ODE: Assignment-4

In this assignment, we will denote:

$$y'' + p(x)y' + q(x)y = r(x), \ x \in I \quad (*)$$

$$y'' + p(x)y' + q(x)y = 0, \quad x \in I \quad (**)$$

where $I \subset \mathbb{R}$ is an interval and $p(x), q(x), r(x)$ are continuous functions on I.

1. (**T**) Let y_1 be the solution of the IVP

   $$y'' + (2x-1)y' + \sin(e^x)y = 0, \quad y(0) = 1, y'(0) = -1;$$

   and y_2 be the solution of the IVP

   $$y'' + (2x-1)y' + \sin(e^x)y = 0, \quad y(0) = 2, y'(0) = -1.$$

   Find the Wronskian of y_1, y_2. What is the general solution of $y''+(2x-1)y'+\sin(e^x)y = 0$?

2. (**T**) Show that the set of solutions of the linear homogeneous equation $(**)$ is a real vector space. Also show that the set of solutions of the linear non-homogeneous equation $(*)$ is not a real vector space. If $y_1(x), y_2(x)$ are any two solutions of $(*)$, obtain conditions on the constants a and b so that $ay_1 + by_2$ is also its solution.

3. Decide if the statements are true or false. If the statement is true, prove it, if it is false, give a counter example showing it is false.

   (i) If $f(x)$ and $g(x)$ are linearly independent functions on an interval I, then they are linearly independent on any larger interval containing I.

   If $f(x)$ and $g(x)$ are linearly independent functions on an interval I, then they are linearly independent on any smaller interval contained in I.

   (ii) If $f(x)$ and $g(x)$ are linearly dependent functions on an interval I, then they are linearly dependent on any subinterval of I.

   If $y_1(x)$ and $y_2(x)$ are linearly dependent functions on an interval I, then they are linearly dependent on any larger interval containing I.

   (iii) If $y_1(x)$ and $y_2(x)$ are linearly independent solution of $(**)$ on an interval I, they are linearly independent on any interval contained in I.

   (iv) If $y_1(x)$ and $y_2(x)$ are linearly dependent solutions of $(**)$ on an interval I, they are linearly dependent on any interval contained in I.

4. Can x^3 be a solution of $(**)$ on $I = [-1, 1]$? Find two 2nd order linear homogeneous ODE with x^3 as a solution.

5. (**T**) Can $x \sin x$ be a solution of a second order linear homogeneous equation with constant coefficients?

6. (**T**) Find the largest interval on which a unique solution is guaranteed to exist of the IVP. $(x+2)y'' + xy' + \cot(x)y = x^2 + 1, \quad y(2) = 11, \quad y'(2) = -2.$

7. Without solving determine the largest interval in which the solution is guaranteed to uniquely exist of the IVP $ty'' - y' = t^2 + t, \quad y(1) = 1, y'(1) = 5$. Verify your answer by solving it explicitly.

8. Find the differential equation satisfied by each of the following two-parameter families of plane curves:
(i) $y = \cos(ax+b)$ (ii) $y = ax + \frac{b}{x}$ (iii) $y = ae^x + bxe^x$

9. Find general solution of the following differential equations given a known solution y_1:
(i) (**T**) $x(1-x)y'' + 2(1-2x)y' - 2y = 0$ $\quad y_1 = 1/x$
(ii) $(1-x^2)y'' - 2xy' + 2y = 0$ $\quad y_1 = x$

10. Verify that $\sin x/\sqrt{x}$ is a solution of $x^2y'' + xy' + (x^2 - 1/4)y = 0$ over any interval on the positive x-axis and hence find its general solution.

11. Solve the following differential equations:
(i) $y'' - 4y' + 3y = 0$ (ii) $y'' + 2y' + (\omega^2 + 1)y = 0, \quad \omega$ is real.

12. Solve the following initial value problems:
(i) (**T**) $y'' + 4y' + 4y = 0$ $\quad y(0) = 1,\ y'(0) = -1$
(ii) $y'' - 2y' - 3y = 0$ $\quad y(0) = 1,\ y'(0) = 3$

13. Reduce the following second order differential equation to first order differential equation and hence solve.
(i) $xy'' + y' = y'^2$ (ii) (**T**) $yy'' + y'^2 + 1 = 0$ (iii) $y'' - 2y' \coth x = 0$

14. Find the curve $y = y(x)$ which satisfies the ODE $y'' = y'$ and the line $y = x$ is tangent at the origin.

15. Are the following functions linearly dependent on the given intervals?

(i)	$\sin 4x,\ \cos 4x$	$(-\infty, \infty)$	(ii)	$\ln x,\ \ln x^3$	$(0, \infty)$
(iii)	$\cos 2x,\ \sin^2 x$	$(0, \infty)$	(iv)(**T**)	$x^3,\ x^2\lvert x\rvert$	$[-1, 1]$

16. (a) Show that a solution to (**) with x-axis as tangent at any point in I must be identically zero on I.

(b) (**T**) Let $y_1(x), y_2(x)$ be two solutions of (**) with a common zero at any point in I. Show that y_1, y_2 are linearly dependent on I.

(c) (**T**) Show that $y = x$ and $y = \sin x$ are not a pair solutions of equation (**), where $p(x), q(x)$ are continuous functions on $I = (-\infty, \infty)$.

17. (a)(**T**) Let $y_1(x), y_2(x)$ be two twice continuously differentiable functions on an interval I.

    (i) Show that the Wronskian $W(y_1, y_2)$ does not vanish anywhere in I if and only if there exists continuous $p(x), q(x)$ on I such that (**) has y_1, y_2 as independent solutions.

    (ii) Is it true that if y_1, y_2 are independent on I then there exists continuous $p(x), q(x)$ on I such that (**) has y_1, y_2 as independent solutions?

    (b) Construct equations of the form (**) from the following pairs of solutions: e^{-x}, xe^{-x} .

18. By using the method of variation of parameters, find the general solution of:

    (i) $y'' + 4y = 2\cos^2 x + 10e^x$ (ii) (**T**) $y'' + y = x \sin x$

    (iii) $y'' + y = \cot^2 x$ (iv) $x^2y'' - x(x + 2)y' + (x + 2)y = x^3, \quad x > 0.$

    [Hint. $y = x$ is a solution of the homogeneous part]

19. Find the general solution of a 7th-order homogeneous linear differential equation with constant coefficients whose characteristic polynomial is $p(m) = m(m^2 - 3)^2(m^2 + m + 2)$.

Initial Value Problem vs. Boundary Value Problem

A second-order *initial value problem* consists of a second-order ordinary differential equation $y''(t) = F(t, y(t), y'(t))$ and initial conditions $y(t_0) = y_0, \quad y'(t_0) = y'_0$ where t_0, y_0, y'_0 are numbers.

It might seem that there are more than one ways to present the initial conditions of a second order equation. Instead of locating both initial conditions $y(t_0) = y_0$ and $y'(t_0) = y'_0$ at the same point t_0, couldn't we take them at different points, for examples $y(t_0) = y_0$ and $y(t_1) = y_1$; or $y'(t_0) = y'_0$ and $y'(t_1) = y'_1$? The answer is NO. **All the initial conditions in an initial value problem must be taken at the same point** t_0. The sets of conditions above where the values are taken at different points are known as *boundary conditions.* A boundary value problem does not have the existence and uniqueness guaranteed.

Example: Every function of the form $y = C\sin(t)$, where C is a real number satisfies the boundary value problem $y'' + y = 0, y(0) = 0,\ y(\pi) = 0$. Therefore, the problem has infinitely many solutions, even though $p(t) = 0, \quad q(t) = 1, \quad r(t) = 0$ are all continuous everywhere.