ODE: Assignment-7

Frobenius method and Bessel function

1. For each of the following, verify that the origin is a regular singular point and find two linearly independent solutions:

   (a) $9x^2y'' + (9x^2+2)y = 0$ (b) $x^2(x^2-1)y'' - x(1+x^2)y' + (1+x^2)y = 0$

   (**T**) (c) $xy'' + (1-2x)y' + (x-1)y = 0$ (d) $x(x-1)y'' + 2(2x-1)y' + 2y = 0$

   Solution:

   (a)

   The given ODE can be written as

$$y'' + \frac{9x^2+2}{9x^2}y = 0$$

Hence $x=0$ is a regular singular point. Let $y = \sum_{n=0} a_n x^{n+r}$, $a_0 \neq 0$. This gives

$$\sum_{n=0} 9(n+r)(n+r-1)a_n x^{n+r} + \sum_{n=0} 9a_n x^{n+r+2} + \sum_{n=0} 2a_n x^{n+r} = 0$$

which can be written as

$$\sum_{n=0} 9(n+r)(n+r-1)a_n x^{n+r} + \sum_{n=2} 9a_{n-2} x^{n+r} + \sum_{n=0} 2a_n x^{n+r} = 0$$

This can be rearranged as (after canceling x^r)

$$\Big(9r(r-1)+2\Big)a_0 + \Big(9r(r+1)+2\Big)a_1 x + \sum_{n=2}\Big(9(n+r)(n+r-1)a_n + 9a_{n-2} + 2a_n\Big)x^n = 0$$

This implies

$$\Big(9r(r-1)+2\Big)a_0 = 0,\ \Big(9r(r+1)+2\Big)a_1 = 0 \quad \text{and} \quad a_n = -\frac{9a_{n-2}}{9(n+r)(n+r-1)+2},\ n \geq 2.$$

Since $a_0 \neq 0$, we have $9r(r-1)+2 = 0 \implies r = 2/3 = r_1$, $r = 1/3 = r_2$. Here $r_1 - r_2 = 1/3$ is not an integer and we have two independent Frobenius series solutions. With $r = r_1$ or $r = r_2$, $9r(r+1)+2 \neq 0 \implies a_1 = 0$. This leads to $a_{2n+1} = 0,\ n \geq 0$. Also,

$$a_n = -\frac{9a_{n-2}}{(3n+3r-2)(3n+3r-1)},\ n \geq 2.$$

With $r = r_1 = 2/3$ we find

$$y_1(x) = x^{2/3}\sum_{n=0} a_{2n}x^{2n}, \qquad a_0 = 1,\ a_{2n} = -\frac{3a_{2n-2}}{2n(6n+1)},\ n \geq 1.$$

With $r = r_1 = 1/3$ we find

$$y_2(x) = x^{1/3}\sum_{n=0} a_{2n}x^{2n}, \qquad a_0 = 1,\ a_{2n} = -\frac{3a_{2n-2}}{2n(6n-1)},\ n \geq 1.$$

(b)

The given ODE can be written as

$$y'' - \frac{1+x^2}{x(x^2-1)}y' + \frac{1+x^2}{x^2(x^2-1)} = 0$$

Hence $x=0$ is a regular singular point. Let $y = \sum_{n=0} a_n x^{n+r},\ a_0 \neq 0$. This gives

$$\sum_{n=0}\Big((n+r)(n+r-1)a_n(x^{n+r+2}-x^{n+r})-(n+r)a_n(x^{n+r}+x^{n+r+2})+a_n(x^{n+r}+x^{n+r+2})\Big) = 0$$

which can be written as

$$\sum_{n=2}\Big((n+r-2)(n+r-3)-(n+r-2)+1\Big)a_{n-2}x^{n+r}-\sum_{n=0}\Big((n+r)(n+r-1)+(n+r)-1\Big)a_n x^{n+r} = 0$$

This can be rearranged as (after canceling x^r)

$$-\Big(r^2-1\Big)a_0 - \Big((r+1)^2-1\Big)a_1 x + \sum_{n=2}\Big((n+r-3)^2 a_{n-2} - \big((n+r)^2-1\big)a_n\Big)x^n = 0$$

This implies

$$\Big(r^2-1\Big)a_0 = 0,\ \Big((r+1)^2-1\Big)a_1 = 0, \quad \text{and} \quad a_n = \frac{(n+r-3)^2}{(n+r)^2-1}a_{n-2},\ n \geq 2.$$

Since $a_0 \neq 0$, we have $r^2 - 1 = 0 \implies r = 1 = r_1,\ r = -1 = r_2$. Here $r_1 - r_2 = 2$ is an integer and we may or may not have two independent Frobenius series solutions.

With $r = r_1$, $(r+1)^2 - 1 \neq 0 \implies a_1 = 0$. Also,

$$a_n = \frac{(n-2)^2}{n(n+2)}a_{n-2},\ n \geq 2 \implies a_n = 0,\ n \geq 1.$$

Hence

$$y_1(x) = x, \qquad a_0 = 1.$$

For the other solution, let $y_2 = y_1 u(x) = xu$ (reduction of order technique)

$$x(x^2-1)u'' + (x^2-3)u' = 0 \implies \frac{u''}{u'} = \frac{1}{1+x} - \frac{1}{1-x} - \frac{3}{x} \implies u' = 1/x^3 - 1/x$$

which integrating again gives

$$u = -\log x - \frac{1}{2x^2}$$

Hence $y_2 = x\ln x + 1/(2x)$ (ignoring the negative sign)

(c)

The given ODE can be written as

$$y'' + \frac{1-2x}{x}y' + \frac{x-1}{x} = 0$$

Hence $x=0$ is a regular singular point. Let $y=\sum_{n=0} a_n x^{n+r},\ a_0\neq 0$. This gives

$$\sum_{n=0}\Big((n+r)(n+r-1)a_n x^{n+r-1}+(n+r)a_n(x^{n+r-1}-2x^{n+r})+a_n(x^{n+r+1}-x^{n+r})\Big)=0$$

which can be written as

$$\sum_{n=2} a_{n-2}x^{n+r-1}-\sum_{n=1}\Big(2(n+r-1)+1\Big)a_{n-1}x^{n+r-1}+\sum_{n=0}\Big((n+r)(n+r-1)+(n+r)\Big)a_n x^{n+r-1}=0$$

This can be rearranged as (after canceling x^{r-1})

$$r^2a_0+\Big((r+1)^2a_1-(2r+1)a_0\Big)x+\sum_{n=2}\Big((n+r)^2a_n-\big(2(n+r-1)+1\big)a_{n-1}+a_{n-2}\Big)x^n=0$$

This implies

$$r^2a_0=0,\ (r+1)^2a_1=(2r+1)a_0,\ (n+r)^2a_n=\big(2(n+r)-1\big)a_{n-1}-a_{n-2},\ n\geq 2$$

Now $a_0\neq 0 \implies r=r_1=0, r=r_2=0$. Since the indicial equation has double roots, the given equation has only one independent Frobenius series solution. We take $r=0$ and this gives $a_1=a_0$. We also have

$$a_n=\frac{2n-1}{n^2}a_{n-1}-\frac{1}{n^2}a_{n-2},\quad n\geq 2.$$

With $a_0=1$ we get $a_1=1$. This leads to $a_2=1/2!$, $a_3=1/3!$. We prove $a_n=1/n!$ by induction. Clearly the induction hypothesis is true for $n=1,2,3$. Let it be true for $n=k$. For $n=k+1$, we have

$$a_{k+1}=\frac{2k+1}{(k+1)^2}a_k-\frac{1}{(k+1)^2}a_{k-1}=\frac{1}{(k+1)^2(k-1)!}\left(\frac{2k+1}{k}-1\right)=\frac{1}{(k+1)!}$$

Hence

$$y_1(x)=\sum_{n=0}\frac{x^n}{n!}=e^x$$

For other solution let $y_2=y_1u(x)=e^xu$. This gives

$$xu''+u'=0 \implies u'=1/x \implies u=\ln x$$

Hence $y_2(x)=e^x\ln x$

(d)

The given ODE can be written as

$$y''+\frac{2(2x-1)}{x(x-1)}y'+\frac{2}{x(x-1)}y=0$$

Hence $x=0$ is a regular singular point. Let $y=\sum_{n=0} a_n x^{n+r},\ a_0\neq 0$. This gives

$$\sum_{n=0}\Big((n+r)(n+r-1)a_n(x^{n+r}-x^{n+r-1})+(n+r)a_n(4x^{n+r}-2x^{n+r-1})+2a_nx^{n+r}\Big)=0$$

which can be written as

$$\sum_{n=1} \Big((n+r-1)(n+r-2)+4(n+r-1)+2\Big)a_{n-1}x^{n+r-1} - \sum_{n=0} \Big((n+r)(n+r-1)+2(n+r)\Big)a_n x^{n+r-1} = 0$$

This can be rearranged as (after canceling x^{r-1})

$$(r^2 + r)a_0 - \sum_{n=1} \Big((n+r)(n+r+1)a_n - \big((n+r-1)(n+r+2)+2\big)a_{n-1}\Big)x^n = 0$$

This implies

$$(r^2 + r)a_0 = 0,\ (n+r)(n+r+1)a_n - \big((n+r-1)(n+r+2)+2\big)a_{n-1} = 0,\ n \geq 1$$

Now $a_0 \neq 0 \implies r = r_1 = 0, r = r_2 = -1$. Hence $r_1 - r_2 = 1$ is an integer and hence the ODE may or may not have two independent Frobenius series solution.

With $r = r_1 = 0$,

$$n(n+1)a_n = \big((n-1)(n+2)+2\big)a_{n-1} \implies a_n = a_{n-1} \implies a_n = a_0,\ n \geq 1.$$

Hence (with $a_0 = 1$)

$$y_1(x) = \sum_{n=0} x^n = \frac{1}{1-x}$$

For the other solution, let $y_2 = y_1 u(x)$. This gives

$$xu'' + 2u' = 0 \implies u' = \frac{1}{x^2} \implies u = -1/x$$

Hence (neglecting the negative sign)

$$y_2(x) = \frac{1}{x(1-x)}$$

We can write

$$y_2(x) = \frac{1}{x} + \frac{1}{1-x}$$

Since the last term is $y_1(x)$, we can take $y_2(x) = 1/x$

Note: If we continue the Frobenius series method with $r = r_2 = -1$, then from the recurrence relation

$$n(n-1)a_n = n(n-1)a_{n-1},\ n \geq 1.$$

For $n = 1$, the relation is automatically satisfied for any value of a_1. We may take $a_1 = 0$. This leads to $a_n = 0$ for $n \geq 1$. Then we again get (taking $a_0 = 1$)

$$y_2(x) = \frac{1}{x}$$

2. Show that $2x^3y'' + (\cos 2x - 1)y' + 2xy = 0$ has only one Frobenius series solution.

   Solution:

We can write the ODE as

$$2x^2y'' + \frac{\cos 2x - 1}{x^2}xy' + 2y = 0$$

Since

$$\lim_{x\to 0} \frac{\cos 2x - 1}{x^2} = -2,$$

the indicial equation is

$$2r(r-1) - 2r + 2 \implies r^2 - 2r + 1 \implies r = 1, 1.$$

Since the indicial equation has double roots, it has only one Frobenius series solution.

3. (**T**) Reduce $x^2y'' + xy' + (x^2 - 1/4)y = 0$ to normal form and hence find its general solution.

Solution:

Suppose $y(x) = u(x)v(x)$. Hence

$$x^2(u''v + 2u'v' + uv'') + x(u'v + uv') + \left(x^2 - \frac{1}{4}\right)uv = 0$$

or

$$x^2vu'' + (2x^2v' + xv)u' + \left((x^2v'' + xv' + \left(x^2 - \frac{1}{4}\right)v\right)u = 0.$$

To make the 2nd term vanish, we set

$$2x^2v' + xv = 0 \implies 2xv' + v = 0 \implies v = \frac{1}{\sqrt{x}}$$

Using this transformation the given ODE reduces to

$$u'' + u = 0.$$

Thus general solution of the reduced equation is $u = A\sin x + B\cos x$. For the original equation, the general solution is

$$y = A\frac{\sin x}{\sqrt{x}} + B\frac{\cos x}{\sqrt{x}}.$$

4. Using recurrence relations, show the following for Bessel function J_n:
(i)(**T**) $J_0''(x) = -J_0(x) + J_1(x)/x$ (ii) $xJ_{n+1}'(x) + (n+1)J_{n+1}(x) = xJ_n(x)$

Solution:

Useful identities for problems with Bessel's functions:

$$\left(x^\nu J_\nu\right)' = x^\nu J_{\nu-1}, \quad \left(x^{-\nu} J_\nu\right)' = -x^{-\nu} J_{\nu+1},$$
$$J_{\nu-1} + J_{\nu+1} = 2\nu J_\nu/x, \quad J_{\nu-1} - J_{\nu+1} = 2J_\nu'.$$

(i)

$$2J_0'(x) = J_{-1}(x) - J_1(x) = -2J_1(x)$$

$$\implies 2J_0''(x) = -2J_1'(x) = J_2(x) - J_0(x) = 2J_1(x)/x - 2J_0(x)$$

$$\implies J_0''(x) = J_1(x)/x - J_0(x)$$

(ii)

$$\left(x^{n+1}J_{n+1}(x)\right)' = x^{n+1}J_n(x) \implies xJ_{n+1}'(x) + (n+1)J_{n+1}(x) = xJ_n(x)$$

5. Express
(i)(**T**) $J_3(x)$ in terms of $J_1(x)$ and $J_0(x)$ (ii) $J_2'(x)$ in terms of $J_1(x)$ and $J_0(x)$
(iii) $J_4(ax)$ in terms of $J_1(ax)$ and $J_0(ax)$

Solution:

(i) Using the identity $J_{\nu+1} = 2\nu J_\nu/x - J_{\nu-1}$ we have

$$\begin{aligned} J_3(x) &= \frac{4}{x}J_2(x) - J_1(x) = \frac{4}{x}\left(\frac{2}{x}J_1(x) - J_0(x)\right) - J_1(x) \\ &= \left(\frac{8}{x^2} - 1\right)J_1(x) - \frac{4}{x}J_0(x) \end{aligned}$$

(ii) Using identities involving Bessel's function, we get

$$2J_2'(x) = J_1(x) - J_3(x) = J_1(x) - \left(\frac{4}{x}J_2(x) - J_1(x)\right) = 2J_1(x) - \frac{4}{x}\left(\frac{2}{x}J_1(x) - J_0(x)\right)$$

$$\text{Hence} \quad J_2'(x) = \frac{2}{x}J_0(x) + \left(1 - \frac{4}{x^2}\right)J_1(x)$$

(iii) Using the identity $J_{\nu+1} = 2\nu J_\nu/x - J_{\nu-1}$, we get

$$\begin{aligned} J_4(ax) &= \frac{6}{ax}J_3(ax) - J_2(ax) = \frac{6}{ax}\left(\frac{4}{ax}J_2(ax) - J_1(ax)\right) - J_2(ax) \\ &= \left(\frac{24}{a^2x^2} - 1\right)J_2(ax) - \frac{6}{ax}J_1(ax) \\ &= \left(\frac{24}{a^2x^2} - 1\right)\left(\frac{2}{ax}J_1(ax) - J_0(ax)\right) - \frac{6}{ax}J_1(ax) \\ &= \frac{1}{ax}\left(\frac{48}{a^2x^2} - 8\right)J_1(ax) - \left(\frac{24}{a^2x^2} - 1\right)J_0(ax) \end{aligned}$$

6. Prove that between each pair of consecutive positive zeros of Bessel function $J_\nu(x)$, there is exactly one zero of $J_{\nu+1}(x)$ and vice versa.

Solution:

Let α and β be two consecutive positive zeros of $J_{\nu+1}$. Let $f(x) = x^{\nu+1}J_{\nu+1}$. Then $f(\alpha) = f(\beta) = 0$. Thus there exists $c \in (\alpha, \beta)$ such that $f'(c) = 0$. Taking $\gamma = \nu + 1$ in $[x^\gamma J_\gamma]' = x^\gamma J_{\gamma-1}$, we see that $J_\nu(c) = 0$. Thus there exists a zero of J_ν between consecutive zeros of $J_{\nu+1}$. Similarly taking $\gamma = \nu$ in $[x^{-\gamma} J_\gamma]' = -x^{-\gamma} J_{\gamma+1}$, we conclude that there exists a zero of $J_{\nu+1}$ between consecutive positive zeros of J_ν. To prove uniqueness, let there exist two zero of J_ν between consecutive zeros α and β of $J_{\nu+1}$. This implies that there exist a zero of $J_{\nu+1}$ between α and β, which contradicts the fact that α and β are consecutive zeroes.

7. Show that the Bessel functions J_ν ($\nu \geq 0$) satisfy

$$\int_0^1 x J_\nu(\lambda_m x) J_\nu(\lambda_n x)\, dx = \frac{1}{2} J_{\nu+1}^2(\lambda_n)\delta_{mn},$$

where λ_i are the positive zeros of J_ν.

Solution:

We know that $y(t) = J_\nu(t)$ satisfies

$$\ddot{y} + \frac{1}{t}\dot{y} + \left(1 - \frac{\nu^2}{t^2}\right) y = 0, \qquad \cdot \equiv \frac{d}{dt}$$

Let $t = \lambda x \implies y(t) = y(\lambda x) = u(x)$. Then $u'(x) = \lambda \dot{y}$ and $u''(x) = \lambda^2 \ddot{y}$. Hence $u(x) = J_\nu(\lambda x)$ satisfies

$$u'' + \frac{1}{x}u' + \left(\lambda^2 - \frac{\nu^2}{x^2}\right) u = 0, \tag{1}$$

Similarly, $v(x) = J_\nu(\mu x)$ satisfies

$$v'' + \frac{1}{x}v' + \left(\mu^2 - \frac{\nu^2}{x^2}\right) v = 0. \tag{2}$$

Multiplying (1) by v and (2) by u and subtracting, we find

$$\frac{d}{dx}\Big[x(u'v - uv')\Big] = \left(\mu^2 - \lambda^2\right) xuv.$$

Integrating from $x = 0$ to $x = 1$, we find

$$\left(\mu^2 - \lambda^2\right) \int_0^1 xuv\, dx = u'(1)v(1) - u(1)v'(1). \tag{3}$$

Now $u(1) = J_\nu(\lambda)$ and $v(1) = J_\nu(\mu)$. Let us choose $\lambda = \lambda_m$ and $\mu = \lambda_n$, where λ_m and λ_n are positive zeros of J_ν. Then $u(1) = v(1) = 0$ and thus find

$$(\lambda_n^2 - \lambda_m^2) \int_0^1 x J_\nu(\lambda_m x) J_\nu(\lambda_n x)\, dx = 0.$$

If $n \neq m$, then

$$\int_0^1 x J_\nu(\lambda_m x) J_\nu(\lambda_n x)\, dx = 0.$$

Now from (3), we find [since $u'(x) = \lambda J_\nu'(\lambda x)$ etc]

$$\begin{aligned}\int_0^1 x J_\nu^2(\lambda x)\, dx &= \lim_{\mu \to \lambda} \frac{\lambda J_\nu'(\lambda) J_\nu(\mu) - \mu J_\nu(\lambda) J_\nu'(\mu)}{\mu^2 - \lambda^2} \\ &= \frac{\lambda\big(J_\nu'(\lambda)\big)^2 - J_\nu(\lambda)J_\nu'(\lambda) - \lambda J_\nu(\lambda) J_\nu''(\lambda)}{2\lambda}\end{aligned}$$

Now if we take $\lambda = \lambda_n$, where λ_n is a positive zero of J_ν, then we find

$$\int_0^1 x J_\nu^2(\lambda_n x)\, dx = \frac{1}{2}\Big(J_\nu'(\lambda_n)\Big)^2.$$

Now from

$$\Big(x^{-\nu} J_\nu(x)\Big)' = -x^{-\nu} J_{\nu+1}(x) \implies J_\nu'(x) - \frac{\nu}{x} J_\nu(x) = -J_{\nu+1}(x),$$

we find by substituting $x = \lambda_n$

$$J_\nu'(\lambda_n) = -J_{\nu+1}(\lambda_n).$$

Thus, finally we get

$$\int_0^1 x J_\nu^2(\lambda_n x)\, dx = \frac{1}{2} J_{\nu+1}^2(\lambda_n).$$

Laplace Transform

1. Let $F(s)$ be the Laplace transform of $f(t)$. Find the Laplace transform of $f(at) \quad (a > 0)$.

   Solution:

$$\mathcal{L}\big(f(at)\big) = \int_0^\infty e^{-st} f(at)\, dt = \frac{1}{a}\int_0^\infty e^{-(s/a)\tau} f(\tau)\, d\tau = \frac{1}{a} F(s/a)$$

2. Find the Laplace transforms:

   (a) $[t]$ (greatest integer function), (b) $t^m \cosh bt \quad (m \in$ non-negative integers),

   (**T**)(c) $e^t \sin at$, (d) $\dfrac{e^t \sin at}{t}$, (e) $\dfrac{\sin t \cosh t}{t}$, (f) $f(t) = \begin{cases} \sin 3t, & 0 < t < \pi, \\ 0, & t > \pi, \end{cases}$

   Solution:

   (a)

$$\begin{aligned} \mathcal{L}\big([t]\big) &= \int_1^2 e^{-st}\, dt + 2\int_2^3 e^{-st}\, dt + 3\int_3^4 e^{-st}\, dt + \cdots \\ &= \frac{e^{-s}}{s}(1 + e^{-s} + e^{-2s} + e^{-3s} + \cdots) = \frac{e^{-s}}{s(1-e^{-s})} \quad (s > 0 \implies 0 < e^{-s} < 1) \end{aligned}$$

   (b)

$$\begin{aligned} \mathcal{L}(t^m) &= \frac{m!}{s^{m+1}} \implies \mathcal{L}(t^m \cosh bt) = \frac{1}{2}\mathcal{L}(e^{bt}t^m + e^{-bt}t^m) \\ &= \frac{m!}{2}\left(\frac{1}{(s-b)^{m+1}} + \frac{1}{(s+b)^{m+1}}\right) \end{aligned}$$

   (c)

$$\mathcal{L}(\sin at) = \frac{a}{s^2 + a^2} \implies \mathcal{L}(e^t \sin at) = \frac{a}{(s-1)^2 + a^2}$$

(d) Use $\mathcal{L}\big(f(t)/t\big) = \int_s^\infty F(s)\,ds$. Now

$$\begin{aligned}
\mathcal{L}(\sin at) &= \frac{a}{s^2+a^2} \\
\implies \mathcal{L}\left(\frac{\sin at}{t}\right) &= a\int_s^\infty \frac{ds}{s^2+a^2} = \frac{\pi}{2} - \tan^{-1}(s/a) \\
\implies \mathcal{L}\left(\frac{e^t \sin at}{t}\right) &= \frac{\pi}{2} - \tan^{-1}\left(\frac{s-1}{a}\right)
\end{aligned}$$

(e) Using result of the previous question

$$\mathcal{L}\left(\frac{\sin t}{t}\right) = \frac{\pi}{2} - \tan^{-1}(s) \implies \mathcal{L}\left(\frac{\cosh t \sin t}{t}\right) = \frac{1}{2}\left(\frac{e^t \sin t}{t} + \frac{e^{-t}\sin t}{t}\right)$$

$$= \frac{1}{2}\left(\pi - \tan^{-1}(s-1) - \tan^{-1}(s+1)\right)$$

(f)

$$\mathcal{L}\big(f(t)\big) = \int_0^\infty e^{-st} f(t)\,dt = \int_0^\pi e^{-st}\sin 3t\,dt = \frac{3(1+e^{-\pi s})}{s^2+9}$$

1. Find the Laplace transforms (Hint: use second shifting theorem):

(a) $f(t) = \begin{cases} 1, & 0 < t < \pi, \\ 0, & \pi < t < 2\pi, \\ \cos t, & t > 2\pi, \end{cases}$

(b) $f(t) = \begin{cases} 0, & 0 < t < 1, \\ \cos(\pi t), & 1 < t < 2, \\ 0, & t > 2 \end{cases}$

Solution:

(a) Consider $g(t) = u(t) - u(t-\pi) + u(t-2\pi)\cos t = u(t) - u(t-\pi) + u(t-2\pi)\cos(t-2\pi)$

$$\mathcal{L}\big(f(t)\big) = \mathcal{L}\big(g(t)\big) = \frac{1}{s} - e^{-\pi s}\frac{1}{s} + e^{-2\pi s}\frac{s}{s^2+1}$$

(b) Consider $g(t) = \big(u(t-1) - u(t-2)\big)\cos(\pi t) = -u(t-1)\cos\pi(t-1) - u(t-2)\cos\pi(t-2)$

$$\mathcal{L}\big(f(t)\big) = \mathcal{L}\big(g(t)\big) = -\left(e^{-s}\frac{s}{s^2+\pi^2} + e^{-2s}\frac{s}{s^2+\pi^2}\right)$$

2. Find the inverse Laplace transforms of

(a) $\tan^{-1}(a/s)$, (b)$\ln\dfrac{s^2+1}{(s+1)^2}$, (**T**)(c)$\dfrac{s+2}{(s^2+4s-5)^2}$, (d) $\dfrac{se^{-\pi s}}{s^2+4}$, (e) $\dfrac{(1-e^{-2s})(1-3e^{-2s})}{s^2}$.

Solution:

(a) Use $\mathcal{L}\big(-tf(t)\big) = F'(s)$. Thus,

$$F'(s) = -\frac{a}{s^2+a^2} \implies \mathcal{L}^{-1}\big(F'(s)\big) = -\sin at \implies f(t) = \frac{\sin at}{t}$$

(b)

$$F'(s) = \frac{2s}{s^2+1} - \frac{2}{s+1} \implies \mathcal{L}^{-1}\big(F'(s)\big) = 2(\cos t - e^{-t}) \implies f(t) = \frac{2(e^{-t} - \cos t)}{t}$$

(c)

$$F(s) = \frac{s+2}{(s^2+4s-5)^2} = \frac{1}{12}\left(\frac{1}{(s-1)^2} - \frac{1}{(s+5)^2}\right)$$

$$F'(s) = \frac{1}{12}\left(\frac{2}{(s+5)^3} - \frac{2}{(s-1)^3}\right) \implies \mathcal{L}^{-1}\big(F'(s)\big) = \frac{t^2e^{-5t} - t^2e^t}{12}$$

Thus,

$$f(t) = t\frac{e^t - e^{-5t}}{12}$$

(d)

$$\frac{se^{-\pi s}}{s^2+4} = e^{-\pi s}\mathcal{L}(\cos 2t) = \mathcal{L}\big(u(t-\pi)\cos 2(t-\pi)\big)$$

Thus,

$$\mathcal{L}^{-1}\left(\frac{se^{-\pi s}}{s^2+4}\right) = u(t-\pi)\cos 2t$$

(e)

$$\frac{(1-e^{-2s})(1-3e^{-2s})}{s^2} = \frac{1}{s^2} - \frac{4e^{-2s}}{s^2} + \frac{3e^{-4s}}{s^2}$$

Thus,

$$f(t) = t - 4u(t-2)(t-2) + 3(t-4)u(t-4)$$

3. Using convolution, find the inverse Laplace transforms:

(**T**)(a) $\frac{1}{s^2-5s+6}$, (b) $\frac{2}{s^2-1}$, (c) $\frac{1}{s^2(s^2+4)}$, (d) $\frac{1}{(s-1)^2}$.

Solution:

(a)

$$F(s) = \frac{1}{s^2-5s+6} = \frac{1}{(s-3)(s-2)}$$

Now

$$\mathcal{L}(e^{3t}) = \frac{1}{s-3}, \quad \mathcal{L}(e^{2t}) = \frac{1}{s-2}.$$

Hence,

$$f(t) = \int_0^t e^{3\tau}e^{2(t-\tau)}d\tau = e^{2t}\int_0^t e^{\tau}\,d\tau = e^{3t} - e^{2t}$$

(b)

$$F(s) = \frac{2}{s^2-1} = \frac{2}{(s+1)(s-1)}$$

Now

$$\mathcal{L}(e^{t}) = \frac{1}{s-1}, \quad \mathcal{L}(e^{-t}) = \frac{1}{s+1}.$$

Hence,

$$f(t) = 2\int_0^t e^{\tau}e^{-(t-\tau)}d\tau = 2e^{-t}\int_0^t e^{2\tau}d\tau = e^t - e^{-t} = 2\sinh t$$

(c)

$$F(s) = \frac{1}{s^2(s^2+4)} = \frac{1}{2}\frac{1}{s^2}\frac{2}{s^2+4}$$

Now

$$\mathcal{L}(t) = \frac{1}{s^2}, \quad \mathcal{L}(\sin 2t) = \frac{2}{s^2+4}.$$

Hence,

$$f(t) = \frac{1}{2}\int_0^t (t-\tau)\sin(2\tau)d\tau = \frac{2t - \sin 2t}{8}$$

(d)

$$F(s) = \frac{1}{(s-1)^2} = \frac{1}{s-1}\frac{1}{s-1}$$

Now

$$\mathcal{L}(e^t) = \frac{1}{s-1}.$$

Hence,

$$f(t) = \int_0^t e^{\tau}e^{t-\tau}d\tau = e^t\int_0^t d\tau = te^t$$

6. Use Laplace transform to solve the initial value problems:

(a) $y'' + 4y = \cos 2t, \qquad y(0) = 0,\ y'(0) = 1.$

(**T**)(b) $y'' + 3y' + 2y = \begin{cases} 4t & \text{if } 0 < t < 1 \\ 8 & \text{if } t > 1 \end{cases} \qquad y(0) = y'(0) = 0$

(c) $y'' + 9y = \begin{cases} 8\sin t & \text{if } 0 < t < \pi \\ 0 & \text{if } t > \pi \end{cases} \qquad y(0) = 0, y'(0) = 4$

(d) $y_1' + 2y_1 + 6\int_0^t y_2(\tau)\,d\tau = 2u(t), \quad y_1' + y_2' = -y_2, \qquad y_1(0) = -5, y_2(0) = 6$

Solution:

(a) Taking Laplace Transform on both sides and simplifying (Y(s)=$\mathcal{L}$[y(t)])

$$Y(s) = s/(s^2+4)^2 + 1/(s^2+4)$$

Using convolution [or any other technique]

$$\begin{aligned} y(t) &= \frac{1}{2}\int_0^t \sin(2\tau)\cos(2(t-\tau))d\tau + \frac{\sin 2t}{2} \\ &= \frac{t\sin 2t}{4} + \frac{\sin 2t}{2} \end{aligned}$$

(b) Let $r(t) = 4\big(u(t) - u(t-1)\big)t + 8u(t-1) = 4u(t-0)t + 4u(t-1)(1-(t-1))$. Taking Laplace Transform on both sides of the ODE, we get

$$(s^2+3s+2)Y(s) = R(s) \implies Y(s) = \frac{4}{s^2(s+1)(s+2)} + e^{-s}\frac{4(s-1)}{s^2(s+1)(s+2)}$$

Using partial fraction and shifting theorem we get

$$Y(s) = \left(-\frac{3}{s} + \frac{2}{s^2} + \frac{4}{s+1} - \frac{1}{s+2}\right) + e^{-s}\left(\frac{5}{s} - \frac{2}{s^2} - \frac{8}{s+1} + \frac{3}{s+2}\right)$$

$$\implies y(t) = -3 + 2t + 4e^{-t} - e^{-2t} + u(t-1)\Big(5 - 2(t-1) - 8e^{-(t-1)} + 3e^{-2(t-1)}\Big)$$

(c) Let $r(t) = 8(u(t) - u(t-\pi))\sin t = 8u(t)\sin t + u(t-\pi)\sin(t-\pi)$. Taking Laplace Transform on both sides of the ODE, we get

$$(s^2+9)Y(s) = R(s) + 4 \implies Y(s) = \frac{4}{s^2+9} + \frac{R(s)}{s^2+9}$$

We can explicitly write $R(s)$ and then use partial fraction technique.

$$Y(s) = \frac{4}{s^2+9} + (1+e^{-\pi s})\frac{8}{(s^2+1)(s^2+9)} = \frac{4}{s^2+9} + (1+e^{-\pi s})\left(\frac{1}{s^2+1} - \frac{1}{s^2+9}\right)$$

This gives

$$\begin{aligned} y(t) &= \frac{4}{3}\sin 3t + \left(\sin t - \frac{1}{3}\sin 3t\right) + u(t-\pi)\left(\sin(t-\pi) - \frac{1}{3}\sin 3(t-\pi)\right) \\ &= \sin t + \sin 3t + u(t-\pi)\left(\frac{1}{3}\sin 3t - \sin t\right) \end{aligned}$$

(Otherwise, use convolution as follows

$$y(t) = \frac{4}{3}\sin 3t + \frac{1}{3}\int_0^t r(\tau)\sin 3(t-\tau)\,d\tau$$

Thus for $0 < t < \pi$, we get

$$y(t) = \frac{4}{3}\sin 3t + \frac{8}{3}\int_0^t \sin\tau\sin 3(t-\tau)\,d\tau = \frac{4}{3}\sin 3t + \sin t - \frac{1}{3}\sin 3t = \sin 3t + \sin t$$

and for $t > \pi$, we get [since $r(t) = 0$]

$$y(t) = \frac{4}{3}\sin 3t + \frac{8}{3}\int_0^\pi \sin\tau\sin 3(t-\tau)\,d\tau + \frac{1}{3}\int_\pi^t 0\,\sin 3(t-\tau)\,d\tau = \frac{4}{3}\sin 3t$$

This solution matches with that obtained earlier.)

(d) Taking Laplace transform, we get

$$\begin{aligned} (s+2)Y_1 + \frac{6Y_2}{s} &= \frac{2}{s} - 5 \\ sY_1 + (s+1)Y_2 &= 1 \end{aligned}$$

Solving

$$\begin{aligned} Y_1(s) &= \frac{1}{s} - \frac{12}{5}\frac{1}{s-1} - \frac{18}{5}\frac{1}{s+4} \\ Y_2(s) &= \frac{6}{5}\frac{1}{s-1} + \frac{24}{5}\frac{1}{s+4} \end{aligned}$$

Thus,

$$\begin{aligned} y_1(t) &= 1 - \frac{12}{5}e^t - \frac{18}{5}e^{-4t} \\ y_2(t) &= \frac{6}{5}e^t + \frac{24}{5}e^{-4t} \end{aligned}$$

7. Solve the integral equations:

(a) $y(t) + \int_0^t y(\tau)\, d\tau = u(t-a) + u(t-b)$

(b) $e^{-t} = y(t) + 2\int_0^t \cos(t-\tau)y(\tau)\, d\tau$

(c) $3\sin 2t = y(t) + \int_0^t (t-\tau)y(\tau)\, d\tau$

Solution:

(a) Taking Laplace Transform, we get

$$Y(s) = \frac{e^{-as}}{s+1} + \frac{e^{-bs}}{s+1} \implies y(t) = u(t-a)e^{-(t-a)} + u(t-b)e^{-(t-b)}$$

(b) Taking Laplace Transform, we get

$$Y(s) = \frac{s^2+1}{(s+1)^3} = \frac{1}{1+s} - \frac{2}{(s+1)^2} + \frac{2}{(s+1)^3}$$

Thus,

$$y(t) = e^{-t}(t-1)^2$$

(c) Taking Laplace Transform, we get

$$Y(s) = -\frac{2}{s^2+1} + \frac{8}{(s^2+4)} \implies y(t) = -2\sin t + 4\sin 2t$$