# ODE: Assignment-5

(For calculations of Particular Integrals by operator method, see Simmons books, page 161, section 23 of the chapter Second order linear equations.)

1. Solve: (i) $x^2y'' + 2xy' - 12y = 0$ (ii)(**T**) $x^2y'' + 5xy' + 13y = 0$ (iii) $x^2y'' - xy' + y = 0$

   [Recall: The ODE of the form $x^2\frac{d^2y}{dx^2} + ax\frac{dy}{dx} + by = 0$, where $a, b$ are constants, is called the Cauchy-Euler equation. Under the transformation $x = e^t$ (when $x > 0$) for the independent variable, the above reduces to $\frac{d^2y}{dt^2} + (a-1)\frac{dy}{dt} + by = 0$, which is an equation with constant coefficients. ]

   **Solution:**

   (i) Using the substitution $x = e^t$, the given equation reduces to

   $$\frac{d^2u}{dt^2}+\frac{du}{dt}-12u = 0 \implies m^2+m-12 = 0 \implies m = -4, 3 \implies u(t) = Ae^{-4t}+Be^{3t} = y(e^t).$$

   The general solution is thus

   $$y(x) = \frac{A}{x^4} + Bx^3.$$

   (ii) Using the substitution $x = e^t$, the given equation reduces to,

   $$\frac{d^2u}{dt^2} + 4\frac{du}{dt} + 13u = 0 \implies m^2 + 4m + 13 = 0 \implies m = -2 \pm 3i.$$

   Thus

   $$u(t) = e^{-2t}(A\cos 3t + B\sin 3t) = y(e^t).$$

   The general solution is

   $$y(x) = \frac{1}{x^2}\big(A\cos(3\ln x) + B\sin(3\ln x)\big).$$

   (iii) Using the substitution $x = e^t$, the given equation reduces to

   $$\frac{d^2u}{dt^2} - 2\frac{du}{dt} + u = 0 \implies m^2 - 2m + 1 = 0 \implies m = 1, 1 \implies u(t) = e^t(A + Bt) = y(e^t)$$

   The general solution is thus

   $$y(x) = e^x(A + B\ln x).$$

2. (Higher order Cauchy-Euler equations) Let us denote $D = \frac{d}{dx}$ and $\mathcal{D} = \frac{d}{dt}$ where $x = e^t$. Show that

   $$xD = \mathcal{D}, \quad x^2D^2 = \mathcal{D}(\mathcal{D}-1), \quad x^3D^3 = \mathcal{D}(\mathcal{D}-1)(\mathcal{D}-2).$$

   Hence conclude that $(x^3D + ax^2D^2 + bxD + c)y = 0, \quad x > 0$ is transformed into constant coefficients ODE $[\mathcal{D}(\mathcal{D}-1)(\mathcal{D}-2) + a\mathcal{D}(\mathcal{D}-1) + b\mathcal{D} + c]y = 0$ by the substitution $x = e^t$.

**Solution:**

Given $x = e^t$, so $\frac{dx}{dt} = e^t = x$. Now, by chain rule, $\frac{d}{dx} = \frac{d}{dt}\frac{dt}{dx} = e^{-t}\frac{d}{dt}$. Thus $xD = \mathcal{D}$. Differentiating this with respect to $x$, we have $xD^2 + D = \mathcal{D}^2\frac{dt}{dx} = \mathcal{D}^2 e^{-t}$, $\implies x^2D^2 + xD = \mathcal{D}^2$, $\implies x^2D^2 = \mathcal{D}^2 - \mathcal{D} = \mathcal{D}(\mathcal{D}-1)$.

Differentiating $x^2D^2 = \mathcal{D}^2 - \mathcal{D}$ with respect to $x$, we have $x^2D^3 + 2xD^2 = [\mathcal{D}^3 - \mathcal{D}^2]e^{-t}$, $\implies x^3D^3 = \mathcal{D}^3 - \mathcal{D}^2 - 2\mathcal{D}(\mathcal{D}-1) = \mathcal{D}(\mathcal{D}-1)(\mathcal{D}-2)$.

3. Find a particular solution of each of the following equations by operator methods and hence find its general solution:

   (i) $y'' + 4y = 2\cos^2 x + 10e^x$ (ii)(**T**) $y'' + y = \sin x + (1+x^2)e^x$

   (**T**) (iii) $y'' - y = e^{-x}(\sin x + \cos x)$ (iv) $y''' - 3y'' - y' + 3y = x^2e^x$

**Solution:**

(i) Characteristic equation $m^2 + 4 = 0 \implies m = \pm 2i$. Hence homogeneous solution $y_h = A\cos 2x + B\sin 2x$. Now $r(x) = 2\cos^2 x + 10e^x = \cos 2x + 1 + 10e^x$. Let $D \equiv d/dx$ and $y_p$ be the particular solution. Then

$$\frac{1}{D^2+4}1 = \frac{1}{D^2+4}e^{0x} = 1/4.$$

$$\frac{1}{D^2+4}10e^x = 10\frac{1}{1^2+4}e^x = 2e^x.$$

$$\frac{1}{D^2+4}e^{2ix} = x\frac{1}{2D}e^{2ix} = x\frac{1}{2.2i}e^{2ix} = xe^{2ix}/4i.$$

Taking real part

$$\frac{1}{D^2+4}\cos 2x = x\sin 2x/4.$$

Adding, we get the particular solution as

$$y_p = \frac{x\sin 2x}{4} + \frac{1}{4} + 2e^x.$$

Thus the general solution is

$$y = A\cos 2x + B\sin 2x + \frac{x\sin 2x}{4} + \frac{1}{4} + 2e^x.$$

(ii) Characteristic equation $m^2 + 1 = 0 \implies m = \pm i$. Hence homogeneous solution $y_h = A\cos x + B\sin x$. Now $r(x) = \sin x + (1+x^2)e^x$. Let $D \equiv d/dx$ and $y_p$ be the particular solution. Then

$$\frac{1}{1+D^2}e^{ix} = x\frac{1}{2D}e^{ix} = x\frac{1}{2i}e^{ix} = \frac{x}{2i}(\cos x + i\sin x).$$

Taking imaginary part

$$\frac{1}{1+D^2}\sin x = -\frac{x\cos x}{2}.$$

$$\frac{1}{1+D^2}(1+x^2)e^x = e^x\frac{1}{(D+1)^2+1}(1+x^2) = e^x\frac{1}{D^2+2D+2}(x^2+1)$$

$$= \frac{e^x}{2}\frac{1}{1+D+D^2/2}(x^2+1) = \frac{e^x}{2}(1-D-D^2/2+(D+D^2/2)^2+\cdots)(x^2+1)$$

$$= \frac{e^x}{2}(1-D-D^2/2+D^2/2+\cdots)(x^2+1) = \frac{e^x}{2}(1-D+D^2/2+\cdots)(x^2+1) = \frac{e^x}{2}(1+x^2-2x+1)$$

Thus the general solution is

$$y = A\cos x + B\sin x - \frac{x\cos x}{2} + \left(1-x+\frac{x^2}{2}\right)e^x$$

(iii) Characteristic equation $m^2-1=0 \implies m=\pm 1$. Hence homogeneous solution $y_h = Ae^x + Be^{-x}$. Now $r(x) = e^{-x}(\sin x + \cos x)$. Let $D \equiv d/dx$ and $y_p$ be the particular solution.

$$\frac{1}{D^2-1}e^{-x}e^{ix} = \frac{e^{-x}e^{ix}}{(i-1)^2-1} = \frac{e^{-x}e^{ix}}{-2i-1} = -\frac{e^{-x}}{5}[\cos x + 2\sin x + i(\sin x - 2\cos x)].$$

Then the particular solution is obtained by adding the real and imaginary parts:

$$y_p(x) = \frac{e^{-x}(\cos x - 3\sin x)}{5}$$

Thus the general solution is

$$y = Ae^x + Be^{-x} + \frac{e^{-x}(\cos x - 3\sin x)}{5}$$

(iv) Characteristic equation $m^3-3m^2-m+3=0 \implies m=-1,1,3$. Hence homogeneous solution $y_h = Ae^{-x} + Be^x + Ce^{3x}$. Now $r(x) = x^2e^x$. Let $D \equiv d/dx$ and $y_p$ be the particular solution. Then

$$\frac{1}{D^3-3D^2-D+3}x^2e^x = \frac{1}{(D-1)^3-4(D-1)}x^2e^x = e^x\frac{1}{D^3-4D}x^2$$

$$= e^x\frac{1}{-4D(1-D^2/4)}x^2 = e^x\frac{1}{-4D}(1+D^2/4+\cdots)x^2 = e^x\frac{1}{-4D}(x^2+\frac{1}{2}) = -\frac{e^x}{4}(\frac{x^3}{3}+\frac{x}{2}).$$

So the particular integral is

$$y_p(x) = -e^x\left(\frac{x}{8}+\frac{x^3}{12}\right).$$

Thus the general solution is

$$y = Ae^{-x} + Be^x + Ce^{3x} - e^x\left(\frac{x}{8}+\frac{x^3}{12}\right).$$

4. Solve $y'' + y' - 2y = e^x$.

**Solution:** Characteristic equation of the homogeneous part is: $m^2 + m - 2 = 0, \quad m = 1, -2$. Solution for the homogeneous part: $c_1 e^x + c_2 e^{-2x}$.

Particular integral:

$$\frac{1}{D^2 + D - 2} e^x = x \frac{1}{2D + 1} e^x = \frac{xe^x}{(2.1 + 1)} = \frac{xe^x}{3}.$$

General solution:

$$c_1 e^x + c_2 e^{-2x} + \frac{xe^x}{3}.$$

5. Solve by using operator method $(D^2 + 9)y = \sin 2x \cos x$.

**Solution:**

Characteristic equation of the homogeneous part is: $m^2 + 9 = 0, \quad m = \pm 3i$. Solution for the homogeneous part: $c_1 \cos 3x + c_2 \sin 3x$.

Particular integral:

$$\frac{1}{D^2 + 9} \sin 2x \cos x = \frac{1}{2(D^2 + 9)} (\sin 3x + \sin x).$$

Now

$$\frac{1}{D^2 + 9} e^{ix} = e^{ix}/(i^2 + 9) = (\cos x + i \sin x)/8.$$

Taking the imaginary part,

$$\frac{1}{2(D^2 + 9)} (\sin x) = \sin x/16.$$

Now

$$\frac{1}{D^2 + 9} e^{3ix} = \frac{xe^{3ix}}{2.3i} = (x \cos 3x + ix \sin 3x)/6i.$$

Taking the imaginary part,

$$\frac{1}{2(D^2 + 9)} (\sin 3x) = -(x \cos 3x)/12.$$

General solution:

$$c_1 \cos 3x + c_2 \sin 3x + \sin x/16 - (x \cos 3x)/12.$$

6. Find a particular integral by operator method: $D^2 - 6D + 9 = 1 + x + x^2$.

**Solution:**

$$P.I = \frac{1}{D^2 - 6D + 9} 1 + x + x^2 = \frac{1}{9(1 + (D^2 - 6D)/9)} 1 + x + x^2$$

$$= \frac{1}{9}[1 - (D^2 - 6D)/9 + (D^2 - 6D)^2/81 - \cdots](1 + x + x^2).$$

$$= \frac{1}{9}[1 + 2D/3 + D^2/3 + \cdots](1 + x + x^2) = (1 + x + x^2 + 2/3 + 4x/3 + 2/3)$$

$$= \frac{1}{9}(7/3 + 7x/3 + x^2).$$

7. Find P.I: $y'' + 9y = x \cos x$.

   **Solution:**

   Consider

$$\frac{1}{D^2+9} x e^{ix} = e^{ix} \frac{1}{(D+i)^2+9} x = e^{ix} \frac{1}{D^2+2iD+8} x = e^{ix} \frac{1}{8(1+D^2/8+iD/4)} x$$

$$= e^{ix} \frac{1}{8}(1 - D^2/8 - iD/4)x = e^{ix} \frac{1}{8}(x - i/4).$$

   Taking the real part:

$$\frac{1}{D^2+9} x \cos x = \frac{x \cos x}{8} + \frac{\sin x}{32}.$$

8. (**T**) Solve $x^2 y'' - 2xy' - 4y = x^2 + 2\log x, \quad x > 0$.

   **Solution:**

   Apply the transformation $x = e^t$ the equation reduces to $y'' - 3y - 4 = e^{2t} + 2t$.

   Solution of the homogeneous part $c_1 e^{4t} + c_2 e^{-t}$.

   Particular integral: $\frac{1}{D^2-3D-4}(e^{2t} + 2t) = -e^{2t}/6 + \frac{1}{D^2-3D-4} 2t = -\frac{1}{6} e^{2t} - \frac{1}{2}(t - 3/4)$.

$$\frac{1}{D^2 - 3D - 4} 2t = 2 \frac{1}{-4(1 - D^2/4 + 3D/4)} t = -\frac{1}{2}[1 + (D^2/4 - 3D/4 + \cdots)]t = \frac{-1}{2}(t - 3/4).$$

   Hence the general solution is:

$$y = c_1 e^{4t} + c_2 e^{-t} - \frac{1}{6} e^{2t} - \frac{1}{2}(t - 3/4) = c_1 x^4 + c_2/x - \frac{1}{6} x^2 - \frac{1}{2}(\ln x - 3/4).$$

9. (**T**) (Higher order variation of parameter) Consider the $n$-th order linear equation

$$y^{(n)} + \sum_1^n a_i(x) y^{(i)} = y^{(n)} + a_{n-1}(x) y^{(n-1)} + \cdots + a_0(x) y = r(x).$$

   Assume that $y_1, \cdots, y_n$ are $n$-independent solutions of the associated homogeneous equation. Prove that a particular integral of the given ODE is

$$y_p = \sum v_i y_i \text{ where } v_i' = \frac{R_i}{W}.$$

   Here $W$ is the wronskian of $y_1, \cdots, y_n$ and $R_i$ is the determinant obtained by replacing $i$-th column of $W$ by $[0, 0, \cdots, 0, r(x)]$.

**Solution:**

Let

$$y_p = \sum v_i y_i ------(1).$$

Differentiating $y_p' = \sum v_i' y_i + \sum v_i y_i'$. Assume $\sum v_i' y_i = 0$ then

$$y_p' = \sum v_i y_i' ----- (2).$$

Differentiating this $y_p'' = \sum v_i' y_i' + v_i y_i''$. Assuming $\sum v_i' y_i' = 0$ we have

$$y_p'' = \sum v_i y_i''. ------(3)$$

Proceeding similarly, we get

$$y_p^{(n-1)} = \sum v_i y_i^{(n-1)} ----(n)$$

if $\sum v_i' y_i^{(n-2)} = 0$.

$$y_p^{(n)} = \sum v_i' y_i^{(n-1)} + \sum v_i y_i^{(n)} ------(n+1).$$

Then

$$y_p^{(n)} + \sum_1^n a_i(x) y_p^{(i)} = \sum v_i' y_i^{(n-1)}.$$

Hence $y_p$ is a solution of the given ODE if

$$\sum v_i' y_i = 0,\ \sum v_i' y_i' = 0,\ \sum v_i' y_i'' = 0,\ \cdots, \sum v_i' y_i^{(n-2)} = 0, \sum v_i' y_i^{(n-1)} = r(x).$$

Solution such system of linear equation is given by $v_i' = \frac{R_i}{W}$ where $W$ is the wronskian of $y_1, \cdots, y_n$ and $R_i$ is the determinant obtained by replacing $i$-th column of $W$ by $[0, 0, \cdots, 0, r(x)]$.

10. (i) Let $y_1(x), y_2(x)$ are two linearly independent solutions of $y''+p(x)y'+q(x)y = 0$. Show that $\phi(x) = \alpha y_1(x) + \beta y_2(x)$ and $\psi(x) = \gamma y_1(x) + \delta y_2(x)$ are two linearly independent solutions if and only if $\alpha\delta \neq \beta\gamma$.

    (ii) Show that the zeros of the functions $a \sin x + b \cos x$ and $c \sin x + d \cos x$ are distinct and occur alternately whenever $ad - bc \neq 0$.

    **Solution:**

    (i) We have $W(\phi, \psi) = (\alpha\delta - \beta\gamma)W(y_1, y_2)$. Since $y_1, y_2$ are fundamental solutions, $W(y_1, y_2) \neq 0$. If $\alpha\delta \neq \beta\gamma$, then $W(\phi, \psi) \neq 0$. Conversely if $W(\phi, \psi) \neq 0$, then $\alpha\delta \neq \beta\gamma$.

    (ii) We know $\sin x, \cos x$ are independent solutions of $y'' + y = 0$. So by part (i) $a \sin x + b \cos x$ and $c \sin x + d \cos x$ are independent solutions whenever $ad - bc \neq 0$. Hence the result follows from Sturm Separation theorem ( Simmons, page 190, Theorem A).

11. (**T**) Show that any nontrivial solution $u(x)$ of $u'' + q(x)u = 0,\ q(x) < 0$ for all $x$, has at most one zero.

**Solution:**

Consider the equation $z'' = 0$. Then $z = 1$ is a solution of the equation. By Strum comparison theorem, between two zeros of $u(x)$ there must be at least one zero of $z(x)$. But $z = 1$ has no zero. Hence $u(x)$ can have at most one zero.

12. Let $u(x)$ be any nontrivial solution of $u'' + [1 + q(x)]u = 0$, where $q(x) > 0$. Show that $u(x)$ has infinitely many zeros.

**Solution:**

Consider

$$v'' + v = 0, \qquad u'' + \big(1 + q(x)\big)u = 0$$

Now $v = \sin x$ is a nontrivial solution of $v'' + v = 0$. Since $1 + q(x) > 1$, by Strum comparison theorem, $u$ must vanish between two zeros of $\sin x$. Since, $\sin x$ has infinitely many zeros, $u$ also has infinitely may zeros.

13. Let $u(x)$ be any nontrivial solution of $u'' + q(x)u = 0$ on a closed interval $[a, b]$. Show that $u(x)$ has at most a finite number of zeros in $[a, b]$.

**Solution:**

Suppose, on the contrary, $u(x)$ has infinite number of zeros in $[a, b]$. It follows that there exists $x_0 \in [a, b]$ and a sequence of zeros $x_n \neq x_0$ such that $x_n \to x_0$. Since $u(x)$ is continuous and differentiable at $x_0$, we have

$$u(x_0) = \lim_{x_n \to x_0} u(x_n) = 0, \qquad u'(x_0) = \lim_{x_n \to x_0} \frac{u(x_n) - u(x_0)}{x_n - x_0} = 0$$

By uniqueness theorem, $u \equiv 0$ which contradicts the fact that $u$ is nontrivial.

14. (**T**) Let $J_p$ be any non-trivial solution of the Bessel equation

$$x^2 y'' + x y' + (x^2 - p^2) y = 0, \quad x > 0.$$

Show that $J_p$ has infinitely many positive zeros.

**Solution:**

The normal form of Bessel equation is

$$u'' + (1 + \frac{1/4 - p^2}{x^2})u = 0.$$

Given $p \geq 0$, we can choose $x_0$ large enough such that $1 + \frac{1/4 - p^2}{x^2} > 1/4$ for all $x \in (x_0, \infty)$. Compare $J_p$ with $\sin(x/2)$ which is solution of $v'' + \frac{1}{4}v = 0$ in $(x_0, \infty)$. Clearly $\sin(x/2)$ has infinitely many zeros in $(x_0, \infty)$. By Sturm comparison theorem, between two consecutive zeros of $\sin(x/2)$ there is a zero of $J_p$. Hence $J_p$ has infinitely many zero in $(x_0, \infty)$.

15. (**T**) Consider $u'' + q(x)u = 0$ on an interval $I = (0, \infty)$ with $q(x) \geq m^2$ for all $t \in I$. Show any non trivial solution $u(x)$ has infinitely many zeros and distance between two consecutive zeros is at most $\pi/m$.

    **Solution:** Compare $u(x)$ with $\sin mx$ which is a solution of $v'' + m^2 v = 0$. By Sturm comparison theorem, between two consecutive zeros of $v(x) = \sin(mx)$ there is a zero of $u(x)$. Hence $u(x)$ has infinitely many zero in $(x_0, \infty)$.

    Let $u(a) = 0$. We will show that $u(x)$ has a zero in $(a, a + \pi/m]$. Consider $v(x) = \sin(mx - ma)$ which is a solution of $v'' + m^2 v = 0$. Clearly $v(a) = v(a + \pi/m) = 0$. Hence by Sturm comparison theorem, there exists at least one zero of $u(x)$ in $(a, a + \pi/m)$. Hence distance between two consecutive zeros of $u(x)$ is at most $\pi/m$.

16. Consider $u'' + q(x)u = 0$ on an interval $I = (0, \infty)$ with $q(x) \leq m^2$ for all $t \in I$. Show that distance between two consecutive zeros is at least $\pi/m$.

    **Solution:**

    Suppose $u(a) = 0$ and $u(b)$ be two consecutive zeros. Consider $v(x) = \sin(mx - ma)$ which is a solution of $v'' + m^2 v = 0$. By Sturm comparison theorem, there exists a zero of $v(x)$ in $(a, b)$. But we know that $v(a) = 0$ and next zero of $v$ is at $a + \pi/m$. So $b > a + \pi/m$.

17. (**T**) Let $J_p$ be any non-trivial solution of the Bessel equation

    $$x^2 y'' + xy' + (x^2 - p^2)y = 0, \quad x > 0.$$

    Show that (i) If $0 \leq p \leq 1/2$, then every interval of length $\pi$ has at least contains at least one zero of $J_p$.

    (ii) If $p = 1/2$ then distance between consecutive zeros of $J_p$ is exactly $\pi$.

    (iii) If $p > 1/2$ then every interval of length $\pi$ contains at most one zero of $J_p$.

    **Solution:** The normal form of Bessel equation is

    $$u'' + (1 + \frac{1/4 - p^2}{x^2})u = 0.$$

    The zeros of $J_p$ and $u(x)$ are same.

    (i)Apply exercise 15 with $m = 1$.

    (ii) Clear from normal form.

    (iii) Apply exercise 16 with $m = 1$.

18. Let $y(x)$ be a non-trivial solution of $y'' + q(x)y = 0$. Prove that if $q(x) > k/x^2$ for some $k > 1/4$ then $y$ has infinitely many positive zeros. If $q(x) < \frac{1}{4x^2}$ then $y$ has only finitely many positive zeros.

    **Solution:**

Consider the Cauchy-Euler equation $y'' + \frac{ky}{x^2} = 0$. With $x = e^t$, it transforms into $y'' - y' + ky = 0$. So characteristic equation $m^2 - m + k = 0$. So $1 - 4k = 0$ implies two equal real roots and so the solution has finitely many zeros. If $1 - 4k < 0$ then complex conjugate roots and solution look like $x^m \sin(\beta x)$ and it has infinitely many zeros. Rest follows from Sturm comparison theorem.