ODE: Assignment-7

Frobenius method and Bessel function

1. For each of the following, verify that the origin is a regular singular point and find two linearly independent solutions:
   (a) $9x^2y'' + (9x^2 + 2)y = 0$ (b) $x^2(x^2 - 1)y'' - x(1 + x^2)y' + (1 + x^2)y = 0$
   (**T**) (c) $xy'' + (1 - 2x)y' + (x - 1)y = 0$ (d) $x(x - 1)y'' + 2(2x - 1)y' + 2y = 0$

2. Show that $2x^3y'' + (\cos 2x - 1)y' + 2xy = 0$ has only one Frobenius series solution.

3. (**T**) Reduce $x^2y'' + xy' + (x^2 - 1/4)y = 0$ to normal form and hence find its general solution.

4. Using recurrence relations, show the following for Bessel function J_n:
   (i)(**T**) $J_0''(x) = -J_0(x) + J_1(x)/x$ (ii) $xJ_{n+1}'(x) + (n + 1)J_{n+1}(x) = xJ_n(x)$

5. Express
   (i)(**T**) $J_3(x)$ in terms of $J_1(x)$ and $J_0(x)$ (ii) $J_2'(x)$ in terms of $J_1(x)$ and $J_0(x)$
   (iii) $J_4(ax)$ in terms of $J_1(ax)$ and $J_0(ax)$

6. Prove that between each pair of consecutive positive zeros of Bessel function $J_\nu(x)$, there is exactly one zero of $J_{\nu+1}(x)$ and vice versa.

7. Show that the Bessel functions J_ν $(\nu \geq 0)$ satisfy

$$\int_0^1 xJ_\nu(\lambda_m x)J_\nu(\lambda_n x)\,dx = \frac{1}{2}J_{\nu+1}^2(\lambda_n)\delta_{mn},$$

   where λ_i are the positive zeros of J_ν.

Laplace Transform

1. Let $F(s)$ be the Laplace transform of $f(t)$. Find the Laplace transform of $f(at)$ $(a > 0)$.

2. Find the Laplace transforms:
   (a) $[t]$ (greatest integer function), (b) $t^m \cosh bt$ ($m \in$ non-negative integers),
   (**T**)(c) $e^t \sin at$, (d) $\dfrac{e^t \sin at}{t}$, (e) $\dfrac{\sin t \cosh t}{t}$, (f) $f(t) = \begin{cases} \sin 3t, & 0 < t < \pi, \\ 0, & t > \pi, \end{cases}$

1. Find the Laplace transforms (Hint: use second shifting theorem):
   (a) $f(t) = \begin{cases} 1, & 0 < t < \pi, \\ 0, & \pi < t < 2\pi, \\ \cos t, & t > 2\pi, \end{cases}$

(b) $f(t) = \begin{cases} 0, & 0 < t < 1, \\ \cos(\pi t), & 1 < t < 2, \\ 0, & t > 2 \end{cases}$

2. Find the inverse Laplace transforms of

   (a) $\tan^{-1}(a/s)$, (b)$\ln \dfrac{s^2+1}{(s+1)^2}$, (**T**)(c)$\dfrac{s+2}{(s^2+4s-5)^2}$, (d) $\dfrac{se^{-\pi s}}{s^2+4}$, (e) $\dfrac{(1-e^{-2s})(1-3e^{-2s})}{s^2}$.

3. Using convolution, find the inverse Laplace transforms:

   (**T**)(a) $\dfrac{1}{s^2-5s+6}$, (b) $\dfrac{2}{s^2-1}$, (c) $\dfrac{1}{s^2(s^2+4)}$, (d) $\dfrac{1}{(s-1)^2}$.

6. Use Laplace transform to solve the initial value problems:

   (a) $y'' + 4y = \cos 2t, \quad y(0) = 0,\ y'(0) = 1.$

   (**T**)(b) $y'' + 3y' + 2y = \begin{cases} 4t & \text{if } 0 < t < 1 \\ 8 & \text{if } t > 1 \end{cases} \quad y(0) = y'(0) = 0$

   (c) $y'' + 9y = \begin{cases} 8\sin t & \text{if } 0 < t < \pi \\ 0 & \text{if } t > \pi \end{cases} \quad y(0) = 0, y'(0) = 4$

   (d) $y_1' + 2y_1 + 6\int_0^t y_2(\tau)\,d\tau = 2u(t), \quad y_1' + y_2' = -y_2, \qquad y_1(0) = -5, y_2(0) = 6$

7. Solve the integral equations:

   (a) $y(t) + \int_0^t y(\tau)\,d\tau = u(t-a) + u(t-b)$

   (b) $e^{-t} = y(t) + 2\int_0^t \cos(t-\tau)y(\tau)\,d\tau$

   (c) $3\sin 2t = y(t) + \int_0^t (t-\tau)y(\tau)\,d\tau$