ODE: Assignment-5

(For calculations of Particular Integrals by operator method, see Simmons books, page 161, section 23 of the chapter Second order linear equations.)

1. Solve: (i) $x^2y'' + 2xy' - 12y = 0$ (ii)(**T**) $x^2y'' + 5xy' + 13y = 0$ (iii) $x^2y'' - xy' + y = 0$

2. (Higher order Cauchy-Euler equations) Let us denote $D = \frac{d}{dx}$ and $\mathcal{D} = \frac{d}{dt}$ where $x = e^t$. Show that

$$xD = \mathcal{D}, \quad x^2D^2 = \mathcal{D}(\mathcal{D}-1), \quad x^3D^3 = \mathcal{D}(\mathcal{D}-1)(\mathcal{D}-2).$$

   Hence conclude that $(x^3D + ax^2D^2 + bxD + c)y = 0, \quad x > 0$ is transformed into constant coefficients ODE $[\mathcal{D}(\mathcal{D}-1)(\mathcal{D}-2) + a\mathcal{D}(\mathcal{D}-1) + b\mathcal{D} + c]y = 0$ by the substitution $x = e^t$.

3. Find a particular solution of each of the following equations by operator methods and hence find its general solution:

   (i) $y'' + 4y = 2\cos^2 x + 10e^x$ (ii)(**T**) $y'' + y = \sin x + (1 + x^2)e^x$

   (**T**) (iii) $y'' - y = e^{-x}(\sin x + \cos x)$ (iv) $y''' - 3y'' - y' + 3y = x^2e^x$

4. Solve $y'' + y' - 2y = e^x$.

5. Solve by using operator method $(D^2 + 9)y = \sin 2x \cos x$.

6. Find a particular integral by operator method: $D^2 - 6D + 9 = 1 + x + x^2$.

7. Find P.I: $y'' + 9y = x\cos x$.

8. (**T**) Solve $x^2y'' - 2xy' - 4y = x^2 + 2\log x, \quad x > 0$.

9. (**T**) (Higher order variation of parameter) Consider the n-th order linear equation

$$y^{(n)} + \sum_1^n a_i(x)y^{(i)} = y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_0(x)y = r(x).$$

   Assume that $y_1, \cdots, y_n$ are n-independent solutions of the associated homogeneous equation. Prove that a particular integral of the given ODE is

$$y_p = \sum v_i y_i \text{ where } v_i' = \frac{R_i}{W}.$$

   Here W is the wronskian of $y_1, \cdots, y_n$ and R_i is the determinant obtained by replacing i-th column of W by $[0, 0, \cdots, 0, r(x)]$.

10. (i) Let $y_1(x), y_2(x)$ are two linearly independent solutions of $y'' + p(x)y' + q(x)y = 0$. Show that $\phi(x) = \alpha y_1(x) + \beta y_2(x)$ and $\psi(x) = \gamma y_1(x) + \delta y_2(x)$ are two linearly independent solutions if and only if $\alpha\delta \neq \beta\gamma$.

    (ii) Show that the zeros of the functions $a\sin x + b\cos x$ and $c\sin x + d\cos x$ are distinct and occur alternately whenever $ad - bc \neq 0$.

11. (**T**) Show that any nontrivial solution $u(x)$ of $u'' + q(x)u = 0,\ q(x) < 0$ for all x, has at most one zero.

12. Let $u(x)$ be any nontrivial solution of $u'' + [1 + q(x)]u = 0$, where $q(x) > 0$. Show that $u(x)$ has infinitely many zeros.

13. Let $u(x)$ be any nontrivial solution of $u'' + q(x)u = 0$ on a closed interval $[a, b]$. Show that $u(x)$ has at most a finite number of zeros in $[a, b]$.

14. (**T**) Let J_p be any non-trivial solution of the Bessel equation

$$x^2y'' + xy' + (x^2 - p^2)y = 0, \quad x > 0.$$

Show that J_p has infinitely many positive zeros.

15. (**T**) Consider $u'' + q(x)u = 0$ on an interval $I = (0, \infty)$ with $q(x) \geq m^2$ for all $t \in I$. Show any non trivial solution $u(x)$ has infinitely many zeros and distance between two consecutive zeros is at most π/m.

16. Consider $u'' + q(x)u = 0$ on an interval $I = (0, \infty)$ with $q(x) \leq m^2$ for all $t \in I$. Show that distance between two consecutive zeros is at least π/m.

17. (**T**) Let J_p be any non-trivial solution of the Bessel equation

$$x^2y'' + xy' + (x^2 - p^2)y = 0, \quad x > 0.$$

Show that (i) If $0 \leq p \leq 1/2$, then every interval of length π has at least contains at least one zero of J_p.

(ii) If $p = 1/2$ then distance between consecutive zeros of J_p is exactly π.

(iii) If $p > 1/2$ then every interval of length π contains at most one zero of J_p.

18. Let $y(x)$ be a non-trivial solution of $y'' + q(x)y = 0$. Prove that if $q(x) > k/x^2$ for some $k > 1/4$ then y has infinitely many positive zeros. If $q(x) < \frac{1}{4x^2}$ then y has only finitely many positive zeros.