MTH102-ODE Assignment-6

1. (**T**) Consider $f(x) = e^{-\frac{1}{x^2}}$ for $x \neq 0$ and $f(0) = 0$. Then:

   (a) Calculate $f', \quad f'', \quad f'''$.

   (b) Prove derivative of $\frac{c}{x^p}e^{-1/x^2}$ consists of sum of terms of similar form. Hence deduce that $f^{(n)}(x)$ consists of sum terms of the form $\frac{c}{x^p}e^{-1/x^2}$ for different $c, p \in \mathbb{N}$.

   (c) Prove that
   $$\lim_{x \to 0} \frac{c}{x^p} e^{-1/x^2} = 0, \quad c, p \in \mathbb{N}.$$

   (d) Deduce that $f^{(n)}(0) = 0$ for all n.

   (e) Thus conclude that f is infinitely differentiable but f is not analytic at 0.

   [Recall: A real valued function is said to be analytic at x_0 if $f(x)$ can be written as a convergent power series $\sum a_n(x - x_0)^n$ on $|x - x_0| < R$ for some $R > 0$. A function is analytic on a domain Ω if it is analytic at each $x_0 \in \Omega$. We know that any analytic function is infinitely differentiable BUT there exists infinitely real differentiable functions which are not analytic.]

2. Prove that if f, g are analytic at x_0 and $g(x_0) \neq 0$ then f/g is analytic at x_0.

3. (**T**)(i) Prove that zeros of an analytic function $f(x)$, which is not identically zero, are isolated points i.e. if x_0 is a zero of $f(x)$ then there exists $\epsilon > 0$ such that $f(x) \neq 0$ for all $0 < |x - x_0| < \epsilon$.

   (**T**)(ii) Deduce that f, g analytic on an interval I and $W(f, g) = 0$ on I then f, g are linearly dependent on I.

   (Compare this with the result we have proved before: if $W(y_1, y_2) = 0$ and they are solution of second order linear homogeneous equation, then y_1, y_2 are linearly dependent.)

4. Locate and classify the singular points in the following:
   (**T**)(i) $x^3(x - 1)y'' - 2(x - 1)y' + 3xy = 0$ (ii) $(3x + 1)xy'' - xy' + 2y = 0$

5. Consider the equation $y'' + y' - xy = 0$.

   (i) Find the power series solutions $y_1(x)$ and $y_2(x)$ such that $y_1(0) = 1, y_1'(0) = 0$ and $y_2(0) = 0, y_2'(0) = 1$.

   (ii) Find the radius of convergence for $y_1(x)$ and $y_2(x)$.

6. (**T**) Consider the equation $(1 + x^2)y'' - 4xy' + 6y = 0$.

   (i) Find its general solution in the form $y = a_0y_1(x) + a_1y_2(x)$, where $y_1(x)$ and $y_2(x)$ are power series.

   (ii) Find the radius of convergence for $y_1(x)$ and $y_2(x)$.

7. Find the first three non zero terms in the power series solution of the IVP

$$y'' - (\sin x)y = 0, \quad y(\pi) = 1, \quad y'(\pi) = 0.$$

8. Using Rodrigues' formula for $P_n(x)$, show that
(T)(i) $P_n(-x) = (-1)^n P_n(x)$ (ii) $P_n'(-x) = (-1)^{n+1} P_n'(x)$
(iii) $\int_{-1}^{1} P_n(x)P_m(x)\,dx = \frac{2}{2n+1}\delta_{mn}$ (iv) $\int_{-1}^{1} x^m P_n(x)\,dx = 0 \quad$ if $n > m$.

9. Expand the following functions in terms of Legendre polynomials over $[-1, 1]$:
(i) $f(x) = x^3 + x + 1$ **(T)**(ii) $f(x) = \begin{cases} 0 & \text{if} \quad -1 \le x < 0 \\ x & \text{if} \quad 0 \le x \le 1 \end{cases}$ (first three nonzero terms)

10. Suppose $m > n$. Show that $\int_{-1}^{1} x^m P_n(x)\,dx = 0$ if $m - n$ is odd. What happens if $m - n$ is even?

11. The function on the left side of

$$\frac{1}{\sqrt{1 - 2xt + t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n$$

is called the generating function of the Legendre polynomial P_n. Assuming this, show that
(T)(i) $(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0$ (ii) $nP_n(x) = xP_n'(x) - P_{n-1}'(x)$
(iii) $P_{n+1}'(x) - xP_n'(x) = (n+1)P_n(x)$; (iv) $P_n(1) = 1, \; P_n(-1) = (-1)^n$
(v) $P_0(0) = 1, \; P_{2n+1}(0) = 0, \; P_{2n}(0) = (-1)^n \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{2^n n!}, \quad n \ge 1$