, 0, \pm\pi, \pm 2\pi, \pm 3\pi, \cdots$. The largest interval that contains $x_0 = 2$ but none of the discontinuities is, therefore, $(0, \pi)$.

7. Without solving determine the largest interval in which the solution is guaranteed to uniquely exist of the IVP $ty'' - y' = t^2 + t, \quad y(1) = 1, y'(1) = 5$. Verify your answer by solving it explicitly.

Solution:

Since p, r are not continuous at 0, the maximum interval of existence and uniqueness of solution of the given IVP is $(0, \infty)$.

Here dependent variable y is missing. Solving it, $y(t) = t^3/3 + 7t^2/4 + t^2(\ln t)/2 - 13/12$ for which the max interval of validity is $(0, \infty)$.

8. Find the differential equation satisfied by each of the following two-parameter families of plane curves:
(i) $y = \cos(ax + b)$ (ii) $y = ax + \frac{b}{x}$ (iii) $y = ae^x + bxe^x$

Solution:

For two arbitrary constants, the order of the ODE will be two. Eliminate constants a and b by differentiating twice.
(i) $y = \cos(ax+b) \implies y' = -a\sin(ax+b), \; y'' = -a^2\cos(ax+b) = -a^2 y$. From this we find

$$\frac{y'^2}{a^2} + y^2 = 1 \implies (1-y^2)a^2 = y'^2 \implies -(1-y^2)\frac{y''}{y} = y'^2 \implies (1-y^2)y'' + yy'^2 = 0$$

(ii) $y = ax + b/x \implies xy = ax^2 + b \implies xy' + y = 2ax \implies y' + y/x = 2a$ which on differentiating again gives $y'' + y'/x - y/x^2 = 0 \implies x^2y'' + xy' - y = 0$.

(iii) $y = ae^x + bxe^x \implies e^{-x}y = a + bx \implies e^{-x}y' - e^{-x}y = b \implies e^{-x}y'' - 2e^{-x}y' + e^{-x}y = 0$ which on simplification gives $y'' - 2y' + y = 0$

9. Find general solution of the following differential equations given a known solution y_1:
(i) (**T**) $x(1-x)y'' + 2(1-2x)y' - 2y = 0$ $\qquad y_1 = 1/x$
(ii) $(1-x^2)y'' - 2xy' + 2y = 0$ $\qquad y_1 = x$

Solution:

(i) Here $y_1 = 1/x$. Substitute $y = u(x)/x$ to get $(1-x)u'' - 2u' = 0$. Thus, $u' = 1/(1-x)^2$ and $u = 1/(1-x)$. Hence, $y_2 = 1/(x(1-x))$ and the general solution is $y = a/x + b/(x(1-x))$.

(ii) Here $y_1 = x$. Substitute $y = xu(x)$ to get $x(1-x^2)u'' = 2(2x^2-1)u'$. Thus,

$$\frac{u''}{u'} = \frac{2(2x^2-1)}{x(1-x^2)} = -\frac{2}{x} - \frac{1}{1+x} + \frac{1}{1-x} \implies u' = \frac{1}{x^2(1-x^2)}$$

Thus,

$$u' = \frac{1}{x^2} + \frac{1}{2}\left(\frac{1}{1+x} + \frac{1}{1-x}\right) \implies u = -\frac{1}{x} + \frac{1}{2}\ln(\left(\frac{1+x}{1-x}\right)$$

Hence,

$$y_2 = -1 + \frac{x}{2}\ln(\left(\frac{1+x}{1-x}\right)$$

and the general solution is

$$y = ax + b\left\{-1 + \frac{x}{2}\ln(\left(\frac{1+x}{1-x}\right)\right\}.$$

10. Verify that $\sin x/\sqrt{x}$ is a solution of $x^2y'' + xy' + (x^2 - 1/4)y = 0$ over any interval on the positive x-axis and hence find its general solution.

Solution:

Verification is straightforward.

Substitute $y = u(x)\,\sin x/\sqrt{x}$ to get

$$y' = \frac{\sin x}{\sqrt{x}}u' + \left(\frac{\cos x}{\sqrt{x}} - \frac{\sin x}{2x^{3/2}}\right)u$$

$$y'' = \frac{\sin x}{\sqrt{x}}u'' + 2\left(\frac{\cos x}{\sqrt{x}} - \frac{\sin x}{2x^{3/2}}\right)u' + \left(-\frac{\sin x}{\sqrt{x}} - \frac{\cos x}{x^{3/2}} + \frac{3}{4}\frac{\sin x}{x^{5/2}}\right)u$$

This leads to

$$\sin x\, u'' + 2\cos x\, u' = 0 \implies u' = \operatorname{cosec}^2 x \implies u = -\cot x$$

Hence, $y_2 = -\cos x/\sqrt{x}$ and the general solution is $y = (a\sin x + b\cos x)/\sqrt{x}$.

11. Solve the following differential equations:
(i) $y'' - 4y' + 3y = 0$ $\qquad$ (ii) $y'' + 2y' + (\omega^2 + 1)y = 0$, $\quad \omega$ is real.

Solution:

(i) Characteristic (or auxiliary) equation: $m^2 - 4m + 3 = 0 \implies m = 1, 3$.
General sol: $y = Ae^x + Be^{3x}$

(ii) Characteristic equation: $m^2 + 2m + (1 + \omega^2) = 0 \implies m = -1 \pm \omega i$.

Case 1: $\omega = 0 \implies$ equal roots $m = -1, -1$ and general sol: $y = (A + Bx)e^{-x}$

Case 2: $\omega \neq 0 \implies$ complex conjugate roots $m = -1 \pm \omega i$ and general sol: $y = e^{-x}(A \sin \omega x + B \cos \omega x)$

12. Solve the following initial value problems:
(i) (**T**) $y'' + 4y' + 4y = 0 \qquad y(0) = 1,\ y'(0) = -1$
(ii) $y'' - 2y' - 3y = 0 \qquad y(0) = 1,\ y'(0) = 3$

Solution:

(i) Assume $y = e^{mx}$ is a solution. Putting in the given equation, we get the characteristic equation: $m^2 + 4m + 4 = 0 \implies m = -2, -2$. General sol: $y = e^{-2x}(A + Bx)$. Using initial conditions:

$$A = 1,\ B - 2A = -1 \implies B = 1 \implies y = (x + 1)e^{-2x}$$

(ii) Characteristic equation: $m^2 - 2m - 3 = 0 \implies m = -1, 3$. General sol: $y = (Ae^{3x} + Be^{-x})$. Using initial conditions:

$$A + B = 1,\ 3A - B = 3 \implies A = 1, B = 0 \implies y = e^{3x}$$

13. Reduce the following second order differential equation to first order differential equation and hence solve.
(i) $xy'' + y' = y'^2 \qquad$ (ii) (**T**) $yy'' + y'^2 + 1 = 0 \qquad$ (iii) $y'' - 2y' \coth x = 0$

Solution:

(i) Dependent variable y absent. Substitute $y' = p \implies y'' = dp/dx$. Thus $xp' + p = p^2$. Solving $p = 1/(1 - ax)$ which on integrating again gives $y = b - \ln(1 - ax)/a$, where a and b are arbitrary constants.

(ii) Independent variable x is absent in $yy'' + y'^2 + 1 = 0$. Substitute $y' = p \implies y'' = p\, dp/dy$. Thus

$$py\frac{dp}{dy} + p^2 = 1 \implies \frac{pdp}{1 + p^2} + \frac{dy}{y} = 0 \implies \ln \sqrt{1 + p^2}y = \ln a \implies 1 + p^2 = \frac{a^2}{y^2}$$

From $p^2 = a^2/y^2 - 1$, we find

$$\frac{ydy}{\sqrt{a^2 - y^2}} = \pm dx \implies -\sqrt{a^2 - y^2} = \pm x + b.$$

Both the solutions can be written as $(x + b)^2 + y^2 = a^2$ where a and b are arbitrary constants..

(iii) $y'' - 2y' \coth x = 0$. Substitute $y' = p \implies y'' = dp/dx$. Thus $dp/dx = 2p \coth x$. Solving $p = a \sinh^2 x$, which on integrating again gives $y = a(\sinh 2x - 2x)/4 + b$ where a and b are arbitrary constants.

14. Find the curve $y = y(x)$ which satisfies the ODE $y'' = y'$ and the line $y = x$ is tangent at the origin.

Solution:

The given conditions lead to the following problem:
Solve $y'' - y' = 0$ with $y(0) = 0,\ y'(0) = 1$. Integrating once gives $y' - y = a$ which on another integration gives $y + a = be^x$. $y(0) = 0$ gives $a = b$. $y'(0) = 1$ gives $b = 1$ and hence solution is $y = e^x - 1$.

15. Are the following functions linearly dependent on the given intervals?

(i)	$\sin 4x,\ \cos 4x$	$(-\infty, \infty)$	(ii)	$\ln x,\ \ln x^3$	$(0, \infty)$
(iii)	$\cos 2x,\ \sin^2 x$	$(0, \infty)$	(iv)(**T**)	$x^3,\ x^2\vert x\vert$	$[-1, 1]$

Solution:

(i) $a\sin 4x + b\cos 4x = 0$. For $x = 0$ we find $b = 0$ and for $x = \pi/8$ we get $a = 0$. Hence they are NOT linearly dependent.

(ii) $\ln x^3 - 3\ln x = 0$ for $x \in (0, \infty)$. Hence linearly dependent.

(iii) $a\cos 2x + b\sin^2 x = 0$. For $x = 0$ we find $a = 0$ and for $x = \pi/2$ we get $b = 0$. Hence they are NOT linearly dependent.

(iv) $ax^3 + bx^2|x| = 0$. For $x < 0$ we find $a - b = 0$ and for $x > 0$ we get $a + b = 0$. Hence $a = b = 0$ and thus they are NOT linearly dependent.

16. (a) Show that a solution to (**) with x-axis as tangent at any point in I must be identically zero on I.

(b) (**T**) Let $y_1(x), y_2(x)$ be two solutions of (**) with a common zero at any point in I. Show that y_1, y_2 are linearly dependent on I.

(c) (**T**) Show that $y = x$ and $y = \sin x$ are not a pair solutions of equation (**), where $p(x), q(x)$ are continuous functions on $I = (-\infty, \infty)$.

Solution:

(a) Let $\xi(x)$ be the solution. Since x axis is a tangent, at $x = x_0$, say, then $\xi(x_0) = \xi'(x_0) = 0$. Clearly $y(x) \equiv 0$ satisfies (**) and the initial conditions $y(x_0) = y'(x_0) = 0$. Since the solution is unique, $\xi(x) \equiv 0$ in $\mathcal{I}$.

(b) If $y_1(x), y_2(x)$ have a common zero at $x = x_0$, say, then $y_1(x_0) = y_2(x_0) = 0$. Hence, $W(y_1, y_2) = 0$ at $x = x_0$ and thus y_1, y_2 are linearly dependent.

(c) $y_1 = x$ and $y_2 = \sin x$ are LI on I. So if they were solution of $(**)$, the wronskian $W(y_1, y_2)$ must never be zero. But $W(y_1, y_2) = 0$ at $x = 0$, a contradiction.

17. (a)(**T**) Let $y_1(x), y_2(x)$ be two twice continuously differentiable functions on an interval I.

(i) Show that the Wronskian $W(y_1, y_2)$ does not vanish anywhere in I if and only if there exists continuous $p(x), q(x)$ on I such that (**) has y_1, y_2 as independent solutions.

(ii) Is it true that if y_1, y_2 are independent on I then there exists continuous $p(x), q(x)$ on I such that (**) has y_1, y_2 as independent solutions?

(b) Construct equations of the form (**) from the following pairs of solutions: e^{-x}, xe^{-x} .

Solution:

(a)(i) Suppose that $W(y_1, y_2)$ does not vanish anywhere in I. We want to find $p(x), q(x)$ such that

$$y_1'' + p(x)y_1' + q(x)y_1 = 0, \quad y_2'' + p(x)y_2' + q(x)y_2 = 0. \tag{3}$$

Solving we get:

$$p(x) = -(y_1y_2'' - y_2y_1'')/W(y_1, y_2) = -\frac{d}{dx}(W(y_1, y_2))/W(y_1, y_2)$$

and $q(x) = (y_1'y_2'' - y_2'y_1'')/W(y_1, y_2)$. They are continuous on I since $W(y_1, y_2)$ never zero on I.

[Note that $q(x)$ can also be written as $q(x) = -\frac{1}{y_1}\left(y_1'' + p(x)y_1'\right)$.]

Converse follows from the fact Wronskian is never zero for independent solutions of $(**)$.

(ii) Not true. Consider $y_1(x) = x^3$ and $y_2(x) = x^2|x|$ on $I = [-1, 1.]$ Then they are independent on I, but they are not solutions of any $(**)$ on I.

(b) Using 8(a): $y_1(x) = e^{-x}$ and $y_2(x) = xe^{-x}$. Hence, $W(y_1, y_2) = e^{-2x}$ and $p(x) = 2$. And $q(x) = -(e^{-x} - 2e^{-x})/e^{-x} = 1$. Hence $y'' + 2y' + y = 0$.

Alternative: Write $y = ay_1(x) + by_2(x)$ and eliminate a and b. $y = e^{-x}(a + bx) \implies e^x y = a + bx$. Differentiating w.r.t. x twice we find

$$e^x(y' + y) = b \implies e^x(y'' + 2y' + y) = 0 \implies y'' + 2y' + y = 0$$

18. By using the method of variation of parameters, find the general solution of:

(i) $y'' + 4y = 2\cos^2 x + 10e^x$ (ii) (**T**) $y'' + y = x \sin x$

(iii) $y'' + y = \cot^2 x$ (iv) $x^2y'' - x(x + 2)y' + (x + 2)y = x^3, \quad x > 0$.

[Hint. $y = x$ is a solution of the homogeneous part]

Solution:

If y_1, y_2 are independent solutions of the homogeneous part of the ODE

$$y'' + p(x)y' + q(x)y = r(x),$$

then the general solution is $y = Ay_1 + By_2 + uy_1 + vy_2$, where A, B are arbitrary constants and

$$u = -\int \frac{ry_2}{W}\,dx, \quad v = \int \frac{ry_1}{W}\,dx, \qquad [W(y_1, y_2) \text{ is the Wronskian}]$$

(i) $y_1 = \cos 2x, y_2 = \sin 2x, W(y_1, y_2) = 2, r(x) = 2\cos^2 x + 10e^x = \cos 2x + 1 + 10e^x$. Now

$$u = -\int y_2 r/W\,dx = \frac{\cos 4x}{16} + \frac{\cos 2x}{4} - e^x(\sin 2x - 2\cos 2x)$$

$$v = \int y_1 r/W\, dx = \frac{\sin 4x}{16} + \frac{x}{4} + \frac{\sin 2x}{4} + e^x(2\sin 2x + \cos 2x)$$

Thus

$$y_p = \frac{\cos 2x}{16} + \frac{x \sin 2x}{4} + \frac{1}{4} + 2e^x$$

General solution: (absorbing first term of y_p in the homogeneous solution)

$$y = A\cos 2x + B\sin 2x + \frac{x\sin 2x}{4} + \frac{1}{4} + 2e^x$$

(ii) $y_1 = \cos x, y_2 = \sin x, W(y_1, y_2) = 1, r(x) = x\sin x$. Now

$$u = -\int y_2 r/W\, dx = -\frac{x^2}{4} + \frac{x\sin 2x}{4} + \frac{\cos 2x}{8}$$

$$v = \int y_1 r/W\, dx = -\frac{x\cos 2x}{4} + \frac{\sin 2x}{8}$$

Thus

$$y_p = \frac{\cos x}{8} + \frac{x\sin x}{4} - \frac{x^2\cos x}{4}$$

General solution: (absorbing first term of y_p in the homogeneous solution)

$$y = A\cos x + B\sin x + \frac{x\sin x}{4} - \frac{x^2\cos x}{4}$$

(iii) (ii) $y_1 = \cos x, y_2 = \sin x, W(y_1, y_2) = 1, r(x) = \cot^2 x$. Now

$$u = -\int y_2 r/W\, dx = -\ln(\text{cosec}x - \cot x) - \cos x$$

$$v = \int y_1 r/W\, dx = -\text{cosec}x - \sin x$$

Thus

$$y_p = -2 - \cos x \ln(\text{cosec}x - \cot x)$$

General solution:

$$y = A\cos x + B\sin x - 2 - \cos x\ln(\text{cosec}x - \cot x)$$

(iv) $y_1 = x$ is a solution of the homogeneous part. To find another linearly independent solution we assume $y = xu$. This gives

$$u'' - u' = 0 \implies u' - u = 1 \implies u = e^x - 1 \implies y = xe^x - x$$

Since $y_1 = x$, we take $y_2 = xe^x$. The nonhomogeneous part is written as

$$y'' - \frac{x+2}{x}y' + \frac{(x+2)}{x^2}y = x.$$

Thus $r(x) = x$ and $W(y_1, y_2) = x^2e^x$. Now

$$u = -\int y_2 r/W\, dx = -x$$

and

$$v = \int y_1 r/W \, dx = -e^{-x}$$

Thus $y_p = -x - x^2$.
General solution: (absorbing first term of y_p in the homogeneous solution)

$$y = x(A + Be^x) - x^2.$$

19. Find the general solution of a 7th-order homogeneous linear differential equation with constant coefficients whose characteristic polynomial is $p(m) = m(m^2-3)^2(m^2+m+2)$.

    Solution:

    $m = 0, \pm\sqrt{3}, \pm\sqrt{3}, -1/2 \pm i\sqrt{7}/2$. So general solution:

    $y = c_1 + c_2 e^{\sqrt{3}x} + c_3 x e^{\sqrt{3}x} + c_4 e^{-\sqrt{3}x} + c_5 e^{-\sqrt{3}x} + c_6 e^{-x/2}\cos(\sqrt{7}x/2) + c_7 e^{-x/2}\sin(\sqrt{7}x/2).$

Initial Value Problem vs. Boundary Value Problem

A second-order *initial value problem* consists of a second-order ordinary differential equation $y''(t) = F(t, y(t), y'(t))$ and initial conditions $y(t_0) = y_0, \quad y'(t_0) = y_0'$ where t_0, y_0, y_0' are numbers.

It might seem that there are more than one ways to present the initial conditions of a second order equation. Instead of locating both initial conditions $y(t_0) = y_0$ and $y'(t_0) = y_0'$ at the same point t_0, couldn't we take them at different points, for examples $y(t_0) = y_0$ and $y(t_1) = y_1$; or $y'(t_0) = y_0'$ and $y'(t_1) = y_1'$? The answer is NO. **All the initial conditions in an initial value problem must be taken at the same point** t_0. The sets of conditions above where the values are taken at different points are known as *boundary conditions*. A boundary value problem does not have the existence and uniqueness guaranteed.

Example: Every function of the form $y = C\sin(t)$, where C is a real number satisfies the boundary value problem $y'' + y = 0, y(0) = 0, \ y(\pi) = 0$. Therefore, the problem has infinitely many solutions, even though $p(t) = 0, \quad q(t) = 1, \quad r(t) = 0$ are all continuous everywhere.